Partition of Unity Methods for Heterogeneous Domain Decomposition

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1 Heterogeneous problems and partition of unity decomposition

We are interested in solving linear PDEs of the form

$$\mathscr{L}(u) = f \text{ in } \Omega,\ u = \widetilde{g} \text{ on } \partial\Omega, \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^d$ with $d = 1,2$, $\mathscr{L}$ is a linear (elliptic) differential operator, f and $\widetilde{g}$ are the data, and u is the solution to (1). The weak form of (1) with a Hilbert space $(V, \langle\cdot,\cdot\rangle)$ of functions $v : \Omega \to \mathbb{R}$ is

$$a(u,v) = \ell(v)\ \forall v \in V_0,\ \text{ with } u = \widetilde{g} \text{ on } \partial\Omega, \tag{2}$$

where $V_0 := \{v \in V\,:\, v = 0 \text{ on } \partial\Omega\}$, $a : V \times V \to \mathbb{R}$ is the bilinear form corresponding to the operator $\mathscr{L}$, and $\ell : V \to \mathbb{R}$ is the linear functional induced by f. We assume that (2) has a unique solution $u \in \{v \in V\,:\, v = \widetilde{g} \text{ on } \partial\Omega\}$, and that u is "heterogeneous", behaving very differently in different parts of Ω. Typical examples are advection-diffusion problems, where there are advection dominated and diffusion dominated regions (subdomains), and the boundaries in between are not clearly defined, see [8, 10] and references therein. Apart from the χ-method [6, 1], there are no methods to determine such subdomain decompositions, and our goal is to present and study a new such method. We thus introduce (see [18, 11])

Definition 1 (Membership function). Let $\Omega \subset \mathbb{R}^d$ be a set. A membership function φ is a map $\varphi : \overline{\Omega} \to [0,1]$, and its support $S \subset \overline{\Omega}$ is $S := \overline{\{x \in \Omega\,:\, \varphi(x) \neq 0\}}$.

Given two membership functions $\varphi_1, \varphi_2 : \overline{\Omega} \to [0,1]$ that form a partition of unity on Ω, $\varphi_1(x) + \varphi_2(x) = 1$ for all $x \in \overline{\Omega}$, their supports provide then a domain decompo-

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sition $\overline{\Omega} = \overline{\mathrm{supp}\varphi_1 \cup \mathrm{supp}\varphi_2}$. We introduce the approximation $u_{dd} := \varphi_1 u_1 + \varphi_1 u_2 \approx u$, where u_1 and u_2 represent two different possible behaviors of u, and we assume that $u_{dd} = \widetilde{g}$ on $\partial\Omega$. We proceed as follows to define the spaces that u_{dd}, u_1 and u_2 are to be sought in: first, we introduce two approximate problems,

$$\mathscr{L}_1(u_1) = f_1 \text{ in } \Omega,\, \mathscr{B}_1(u_1,\widetilde{g}) = g, \quad \text{and} \quad \mathscr{L}_2(u_2) = f_2 \text{ in } \Omega,\, \mathscr{B}_2(u_2,\widetilde{g}) = 0. \quad (3)$$

Here $\mathscr{L}_j$ are approximation operators of $\mathscr{L}$, f_j are approximations of f, and $\mathscr{B}_j$ are operators to define the boundary conditions of (3), see Section 3 for concrete examples. The function g represents a control and belongs to an appropriate Hilbert space W. Notice that g is different from the actual boundary data $\widetilde{g}$: the latter is defined on $\partial\Omega$, while we will define the former only on a subset of $\partial\Omega$. We assume[1] that (3) (left) is uniquely solvable in V for any $g \in W$ and (3) (right) has a unique solution $u_2 \in V$. To reformulate (3) (left), we introduce two operators $A : V \to V^*$ and $B_{\widetilde{g}} : W \to V^*$, such that (3) (left) becomes $Au_1 = B_{\widetilde{g}} g + f_1$. Notice that $B_{\widetilde{g}}$ represents the boundary conditions of (3) (left) and takes into account also $\widetilde{g}$. This problem is formally solved by $u_1 = A^{-1}B_{\widetilde{g}} g + A^{-1} f_1$, where A^{-1} is well defined if (3) (left) is well posed. Now, we define the spaces $V_1 := \{v \in V : v = A^{-1}(B_{\widetilde{g}} q + f_1),\, q \in W\}$, and $V_2 := \{u_2\}$. Here V_1 represents the space of all possible solutions to the first problem in (3) generated by all the possible (control) functions in W, while V_2 is a singleton containing only the unique solution u_2 to (3) (right). Finally, we use the definition of a "partition of unity method" space (PUM-space [2, 13]) $V_{PUM} := \varphi_1 V_1 + \varphi_2 V_2 \subset V$, where φ_1, φ_2 are membership functions. V_{PUM}, V_1 and V_2 are the spaces that the approximations u_{dd}, u_1 and u_2 have to be sought in. In particular, for the approximation u_{dd} the functions φ_1, φ_2 and g have to be computed. These are defined as solutions to optimal control problems, as described in Section 2. Here we need to remark that our approach could be computationally expensive. However, it is motivated by applications in astrophysics governed by hyperbolic equations like the Boltzmann equation. In many cases, like for supernova explosion, physical phenomena are modeled using two different (limiting) regimes. However, this would require an a-priori knowledge of the transition regime; see, e.g. [8, 3, 11] and references therein. This is exactly the role of the partition of unity functions obtained by our computational framework. In practice, one could use our computationally expensive approach to obtain the partition of unity functions for one representative case and then reuse them (as approximations) in a domain decomposition fashion to compute approximate solutions of other cases of interest.

