Optimized Schwarz methods for elliptic optimal control problems

Bérangère Delourme[1], Laurence Halpern[1], Binh Thanh Nguyen[1]

Abstract
The present paper deals with the design of optimized Robin-Schwarz methods for the algorithm of optimal control proposed in [1]. In both overlapping and non-overlapping cases, a full analysis of the problem is provided, and is illustrated with numerical tests.

1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^2$, $z \in L^2(\Omega)$, and $\nu > 0$. We consider the following elliptic control problem described in [1] (see also [9, Chapter 2])

$$\min_{u \in L^2(\Omega)} \int_{\Omega} |y(u) - z|^2 dx + \nu \int_{\Omega} |u|^2 dx, \tag{1}$$

where, for a given function $f \in L^2(\Omega)$, $y(u)$ is the unique $H_0^1(\Omega)$ solution to

$$-\Delta y = f + u \text{ in } \Omega, \quad y = 0 \text{ on } \partial\Omega. \tag{2}$$

It is well known that the optimal control u (solution to (1)) is related to the adjoint state p by $u = -\frac{p}{\nu}$, and $(y, p) \in H_0^1(\Omega)^2$ is solution of the coupled problem

$$-\Delta y = f - \frac{p}{\nu} \quad -\Delta p = y - z \tag{3}$$

Introducing the new unknown $w = y + \frac{i}{\sqrt{\nu}} p$ (see [1]), Problem (3) is equivalent to the complex Helmholtz problem: find $w \in H_0^1(\Omega)$ such that

University Paris 13, Villetaneuse, France delourme@math.univ-paris13.fr

$$-\Delta w - \frac{i}{\sqrt{\nu}} w = g \text{ in } \Omega \qquad g = f - \frac{i}{\sqrt{\nu}} z. \tag{4}$$

In [2], Benamou and Després proposed a Robin's non-overlapping domain decomposition algorithm. Let us describe this algorithm (written here also for overlapping subdomains like in the original Schwarz algorithm). We consider the case where $\Omega = \mathbb{R}^2$ is split into two subdomains $\Omega_1 =]-\infty, \frac{L}{2}[\times\mathbb{R}$ and $\Omega_2 =]-\frac{L}{2}, +\infty[\times\mathbb{R}$. Here, L is a non-negative parameter that corresponds to the width of the overlapping zone between Ω_1 and Ω_2. We denote by n_j the outward unit normal vector to Ω_j, ∂_{n_j} the normal derivative on the boundary of Ω_j. Letting $\lambda^0 \in H^{1/2}(\partial\Omega_1)$ and $\ell \in \mathbb{C}$, we construct iteratively the sequences $(w_1^n)_{n\in\mathbb{N}}$, $(w_2^n)_{n\in\mathbb{N}}$ as follows: for any $n \in \mathbb{N}\backslash\{0\}$, find $w_1^n \in H^1(\Omega_1)$ and $w_2^n \in H^1(\Omega_2)$ such that

$$\begin{cases} -\Delta w_1^n - \frac{i}{\sqrt{\nu}} w_1^n = g & \text{in } \Omega_1, \\ \partial_{n_1} w_1^n + \ell w_1^n = \lambda^{n-1} & \text{on } \partial\Omega_1, \end{cases} \qquad \begin{cases} -\Delta w_2^n - \frac{i}{\sqrt{\nu}} w_2^n = g & \text{in } \Omega_2, \\ \partial_{n_2} w_2^n + \ell w_2^n = \partial_{n_2} w_1^n + \ell w_1^n & \text{on } \partial\Omega_2, \end{cases}$$
$$\lambda^n = \partial_{n_1} w_2^n + \ell w_2^n \Big|_{\partial\Omega_1}. \tag{5}$$

It is easily seen ([1, Theorem 1]) that the problems defining w_1^n and w_2^n are well-posed if ℓ belongs to the angular sector $\mathcal{A}$ defined by

$$\mathcal{A} = \{z \in \mathbb{C} \text{ such that } \mathrm{Im}(z) < 0,\ \mathrm{Im}(z) + \mathrm{Re}(z) > 0\}. \tag{6}$$

Moreover, it is proved in [1, Theorem 2] (see also [2]), in the non-overlapping case, that the algorithm (5) converges, namely the sequence w_1^n (resp. w_2^n) tends to w (solution to (4)) in $H^1(\Omega_1)$ (resp. w in $H^1(\Omega_2)$).

