A Crank-Nicholson domain decomposition method for optimal control problem of parabolic partial differential equation

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Abstract A parallel domain decomposition algorithm is considered for solving an optimal control problem governed by a parabolic partial differential equation. The proposed algorithm relies on non-iterative and non-overlapping domain decomposition, which uses some implicit sub-domain problems and explicit flux approximations at each time step in every iteration. In addition, outer iterations are introduced to achieve the parallelism. Numerical experiments are supplied to show the efficiency of our proposed method.

1 Introduction

In [1], Dawson and Dupont presented non-overlapping domain decomposition schemes to solve parabolic equation by some explicit flux exchange on inner boundaries and implicit conservative Galerkin procedures in each sub-domain. Here, explicit flux prediction are simple to compute for the unit outward normal vector (see definition in Section 2). A time step limitation, which is less severe than that of a fully explicit method, is induced to maintain stability because of the explicit prediction. Recently, an improved strategy was considered in [2] to avoid the loss of $H^{-\frac{1}{2}}$ factor for space variable in the work of Dawson and Dupont. We would like to mention that another two calculation methods on inner boundaries were studied by Ma and Sun (see [6] and sequent research papers) based on the integral mean value or extrapolation. In previous work [3], we have shown that explict/implict domain

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decomposition method in [2] could be applied in optimal control problems governed by partial differential equations. The main goal of this paper is to develop the corresponding results for second order procedures based on the analysis and schemes designed to solve single PDE in [4].

2 Model problem and optimality condition

We consider the following distributed convex optimal control problems

$$\min_{u\in\mathscr{K}}\left\{\int_0^T \left(\|y-y_d\|^2_{L^2(\Omega)}+\|u\|^2_{L^2(\Omega)}\right)dt\right\} \tag{1}$$

subject to

$$\begin{cases} \partial_t y-\Delta y=f+u, & \text{in } \Omega, \quad 0<t\le T;\\ y=0, & \text{on } \partial\Omega, \quad 0<t\le T;\\ y=y_0, & \text{in } \Omega, \quad t=0, \end{cases} \tag{2}$$

where $u\in\mathscr{K}$ is the control and $\mathscr{K}$ is a convex admissible set for control, y is the state variable, y_d is the observation, y_0 is the initial function. Fix $V=H_0^1(\Omega)$ and $U=L^2(\Omega)$. In the following, we will write state space $\mathscr{W}=\{y\in L^2(0,T;V);\ y_t\in L^2(0,T;\ H^{-1}(\Omega))\}$ and the control space $\mathscr{U}=L^2(0,T;U)$. In addition, K is a closed convex set in U and $\mathscr{K}=L^2(0,T;K)$ is a closed convex set in the space $\mathscr{U}$.

2.1 Optimality Condition and discretization

We use standard notation for Sobolev spaces. Define $A(u,v):V\times V\to\mathbb{R}$ to be a bilinear form satisfying

$$A(u,v)=(\nabla u,\nabla v)\quad \forall\ u,v\in V. \tag{3}$$

Then the optimal control problem can be transformed into optimality condition in the following lemma:

Theorem 1. *A pair (y,u) in $\mathscr{W}\times\mathscr{K}$ is the solution of* (1)-(2) *if and only if there is a co-state $p\in\mathscr{W}$ such that the triplet (y,p,u) in $\mathscr{W}\times\mathscr{W}\times\mathscr{K}$ satisfies the following optimality conditions:*

$$\begin{cases} (\partial_t y,w)+A(y,w)=(f+u,w),\ \ \forall\ w\in V;\\ y|_{t=0}=y_0; \end{cases} \tag{4}$$

$$\begin{cases} -(\partial_t p,q)+A(q,p)=(y-y_d,q),\ \ \forall\ q\in V;\\ p|_{t=T}=0; \end{cases} \tag{5}$$

$$\int_0^T (u+p, v-u) \geq 0,\ \ \forall\, v \in \mathscr{K}. \tag{6}$$

Here only the case $K = \{u \geq 0\}$ are considered. Therefore, the third inequality in the optimality conditions is equivalent to