[1] This specific approximation is motivated by asymptotic expansion techniques providing in general two problems, one that is uniquely determined and a second one that is determined up to some constants for asymptotic matching [15].

2 Optimal control approaches

To compute φ_1, φ_2 and g, we embed the PUM formulation into an optimal control framework. We begin by inserting u_{dd} into (2) and obtain the bounded linear functional $r : V \to \mathbb{R}$ defined by $r(v) := a(\varphi_1 u_1 + \varphi_2 u_2, v) - \ell(v)$, where $v \in V$. In the case that $v = \varphi_1 w$ and $v = \varphi_2 w$ with $w \in V$, we get the functionals

$$r_j(w) := r(\varphi_j w) = a(\varphi_1 u_1 + \varphi_2 u_2, \varphi_j w) - \ell(\varphi_j w) \quad \forall w \in V, \text{ for } j = 1,2.$$

Since $w \in V \mapsto r_j(w)$, $j = 1,2$, are bounded linear functionals, they are elements in V^*, and by the Riesz representation theorem [7], there exist R_1 and R_2 in V such that

$$\langle R_j, v\rangle = a(\varphi_1 u_1 + \varphi_2 u_2, \varphi_j v) - \ell(\varphi_j v) \quad \forall v \in V_0, \quad j = 1,2, \tag{4}$$

where we used V_0, since u_{dd} is exact on $\partial\Omega$ and thus R_1 and R_2 must vanish there. Now, we define $\varphi := \varphi_1$ with $\varphi_2 = 1 - \varphi$, and recall that $\|r_j\|_{V^*} = \|R_j\|_V$. Minimizing the norms of the residuals $\|R_j\|_V$ leads to the optimal control problem

$$\begin{aligned} \min_{R_1,R_2,u_1,g,\varphi} \ & J(R_1,R_2,g,\varphi) := \frac{1}{2}\|R_1\|_V^2 + \frac{1}{2}\|R_2\|_V^2 + \frac{\alpha}{2}\|\varphi\|_V^2 + \frac{\beta}{2}\|g\|_W^2 \\ \text{s.t. } & \langle R_1, v\rangle = a(\varphi u_1 + (1-\varphi)u_2, \varphi v) - \ell(\varphi v) \quad \forall v \in V_0, \\ & \langle R_2, v\rangle = a(\varphi u_1 + (1-\varphi)u_2, (1-\varphi)v) - \ell((1-\varphi)v) \quad \forall v \in V_0, \\ & Au_1 = B_{\tilde{g}} g + f_1,\ g \in W,\ u_2 \in V_2,\ \varphi \in V,\ 0 \le \varphi \le 1 \text{ a.e. in } \Omega, \end{aligned} \tag{5}$$

where $\alpha, \beta > 0$ are two regularization parameters used to tune the cost of φ and g, and f_1 is the same approximation to f introduced in (3).