The objective of the present work is to find a parameter $\ell \in \mathcal{A}$ that optimizes the rate of convergence of this algorithm. In the case of strongly elliptic real equation, this problem has been solved in [7] for Robin and Ventcel transmission conditions. In the former case, explicit values of the coefficients were given, whereas in the Ventcel case, only asymptotic formulas in terms of the mesh size are available. Extension to real Helmholtz equations were given in [6, 8]. Following these approaches, we consider the errors $e_1^n = w_1^n - w$ and $e_2^n = w_2^n - w$ and we denote by $\hat{e}_1^n$ and $\hat{e}_2^n$ their Fourier transform with respect to y, with Fourier variable k. It is easily seen that $\hat{e}_1^n$ and $\hat{e}_2^n$ follow a geometrical progression: more specifically, there exists two complex constants a_1 and a_2 such that

$$\hat{e}_j^n = a_j\, \delta(\ell, k)^{2n}\, e^{-\omega(k)|x|}, \ \ \delta(\ell, k) = e^{-\omega(k)L}\ \frac{\omega(k) - \ell}{\omega(k) + \ell}, \omega(k) = \sqrt{k^2 - \frac{i}{\sqrt{\nu}}},$$

In the previous formulas, . Moreover, here and all over the text, the complex number $\sqrt{z}$ corresponds to the square root of z belonging to $\mathcal{A}$. As a result, it suffices to minimize the modulus of δ (the square root of the convergence factor) in order to accelerate the convergence of the domain decomposition algorithm (5). As explained in [7, Section 4], we are interested in optimizing δ over a bounded interval $[k_{\min}, k_{\max}]$ (i.e. $k \in [k_{\min}, k_{\max}]$). In practice, the interval depends on the geometry of the domain and the mesh size ($k_{\max} = \frac{\pi}{h}$ where h denotes the characteristic length of the mesh). It leads us to investigate the following homographic best approximation problem (see [7, Section 4.2], [3] for the name in a time-dependent context): find $\delta^* \in \mathbb{R}$ such that

$$\delta^* = \inf_{\ell \in \mathbb{C}} \sup_{k \in [k_{\min}, k_{\max}]} |\delta(\omega(k), \ell)| \tag{7}$$

2 General results of well-posedness

The existence and uniqueness of an optimal parameter ℓ^* are direct consequences of the general results of [3, 4]:

Theorem 1 *For L sufficiently small, there exists a unique $\ell^* \in \mathcal{A}$ such that*

$$\delta^* = \inf_{\ell \in \mathbb{C}} \sup_{k \in [k_{\min}, k_{\max}]} |\delta(\omega(k), \ell)| = \max_{k \in [k_{\min}, k_{\max}]} |\delta(\omega(k), \ell^*)|. \tag{8}$$

Moreover, there exists at least two distinct real numbers $(k_1, k_2) \in [k_{\min}, k_{\max}]^2$ such that

$$\max_{k \in [k_{\min}, k_{\max}]} |\delta(\omega(k), \ell^*)| = |\delta(\omega(k_1), \ell^*)| = |\delta(\omega(k_2), \ell^*)|. \tag{9}$$

Proof (Sketch of the proof of Theorem 1). By contradiction, one can verify that if there exists $\ell^* \in \mathbb{C}$ satisfying (8), then $\ell^* \in \mathcal{A}$ (see e.g. [3, Lemma 4.5] for a similar proof). Then, the existence of ℓ^* ([3, Theorem 2.2 and Theorem 2.8]) results from a compactness argument (k belongs to the compact set $[k_{\min}, k_{\max}]$). Finally, in the non-overlapping case ($L = 0$), the uniqueness is proved in [3, Theorem 2.6]. For $L \neq 0$ and sufficiently small, the uniqueness proof results from an adaptation of [4, Theorem 8]. In both cases, the uniqueness is a consequence of convexity properties and the equi-oscillation property (9)([3, Theorem 2.5 and Theorem 2.11]).