$$(u+p, v-u) \geq 0,\ \ \forall\, v \in K,\ \ 0 \leq t \leq T. \tag{7}$$

In general, for time-dependent optimal control problems, optimality condition, which is a large scale of nonlinear coupled system with respect to time and spacial variables, contains forward and backward PDEs with the variational inequality under consideration. It is very difficult and challenging to solve directly this non-linear system. Domain decomposition method, which could save huge time in calculation by solving the question at the same time, is especially suitable for this kind of complicated problem. To use domain decomposition method, we divide Ω into many non-overlapping sub-domains $\{\Omega_i\}_{i=1}^I$ such that $\bar{\Omega} = \bigcup_{i=1}^I \bar{\Omega}_i$. Set $\Gamma_i = \partial\Omega_i \backslash \partial\Omega$ and $\Gamma = \bigcup_{i=1}^I \Gamma_i$, which is the set of inner boundaries of sub-domains. We recall some definitions which are necessary for deriving the discrete form of (4)-(6). Introduce

$$\phi(x) = \begin{cases} (x-2)/12, & 1 \leq x \leq 2, \\ -5x/4 + 7/6, & 0 \leq x \leq 1, \\ 5x/4 + 7/6, & -1 \leq x \leq 0, \\ -(x+2)/12, & -2 \leq x \leq -1, \\ 0. & |x| > 2. \end{cases}$$

For some $H > 0$, define

$$\phi(\tau) = H^{-1}\varphi\left(\frac{\tau}{H}\right),\ \tau \in \mathbb{R}^1.$$

where H is the width of the local averaging interval, which plays an important role for stability of explicit/implicit scheme. Following Dawson-Dupont's idea, we do not use the exact normal derivative along inner boundaries. A proper approximation is (see [1, 2]):

$$B(\psi)(\boldsymbol{x}) = -\int_{-2H}^{2H} \phi'(\tau)\psi(\boldsymbol{x} + \tau \boldsymbol{n}_\Gamma)\mathrm{d}\tau,\ \ x \in \Gamma_i \cap \Gamma_j,\ \ 1 \leq i < j \leq I. \tag{8}$$

From definitions above, we note that function v has a well-defined jump

$$[v](\boldsymbol{x}) = v(\boldsymbol{x}^+) - v(\boldsymbol{x}^-), \quad \forall\, \boldsymbol{x} \text{ on } \Gamma \tag{9}$$

where

$$v(\boldsymbol{x}^\pm) \triangleq \lim_{t \to 0^\pm} v(\boldsymbol{x} + t\boldsymbol{v}_\Gamma) \tag{10}$$

Make a time partition: $0 = t^0 < t^1 < \cdots < t^N = T$ and set $\Delta t^n = t^n - t^{n-1}$ and $\Delta t = \max\limits_{1 \leq n \leq N} \Delta t^n$. For simplicity, we may take $\Delta t^n = \Delta t$ for $n = 1, 2, \ldots, N$. For a given function $g(\boldsymbol{x}, t)$, let $g^n = g(\boldsymbol{x}, t^n)$ and

$$\bar{\partial}_t g^n = \frac{g^n - g^{n-1}}{\Delta t}, \;\; \bar{g}^{n-\frac{1}{2}} = \frac{g^n + g^{n-1}}{2},$$

$$\hat{g}^{n-\frac{1}{2}} = 2\bar{g}^{n-\frac{3}{2}} - \bar{g}^{n-\frac{5}{2}}, \;\; \tilde{g}^{n+\frac{1}{2}} = 2\bar{g}^{n+\frac{3}{2}} - \bar{g}^{n+\frac{5}{2}}.$$