Solving (5) by an iterative procedure [5, 17] requires at each iteration to solve the two equations (4) for R_1 and R_2, and (3) for u_1. A less expensive optimal control problem is obtained by summing (4) for $j = 1,2$, and we obtain with $R := R_1 + R_2$

$$\langle R, v\rangle = a(\varphi u_1 + (1-\varphi)u_2, v) - \ell(v) \quad \forall v \in V_0, \tag{6}$$

which is a Petrov-Galerkin type equation that we could have obtained directly applying a Petrov-Galerkin method to (2) using V_{PUM} and V as trial and test spaces. Using (6), we get the less expensive optimal control problem

$$\begin{aligned} \min_{R,u_1,g,\varphi} \ & J(R,g,\varphi) := \frac{1}{2}\|R\|_V^2 + \frac{\alpha}{2}\|\varphi\|_V^2 + \frac{\beta}{2}\|g\|_W^2 \\ \text{s.t. } & \langle R, v\rangle = a(\varphi u_1 + (1-\varphi)u_2, v) - \ell(v) \quad \forall v \in V_0, \\ & Au_1 = B_{\tilde{g}} g + f_1,\ g \in W,\ u_2 \in V_2,\ \varphi \in V,\ 0 \le \varphi \le 1 \text{ a.e. in } \Omega. \end{aligned} \tag{7}$$

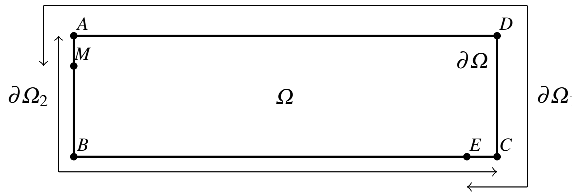

Fig. 1 Example of a boundary decomposition $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$.

3 Optimal control for elliptic boundary-layer problems

As main test cases we consider elliptic problems of the form

$$\mathscr{L}(u) := -\mu\Delta u + \mathbf{a}\cdot\nabla u + cu = f \text{ in } \Omega,\ u = \widetilde{g} \text{ on } \partial\Omega, \tag{8}$$

where Ω is a bounded domain in $\mathbb{R}^d$, for $d = 1,2$, $\widetilde{g} \in C(\partial\Omega)$, f is sufficiently smooth, and the components of $\mathbf{a}$ are assumed to be strictly-positive. The assumption on $\mathbf{a}$ is restrictive, but it simplifies the presentation below and can be relaxed. The corresponding weak problem is to find a $u \in \left\{v \in H^1(\Omega)\,|\, v = \widetilde{g} \text{ on } \partial\Omega\right\}$ such that

$$a(u,v) := \int_\Omega \mu\nabla u\cdot\nabla v + \mathbf{a}\cdot\nabla u\, v + c\, u\, v\, d\mathbf{x} = \int_\Omega f\, v\, d\mathbf{x} =: \ell(v)\ \ \forall v \in H_0^1(\Omega).$$

We also assume that Ω is such that the boundary $\partial\Omega$ can be decomposed into $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$, where the intersection $\partial\Omega_1 \cap \partial\Omega_2$ has a non-zero measure, as illustrated in Figure 1. To obtain $u_{dd} = \varphi u_1 + (1-\varphi)u_2 \approx u$, we define $\Gamma := \partial\Omega \setminus \partial\Omega_1$ and introduce the operator $\mathscr{L}_1 := -\mu\Delta + c$. Then, as in (3), for any choice of the control $g \in H_0^1(\Gamma)$ the corresponding approximate problem for u_1 is

$$\begin{gathered}\int_\Omega \mu\nabla u_1\cdot\nabla v + c\, u_1\, v\, d\mathbf{x} = 0 \qquad \forall v \in H_0^1(\Omega),\\ u_1 \in \Big\{w \in H^1(\Omega)\,|\, w = \widetilde{g} \text{ on } \partial\Omega_1,\, w = \widetilde{g} + g \text{ on } \Gamma,\, \tau(w) \in C(\partial\Omega)\Big\},\end{gathered} \tag{9}$$

where τ is the trace operator on $\partial\Omega$. Notice that we have chosen $f_1 = 0$. As before, we introduce the operator $A : H^1(\Omega) \to H^{-1}(\Omega)$ defined as $\langle Au, v\rangle_{H^{-1},H^1} := \int_\Omega \mu\nabla u\cdot\nabla v + c\, u\, v\, d\mathbf{x}$ for all $v \in H_0^1(\Omega)$, and the operator $B_{\widetilde{g}} : H_0^1(\Gamma) \to H^{-1}(\Omega)$ such that $v \mapsto (B_{\widetilde{g}} g)(v)$ is a bounded linear functional in $H^{-1}(\Omega)$. The operator $B_{\widetilde{g}}$ represents the Dirichlet boundary conditions of (9). $Au_1 = B_{\widetilde{g}} g$ is then equivalent to (9). The corresponding set V_1 is given by

$$V_1 = \{v \in H^1(\Omega) : Av = B_{\widetilde{g}} q \text{ for any } q \in H_0^1(\Gamma)\}.$$

Now, consider the operator $\mathscr{L}_2 := \mathbf{a}\cdot\nabla + c$ and $f_2 = f$. The problem for u_2 is then

$$\mathscr{L}_2(u_2) = \mathbf{a}\cdot\nabla u_2 + c\, u_2 = f \text{ in } \Omega,\ u_2 = \widetilde{g} \text{ on } \partial\Omega_2, \tag{10}$$

which we assume uniquely solvable in $H^1(\Omega)\cap C(\overline{\Omega})$. Notice that (10) is a pure advection problem and the subset $\partial\Omega_2$ is given as the set of points where the characteristic curves enter the domain Ω. This is the main assumption we make on $\partial\Omega_2$ for the problem (10) to be well posed. The set V_2 contains only the solution to (10), i.e. $V_2 = \{u_2\}$. The approximation $u_{dd} \approx u$ is then obtained as $u_{dd} = \varphi_1 u_1 + \varphi_2 u_2$, where the membership functions $\varphi_1 = \varphi, \varphi_2 = 1-\varphi \in H^1(\Omega)$ form a partition of unity, and φ is such that

$$\varphi(\mathbf{x}) \in \begin{cases} \{1\} & \text{if } \mathbf{x} \in \partial\Omega \setminus \partial\Omega_2, \\ [0,1] & \text{if } \mathbf{x} \in \partial\Omega_1 \cap \partial\Omega_2, \\ \{0\} & \text{if } \mathbf{x} \in \Gamma, \end{cases} \tag{11}$$

with $\tau(\varphi) \in C(\partial\Omega)$. Notice that this definition of φ makes u_{dd} exact on the boundary $\partial\Omega$, $\tau(u_{dd}) = \tau(\varphi_1 u_1 + \varphi_2 u_2) = \widetilde{g}$.

In what follows, we study the control problem (7) ((5) would have a similar structure) to optimize φ and g for computing the approximation u_{dd} to the solution to (8). In particular, we first show well-posedness, and then we derive the first-order optimality system. We consider directly a 2-dimensional problem ($d = 2$), since the analysis of the 1-dimensional version is simpler and relies on the same arguments. To define our optimal control problem, as in (7), we consider the cost functional $J(R,g,\varphi) := \frac{1}{2}\|R\|^2_{H^1(\Omega)} + \frac{\alpha}{2}\|\varphi\|^2_{H^1(\Omega)} + \frac{\beta}{2}\|g\|^2_{H^1(\Gamma)}$. Now, we introduce the control-to-state maps $g \mapsto u_1(g)$ and $(g,\varphi) \mapsto R(u_1(g),\varphi)$, where $u_1(g)$ and $R(u_1(g),\varphi)$ solve (9) and

$$\langle R,v\rangle_{H^1(\Omega)} = \int_\Omega \mu\nabla u_{dd}\cdot\nabla v + \mathbf{a}\cdot\nabla u_{dd}\, v + c\, u_{dd}\, v - f\, v\, d\mathbf{x} \quad \forall v \in H^1_0(\Omega). \tag{12}$$

Notice that the left-hand side of (12), that is $\langle R,v\rangle_{H^1(\Omega)} = \int_\Omega \nabla R\cdot\nabla v + Rv\, d\mathbf{x}$, is of a similar form to the left-hand side in (9). These maps are well defined according to the lemmas below and allow us to define the reduced cost functional $\widetilde{J}(g,\varphi) := J(R(u_1(g),\varphi),g,\varphi)$ and the optimal control problem

$$\min_{g,\varphi} \widetilde{J}(g,\varphi) \text{ s.t. } 0 \le \varphi(\mathbf{x}) \le 1 \text{ in } \Omega \text{ and (11) holds.} \tag{13}$$

For well-posedness of this optimization problem, we need four Lemmas:

Lemma 1. *Let $z \in H^1(\partial\Omega)$ with $\Omega \subset \mathbb{R}^2$ convex and $\partial\Omega$ Lipschitz. Then the problem*

$$\int_\Omega \mu\nabla u_1\cdot\nabla v + c\, u_1\, v\, d\mathbf{x} = 0 \quad \forall v \in H^1_0(\Omega) \tag{14}$$

with $u_1 = z$ on $\partial\Omega$ is uniquely solvable by $u_1 \in H^1(\Omega)\cap C(\overline{\Omega})$, and there exists a positive constant c such that $\|u_1\|_{H^1(\Omega)} \le c\|z\|_{H^1(\partial\Omega)}$.