For $i = 1, 2, \ldots, I$, denote $M_i^h \subset V$ be the corresponding continuous piecewise linear finite element space associated with conforming tragualtion $\mathscr{T}_i^h$. Let M^h be the subspace of V such that $w_h \in M^h$ if and only if $w_h|_{\Omega_i} \in M_i^h$ for each $1 \le i \le I$. Similarly, we can define piecewise constant finite element space $U^{h_U} \subset U$ for control variable u. Let $K^{h_U} = K \bigcap U^{h_U}$. Then the discrete form that we want to solve is:

$$\begin{cases} Y^0 = y_0; \quad Y^1 = y_0 + \Delta t(f^0 + \Delta y_0 + U^0); \quad Y^2 = y_0 + 2\Delta t(f^0 + \Delta y_0 + U^0); \\ (\bar{\partial}_t Y^n, V) + A(Y^{n-\frac{1}{2}}, V) - (B(\hat{Y}^{n-\frac{1}{2}}), [V])_\Gamma - (B(V), [\hat{Y}^{n-\frac{1}{2}}])_\Gamma \\ \quad = (\bar{f}^{n-\frac{1}{2}} + \bar{U}^{n-\frac{1}{2}}, V), \;\; \forall\, V \in M^h, \;\; n = 3, 4, \ldots, N; \end{cases} \tag{11}$$

$$\begin{cases} P^N = 0; \quad P^{N-1} = \Delta t(Y^N - y_d^N); \quad P^{N-2} = 2\Delta t(Y^N - y_d^N); \\ -(\bar{\partial}_t P^{n-2}, V) + A(V, \bar{P}^{n-\frac{5}{2}}) - (B(\tilde{P}^{n-\frac{5}{2}}), [V])_\Gamma - (B(V), [\tilde{P}^{n-\frac{5}{2}}])_\Gamma \\ \quad = (\bar{Y}^{n-\frac{5}{2}} - \bar{y}_d^{n-\frac{5}{2}}, V), \;\; \forall\, V \in M^h, \;\; n = N, N-1, \ldots, 3; \end{cases} \tag{12}$$

$$(\bar{U}^{n-\frac{1}{2}} + \bar{P}^{n-\frac{5}{2}}, \bar{Z}^{n-\frac{1}{2}} - \bar{U}^{n-\frac{1}{2}}) \ge 0, \quad \forall\, Z \in K^{h_U}, \; n = 3, 4, \ldots, N; \tag{13}$$

$$U^0 = \max\{0, -P^0\}, \quad U^1 = \max\{0, -P^1\}, \quad U^2 = \max\{0, -P^2\}. \tag{14}$$

We see that the original optimal control problem (4)-(6), which is normally large in size, is now decomposed into a set of subproblems with much smaller sizes. In fact, discrete solution of (11)-(14) does not always exist. One could use contraction mapping principle to ensure the existence and uniqueness of system. Taking the limitation of the length into consideration, we will give a rigorous analysis on this and convergence of the following iterative algorithm in a forthcoming paper [5]. In addition, a priori estimates will also be included.

2.2 *Parallel iterative algorithm*

We note that discrete system (11)-(14) is still a nonlinear system of a forward system for the state variable and a backward system for the co-state variable, which are coupled by the control variable. We introduce outer iterations to decouple the system. Thus, the proposed algorithm could be performed in parallel once domain decomposition is used. Then, fully parallel iterative algorithm is formulated as follows:

PARALLEL DOMAIN DECOMPOSITION ITERATIVE ALGORITHM (PDDIA)

Step 1. *Given initial approximation* $\{U_0^n\}_{n=1}^N \subset U^{h_U}$ *and* $Y^0 \in M^h$*. Take the* $\varepsilon > 0$ *as a tolerance and set* $k := 0$*.*

Step 2. *Update* $\{Y_{k+1}^n\}_{n=0}^N \subset M^h$ *in parallel on each* Ω_i *for* $1 \le i \le I$*:*

$$\begin{cases} Y_{k+1}^0 = Y^0;\ Y_{k+1}^1 = Y_0 + \Delta t(f^0 + \Delta Y^0 + U_k^0);\ Y_{k+1}^2 = Y_0 + 2\Delta t(f^0 + \Delta Y^0 + U_k^0); \\ (\bar{\partial}_t Y_{k+1}^n, V) + A(Y_{k+1}^{n-\frac{1}{2}}, V) - (B(\hat{Y}_{k+1}^{n-\frac{1}{2}}), [V])_\Gamma - (B(V), [\hat{Y}_{k+1}^{n-\frac{1}{2}}])_\Gamma \\ \quad = (\bar{f}^{n-\frac{1}{2}} + \bar{U}_k^{n-\frac{1}{2}}, V), \qquad \forall\, V \in M^h, \quad n = 3, 4, \ldots, N; \end{cases} \tag{15}$$