Proof. To show that there exists a unique $u_1 \in C(\overline{\Omega})$, we define w as the harmonic extension of z in Ω. Recalling the embedding $H^1 \hookrightarrow C$ for one-dimensional domains, we have that $z \in C(\partial\Omega)$. Therefore, since Ω is a Lipschitz domain,

$w \in C^2(\Omega) \cap C(\overline{\Omega})$; see, e.g., [12]. Now, consider the problem $-\mu\Delta v + cv = -cw$ in Ω with $v = 0$ on $\partial\Omega$. Since Ω is convex, Theorems 3.2.1.2-3 in [14] ensure that this problem is uniquely solved by $v \in H^2(\Omega) \cap H_0^1(\Omega)$. Since $\Omega \subset \mathbb{R}^2$, the Sobolev embedding $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$ [7] ensures that $v \in C(\overline{\Omega})$. Noticing that the function $w + v$ solves (14), $u_1 \in C(\overline{\Omega})$ and is unique by the linearity of (14). Next, we show that $u_1 \in H^1(\Omega)$ with $\|u_1\|_{H^1(\Omega)} \leq c\|z\|_{H^1(\partial\Omega)}$. Consider the trace operator $\tau : H^1(\Omega) \to H^{1/2}(\partial\Omega)$. Since Ω is a Lipschitz domain, by [16, Theorem 3.37, page 102] this operator has a bounded right-inverse $\tau^{-1} : H^{1/2}(\partial\Omega) \to H^1(\Omega)$. Now, we define $w := \tau^{-1}z$ and note that $w \in H^1(\Omega)$. So, if we decompose u_1 as $u_1 = w + \widetilde{v}$, then $\widetilde{v}$ must solve in a weak sense the problem $-\mu\Delta\widetilde{v} + c\widetilde{v} = -(-\mu\Delta w + cw)$ in Ω with $\widetilde{v} = 0$ on $\partial\Omega$. By the Lax-Milgram theorem we have that the unique solution is $\widetilde{v} \in H_0^1(\Omega)$ and there exists a constant C such that $\|\widetilde{v}\|_{H^1(\Omega)} \leq C\|w\|_{H^1(\Omega)}$. Therefore, $u_1 \in H^1(\Omega)$ and using the decomposition $u_1 = w + \widetilde{v}$ we get $\|u_1\|_{H^1(\Omega)} \leq (1 + C)\|w\|_{H^1(\Omega)} = (1 + C)\|\tau^{-1}z\|_{H^1(\Omega)} \leq K\|z\|_{H^1(\partial\Omega)}$, for some positive constant K, where we used the boundedness of τ^{-1} [16].

Lemma 2. *Let $\varphi \in H^1(\Omega)$ such that $0 \leq \varphi(\mathbf{x}) \leq 1$ a.e. in Ω. Then for any function $v \in H^1(\Omega) \cap C(\overline{\Omega})$ it holds that $\varphi v \in H^1(\Omega)$.*

Proof. An application of Theorem 1 in [9, page 247] shows that $\nabla(v\varphi) = v\nabla\varphi + \varphi\nabla v$. Then a simple estimate of the norm $\|\nabla(v\varphi)\|_{L^2(\Omega)}$ allows us to obtain the result.

Lemma 3. *Let $\{z_n\}_n$ be a sequence that converges weakly in $H^1(\partial\Omega)$ to a weak limit $\widehat{z} \in H^1(\partial\Omega)$, i.e. $z_n \rightharpoonup \widehat{z}$ in $H^1(\partial\Omega)$. Define the sequence $\{u_{1,n}\}_n$ by $u_{1,n} := u_1(z_n)$, where $u_1(z_n)$ solves (14) with $u_1 = z_n$ on $\partial\Omega$. Then there exists a subsequence u_{1,n_j} that converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to the limit $\widehat{u}_1 = u_1(\widehat{z}) \in H^1(\Omega)$, i.e., $u_{1,n_j} \rightharpoonup \widehat{u}_1$ in $H^1(\Omega)$ and $u_{1,n_j} \to \widehat{u}_1$ in $L^2(\Omega)$.*

Proof. Since the sequence $\{z_n\}_n$ converges weakly in $H^1(\partial\Omega)$, it is bounded in the norm $\|\cdot\|_{H^1(\partial\Omega)}$. By Lemma 1, we have that $\|u_{1,n}\|_{H^1(\Omega)} \leq c\|z_n\|_{H^1(\partial\Omega)} \leq K$, for some positive constant K, and the sequence $u_{1,n}$ is bounded in $H^1(\Omega)$. Since $H^1(\Omega)$ is reflexive, there exists a weakly convergent subsequence $u_{1,n_j} \rightharpoonup \widehat{u}_1$ in $H^1(\Omega)$. Now, from (14), we have that for any $v \in H_0^1(\Omega)$

$$\int_\Omega \mu\nabla u_{1,n_j} \cdot \nabla v + c\,u_{1,n_j}\,v\,d\mathbf{x} \to \int_\Omega \mu\nabla\widehat{u}_1 \cdot \nabla v + c\,\widehat{u}_1\,v\,d\mathbf{x}.$$

Moreover, the weak convergence $z_{n_j} \rightharpoonup \widehat{z}$ and the continuity of the trace operator $\tau : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ [16, Theorem 3.37] implies that $z_{n_j} = \tau(u_{1,n_j}) \rightharpoonup \tau(\widehat{u}_1) = \widehat{z}$, weakly in $H^{1/2}(\partial\Omega)$. Therefore, $\widehat{u}_1 = u_1(\widehat{z})$. We conclude by recalling the Sobolev compact embedding $H^1(\Omega) \Subset L^2(\Omega)$; see, e.g., [7].

Lemma 4. *Let $\{u_{1,n}\}_n$ be the sequence defined in Lemma 3 such that $u_{1,n_j} \rightharpoonup \widehat{u}_1$ (weakly) in $H^1(\Omega)$. Consider a sequence $\{\varphi_n\}_n$ in $H^1(\Omega)$ such that $0 \leq \varphi_n(\mathbf{x}) \leq 1$*

and $\varphi_n \rightharpoonup \widehat{\varphi}$ *(weakly) in* $H^1(\Omega)$ *with* $0 \leq \widehat{\varphi}(\mathbf{x}) \leq 1$. *Then there exist two subsequences* $\{\varphi_{n_j}\}_j$ *and* $\{u_{1,n_j}\}_j$ *such that* $\varphi_{n_j} \to \widehat{\varphi}$ *and* $u_{1,n_j} \to \widehat{u}_1$ *(strongly) in* $L^2(\Omega)$*, and for any* $v \in H_0^1(\Omega)$

$$\int_\Omega \nabla(\varphi_{n_j} u_{1,n_j}) \cdot \nabla v + \varphi_{n_j} u_{1,n_j} v d\mathbf{x} \to \int_\Omega \nabla(\widehat{\varphi}\widehat{u}_1) \cdot \nabla v + \widehat{\varphi}\widehat{u}_1 v d\mathbf{x}.$$

Proof. The existence of the subsequences $\{\varphi_{n_j}\}_j$ and $\{u_{1,n_j}\}_j$ such that $\varphi_{n_j} \to \widehat{\varphi}$ and $u_{1,n_j} \to \widehat{u}_1$ (strongly) in $L^2(\Omega)$ follows from the fact that $\varphi_n \rightharpoonup \widehat{\varphi}$ (weakly in $H^1(\Omega)$), Lemma 3, and the Sobolev (compact) embedding $H^1(\Omega) \Subset L^2(\Omega)$ [7]. Now, recalling Lemma 1 and according to the proof of Lemma 2 it holds that $\nabla(u_{1,n_j}\varphi_{n_j}) = u_{1,n_j}\nabla\varphi_{n_j} + \varphi_{n_j}\nabla u_{1,n_j}$. Therefore, to treat the products of sequences $\widehat{u}_{1.n_j}\nabla\widehat{\varphi}_{n_j}$, $\varphi_{n_j}\nabla u_{1,n_j}$, and $\varphi_{n_j}u_{1,n_j}$, we use [7, Theorem 5.12-4] to obtain for any $v \in H_0^1(\Omega)$ that

$$\int_\Omega \nabla(\varphi_{n_j} u_{1,n_j})\nabla v + \varphi_{n_j} u_{1,n_j} v d\mathbf{x} = \int_\Omega u_{1,n_j} \nabla\varphi_{n_j} \nabla v + \varphi_{n_j} \nabla u_{1,n_j} \nabla v + \varphi_{n_j} u_{1,n_j} v d\mathbf{x}$$
$$\to \int_\Omega \widehat{u}_1 \nabla\widehat{\varphi}\nabla v + \widehat{\varphi}\nabla\widehat{u}_1 \nabla v + \widehat{u}_1 \widehat{\varphi} v d\mathbf{x} = \int_\Omega \nabla(\widehat{\varphi}\widehat{u}_1) \cdot \nabla v + \widehat{\varphi}\widehat{u}_1 v + \widehat{u}_1 \widehat{\varphi} v d\mathbf{x}.$$

We are now ready to prove that (13) is well posed.