Step 3. *Update* $\{P_{k+1}^n\}_{n=0}^N \subset M^h$ *in parallel on each* Ω_i *for* $1 \le i \le I$*:*

$$\begin{cases} P_{k+1}^N = 0;\ P_{k+1}^{N-1} = \Delta t(Y^N - y_d^N);\ P_{k+1}^{N-2} = 2\Delta t(Y^N - y_d^N); \\ -(\bar{\partial}_t P_{k+1}^{n-2}, V) + A(V, \bar{P}_{k+1}^{n-\frac{5}{2}}) - (B(\tilde{P}_{k+1}^{n-\frac{5}{2}}), [V])_\Gamma - (B(V), [\tilde{P}_{k+1}^{n-\frac{5}{2}}])_\Gamma \\ \quad = (\bar{Y}_{k+1}^{n-\frac{5}{2}} - \bar{y}_d^{n-\frac{5}{2}}, V), \quad \forall\, V \in M^h, \quad n = N, N-1, \ldots, 3; \end{cases} \tag{16}$$

Step 4. *Update* $\{\bar{U}_{h_U,k+1}^{n-\frac{1}{2}}\}_{n=1}^N \subset U^{h_U}$ *such that*

$$\begin{cases} \bar{U}_{k+\frac{1}{2}}^{n-\frac{1}{2}} = (1-\rho)\bar{U}_k^{n-\frac{1}{2}} - \rho \bar{P}_{k+1}^{n-\frac{5}{2}}, \\ \bar{U}_{k+1}^{n-\frac{1}{2}} = Q^{h_U} \bar{U}_{k+\frac{1}{2}}^{n-\frac{1}{2}}. \end{cases} \qquad n = 3, 4, \ldots, N; \tag{17}$$

where ρ *is a constant with* $0 < \rho < 1$ *and* Q^{h_U} *is the projection from* U^{h_U} *to* K^{h_U}*.*

Define U_{k+1}^0, U_{k+1}^1 *and* U_{k+1}^2 *such that*

$$U_{k+1}^0 = \max\{0, -P_{k+1}^0\}, \quad U_{k+1}^1 = \max\{0, -P_{k+1}^1\}, \quad U_{k+1}^2 = \max\{0, -P_{k+1}^2\}, \tag{18}$$

Step 5. *Compute the iterative error:*

$$\text{eps} = \sum_{n=0}^{N} \big(\|\bar{U}_k^{n-\frac{1}{2}} - \bar{U}_{k+1}^{n-\frac{1}{2}}\|_{L^2(\Omega)} + \|\bar{Y}_k^{n-\frac{1}{2}} - \bar{Y}_{k+1}^{n-\frac{1}{2}}\|_{L^2(\Omega)} + \|\bar{P}_k^{n-\frac{1}{2}} - \bar{P}_{k+1}^{n-\frac{1}{2}}\|_{L^2(\Omega)}$$

If $eps \le \varepsilon$*, then stop the iteration and output*

$$U^n = U_{k+1}^n, \ \ Y^n = Y_{k+1}^n, \ \ P^n = P_{k+1}^n, \quad n = 0, 1, 2, \ldots, N. \tag{19}$$

Else set $k := k+1$ *and return step 2 to restart new iteration.*

Compared to first order scheme proposed in [3], the computation on Γ requires explicitly the value of three-level solutions, while only little computational cost will be added. We also remark that the algorithm PDDIA is fully parallel.

3 Numerical experiments

In this section, we test the performance and convergence of the proposed PDDIA with respect to the exact solutions:

$$\begin{cases} y = \sin(2\pi x)\sin(2\pi y)t, \\ p = \sin(2\pi x)\sin(2\pi y)(T-t), \\ u = \max(-p,0), \\ y_d = y + \dfrac{\partial p}{\partial t} + \Delta p, \\ f = -u + \dfrac{\partial y}{\partial t} - \Delta y. \end{cases}$$

Let $T = 0.5$. Domain $\Omega = [0,2]\times[0,1]$ is partitioned into two uniform non-overlapping areas with the inner-domain boundary are $\Gamma = \{1\}\times[0,1]$. The mesh in the x-axis and y-axis varies uniformly from $1/36$, $1/49$, $1/64$ to $1/81$ in each sub-domain, respectively.