Theorem 1. *Let* $\alpha, \beta > 0$*, then there exists a solution to problem* (13).

Proof. Consider a minimizing sequence $\{(R_n, \varphi_n, u_{1,n}, g_n)\}_n$, where g_n is extended by zero on $\partial\Omega$. Since J is coercive in φ and g we have the bounds $\|\varphi_n\|_{H^1(\Omega)} \leq c$ and $\|g_n\|_{H^1(\partial\Omega)} \leq c'$, for two positive constants c, c'; see, e.g., [17]. The reflexivity of $H^1(\Omega)$ and $H^1(\partial\Omega)$ ensures the existence of weakly convergent subsequences: $\varphi_{n_j} \rightharpoonup \widehat{\varphi}$ in $H^1(\Omega)$ and $g_{n_j} \rightharpoonup \widehat{g}$ in $H^1(\partial\Omega)$. By the Sobolev (compact) embedding $H^1(\Omega) \Subset L^2(\Omega)$ [7], the sequence $\{\varphi_{n_j}\}_j$ converges strongly in $L^2(\Omega)$ to $\widehat{\varphi}$. Since the set $\{v \in L^2(\Omega) : 0 \leq v(\mathbf{x}) \leq 1 \text{ a.e. in } \Omega\}$ is (weakly) closed in $L^2(\Omega)$ [17], we have $0 \leq \widehat{\varphi}(\mathbf{x}) \leq 1$. Consider now the sequence $\{u_{1,n}\}_n$ and the corresponding subsequence $u_{1,n_j} = u_1(g_{n_j})$. By Lemma 3, we have that $u_{1,n_j} \rightharpoonup \widehat{u_1} = u_1(\widehat{g})$ weakly in $H^1(\Omega)$ and $u_{1,n_j} \to \widehat{u_1} = u_1(\widehat{g})$ strongly in $L^2(\Omega)$. Consider the sequence $\{R_n\}_n$. Since R_n satisfies

$$\langle R_n, v\rangle_{H^1(\Omega)} = \int_\Omega \mu\nabla u_{dd,n} \cdot \nabla v + \mathbf{a} \cdot \nabla u_{dd,n} v + c\, u_{dd,n} v - f\, v\, d\mathbf{x} \ \forall v \in H_0^1(\Omega),$$

where $u_{dd,n} = \varphi_n u_{1,n} + (1 - \varphi_n)u_2$, from the Lax-Milgram theorem we have that $\|R_n\|_{H^1(\Omega)} \leq K(\|u_{1,n}\|_{H^1(\Omega)}, \|\varphi_n\|_{H^1(\Omega)})$, where the constant K depends on $\|u_{1,n}\|_{H^1(\Omega)}$ and $\|\varphi_n\|_{H^1(\Omega)}$, which are bounded. Therefore, R_n is bounded as well, and by Lemma 4, one can show that $R_{n_j} \rightharpoonup \widehat{R} = R(\widehat{u}_1, \widehat{\varphi})$ weakly in $H^1(\Omega)$. Now, the weak-lower semi-continuity of J implies the claim [17, 4].

To obtain the first-order optimality system, we rely on the Lagrange multiplier approach and work in the reduced space of solutions of constraint and adjoint equations; see, e.g., [5, 17]. We first recall the control-to-state maps $g \mapsto u_1(g)$ and $(g,\varphi) \mapsto R(u_1(g),\varphi)$ and the reduced cost functional $\widetilde{J}(g,\varphi)$. Then we notice that its derivatives, for $\delta g \in H_0^1(\Gamma)$ and $\delta\varphi \in H_0^1(\Omega)$, are

$$D_g\widetilde{J}(g,\varphi)(\delta g)=\langle \beta g+R_g,\delta g\rangle_{H^1(\Gamma)}, \quad D_\varphi\widetilde{J}(w,\varphi)(\delta\varphi)=\langle \alpha\varphi+R_\varphi,\delta\varphi\rangle_{H^1(\Omega)}. \tag{15}$$

Here R_g is the solution of the problem

$$\langle R_g,\delta g\rangle_{H_0^1(\Gamma)} = \langle B_{\widetilde{g}}\delta g,\lambda\rangle_{H^{-1},H^1}, \tag{16}$$

where $\langle\cdot,\cdot\rangle_{H^{-1},H^1} : H^{-1}(\Omega)\times H_0^1(\Omega) \to \mathbb{R}$ denotes the duality pairing, and R_φ is the Riesz representative of the linear functional

$$\delta\varphi \mapsto \int_\Omega \mu\nabla\big[(u_1-u_2)\delta\varphi\big]\cdot\nabla R\,d\mathbf{x} + \mathbf{a}\cdot\nabla\big[(u_1-u_2)\delta\varphi\big]R + c\,(u_1-u_2)\delta\varphi\,R\,d\mathbf{x}.$$

In (16), $\lambda \in H_0^1(\Omega)$ is a Lagrange multiplier that solves the adjoint equation

$$\int_\Omega \nabla\lambda\cdot\nabla v + c\,\lambda\,v\,d\mathbf{x} = \int_\Omega \mu\nabla(v\varphi)\cdot\nabla R + \mathbf{a}\cdot\nabla(v\varphi)\,R + c\,v\,\varphi\,R\,d\mathbf{x}, \tag{17}$$

for all $v \in H_0^1(\Omega)$. Therefore, the first-order optimality system is given by (9), (12), (17) and (16) together with the conditions [4, 17]

$$D_g\widetilde{J}(g,\varphi)(\delta g) = 0,$$

for all $\delta g \in H_0^1(\Gamma)$, and for any arbitrary $\theta > 0$

$$\varphi = \mathbb{P}_{V_{ad}}\Big(\varphi - \theta\big(\alpha\varphi + R_\varphi\big)\Big),$$

where $\mathbb{P}_{V_{ad}}$ is the projection onto $V_{ad} := \{v \in H^1(\Omega) : 0 \le v(\mathbf{x}) \le 1 \text{ a.e. in } \Omega\}$.

4 Numerical experiments

We present now numerical experiments for the one-dimensional elliptic problem

$$-\mu\partial_{xx}u - \partial_x u = 1 \text{ in } (0,1), \text{ with } u(0)=0,\, u(1)=0, \tag{18}$$

for given $\mu = 0.01$, computing $u_{dd} = \varphi_1 u_1 + \varphi_2 u_2$, with

$$\begin{aligned} -\mu\partial_{xx}u_1 &= 0 \text{ in } (0,1), \\ u_1(0)=0,\, u_1(1)&=g, \end{aligned} \qquad \text{and} \qquad \begin{aligned} -\partial_x u_2 &= 1 \text{ in } [0,1), \\ u_2(1) &= 0. \end{aligned}$$

We solve both the PUM and Petrov-Galerkin optimality systems discretized by linear finite-elements with a projected-LBFGS method with stopping tolerance $5\cdot 10^{-5}$

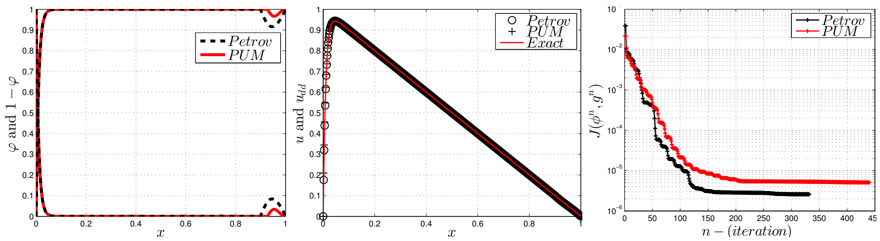

Fig. 2 Comparison of the Petrov and PUM approaches: Left: partition of unity functions φ and $1-\varphi$. Middle: exact solution and approximations. Right: Decay of the cost functional.

on the (relative) residual norm. The regularization parameters are $\alpha = \beta = 10^{-7}$. In Figure 2 (left) we see that the φ and $1-\varphi$ obtained by the two approaches are very similar, and catch well the boundary layer on the left. The small bumps in the right part (close to $x = 1$) are due numerical effects and we checked that they disappear for smaller tolerances. In Figure 2 (middle) the exact solution is compared with the two approximations u_{dd}, and we see good agreement. In Figure 2 (right), we show the decay of the cost functional with respect to the number of iterations, and we see that the Petrov-Galerkin approach converges a bit faster.

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