Table 1 $L^2(0,T;L^2(\Omega))$-norm error for PDDIA ($r=1$)

Grids	$y-Y$	order	$u-U$	order	$p-P$	order
36×36	1.625×10^{-3}		7.324×10^{-3}		1.597×10^{-3}	
49×49	8.770×10^{-4}	2.00	5.382×10^{-3}	0.99	8.473×10^{-4}	2.06
64×64	5.294×10^{-4}	1.89	4.153×10^{-3}	0.97	5.020×10^{-4}	1.96
81×81	3.376×10^{-4}	1.91	3.283×10^{-3}	1.00	3.129×10^{-4}	2.01

For domain decomposition, we set $\Delta t = 0.1h$ and $H^2 = rh$ to balance error accuracy, where parameter r is a constant. The algorithm stops after that error of adjacent iterative step defined in step 5 of the algorithm is less than 10^{-6}.

In all of the numerical tests, the state variable y and co-state variable p are approximated by using piecewise linear functions while control solution u are treated with piecewise constant functions. Compared to the scheme proposed in [3], the number presented in Table 1 to Table 3 are the sum of average value of two neighbouring layer, which is a good approximation for exact solution evaluating at the middle of two adjacent time layer. We present numerical simulations in Table 1 for $r=1$. The L^2-norm error of the numerical solutions are listed in Table 2 for $r=4$. We present the corresponding results when $r=9$ in Table 3.

Table 2 $L^2(0,T;L^2(\Omega))$-norm error for PDDIA ($r=4$)

Grids	$y-Y$	order	$u-U$	order	$p-P$	order
36×36	4.835×10^{-3}		8.069×10^{-3}		4.856×10^{-3}	
49×49	2.525×10^{-3}	2.11	5.654×10^{-3}	1.15	2.523×10^{-3}	2.12
64×64	1.389×10^{-3}	2.24	4.258×10^{-3}	1.06	1.379×10^{-3}	2.26
81×81	8.077×10^{-4}	2.30	3.325×10^{-3}	1.05	7.952×10^{-4}	2.34

Inferred from the tables, we can see that the error of the state variable y and co-state variable p are the second order accuracy with respect to the time and space sizes, whereas the error of the control variable u is only first order to the spatial variable because of the modeling space.

Table 3 $L^2(0,T;L^2(\Omega))$-norm error for PDDIA ($r=9$)

Grids	$y-Y$	order	$u-U$	order	$p-P$	order
36×36	1.939×10^{-2}		1.588×10^{-2}		1.964×10^{-2}	
49×49	1.173×10^{-2}	1.63	1.006×10^{-2}	1.48	1.186×10^{-2}	1.63
64×64	7.137×10^{-3}	1.86	6.623×10^{-3}	1.57	7.202×10^{-3}	1.87
81×81	4.429×10^{-3}	2.03	4.678×10^{-3}	1.47	4.453×10^{-3}	2.04

In addition, we could get a brief relationship about the Δt-H constraint. Because one can take more larger H than h for keeping the optimal order accuracy for the spatial variable, the constraint $\Delta t = O(H^2)$ is less severe than that for fully explicit algorithms.

4 Conclusion

In this paper, an efficient domain decomposition algorithm for an optimal control problem governed by a linear parabolic partial differential equation has been proposed. The algorithm can solve coupled optimality condition accurately and efficiently based on the non-overlapping domain decomposition scheme given in [4]. The efficient calculation strategy on the inner boundaries and the outer iterations enable excellent extensibility and usability in parallel. Because of the implict/explict strategy, it is necessary to preserve stability from the explicit prediction, but less severe than that for fully explicit algorithms. Further, second order convergence in time allow us to use larger time step in calculations.

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