3 Characterization of the optimal parameter in the non-overlapping case

Theorem 2 *The best parameter ℓ^* defined by (8) is given by*

$$\ell^* = \sqrt{\omega_{\min}\omega_{\max}}, \quad \delta^* = \left| \frac{\sqrt{\omega_{\min}} - \sqrt{\omega_{\max}}}{\sqrt{\omega_{\min}} + \sqrt{\omega_{\max}}} \right| \tag{10}$$

where $\omega_{\min} = \omega(k_{\min})$ *and* $\omega_{\max} = \omega(k_{\max})$*. Moreover, if* $k_{\max} = \frac{\pi}{h}$*,* δ^* *and* ℓ^* *admit the following asymptotic expansion*

$$\delta^* = 1 \; - \; 2\,h^{1/2}\,\frac{\mathrm{Re}(\sqrt{\omega_{\min}})}{\sqrt{\pi}} \; + \; o(h^{1/2}), \;\; \ell^* \; = \; h^{-1/2}\,\left(\sqrt{\pi}\,\sqrt{\omega_{\min}} \; + \; o(1)\right). \tag{11}$$

We remark that Formula (10) is the same as in the real positive case (see [7, Theorem 4.4]). The reminder of this section is dedicated to the proof of Theorem 2. First, we remark that in the non-overlapping case (and as in the real case), the equi-oscillation property (9) holds for exactly two points that are nothing but $k_{\min}$ and $k_{\max}$ (the proof of this result may be done using either a geometrical argument or a direct investigation of the derivative of $|\delta(k,\ell)|^2$ with respect to k, see [5]):

Lemma 1 *Let* ℓ^* *be defined by* (8)*. Then,*

$$\max_{k\in[k_{\min},k_{\max}]} |\delta(\omega,\ell^*)| = |\delta(\omega_{\min},\ell^*)| = |\delta(\omega_{\max},\ell^*)|, \tag{12}$$

and, for any $k \in]k_{\min}, k_{\max}[$*,* $|\delta(\omega(k),\ell^*)| < |\delta(\omega_{\min},\ell^*)|$*.*

The previous lemma motivates us to consider the curve of equioscillation Π defined by

$$\Pi = \left\{\ell = re^{i\theta} \in \mathcal{A} \text{ such that } |\delta(\omega_{\min},\ell)| = |\delta(\omega_{\max},\ell)|\right\}, \tag{13}$$

so that the optimization problem (8) can then be rewritten as follows: find $\ell^* \in \Pi$ such that

$$\delta^* = \min_{\ell\in\Pi} |\delta(\omega_{\min},\ell)| = \min_{\ell\in\Pi} |\delta(\omega_{\max},\ell)|. \tag{14}$$

Note that, unlike in the real case, the set Π is not reduced to the singleton $\{p = \sqrt{\omega_{\min}\omega_{\max}}\}$. Nevertheless, $\sqrt{\omega_{\min}\omega_{\max}}$ still belongs to Π. To continue the proof, it is useful to introduce the perpendicular bisector Δ of the segment $[\omega_{\min},\omega_{\max}]$, i.e. $\Delta = \{z = x + iy \in \mathbb{C} \text{ s.t. } y = ax + b\}$ where $a = -\frac{\mathrm{Re}(\omega_{\max}-\omega_{\min})}{\mathrm{Im}(\omega_{\max}-\omega_{\min})}$ and $b = \frac{|\omega_{\max}|^2-|\omega_{\min}|^2}{2\mathrm{Im}(\omega_{\max}-\omega_{\min})}$. For any $\ell \in \mathbb{C}$, we also consider the signed distance between ℓ and Δ, namely the function $d(\ell) = \frac{a\mathrm{Re}(\ell)-\mathrm{Im}(\ell)+b}{\sqrt{1+a^2}}$. Using the intercept theorem, it is easily seen that the best parameter ℓ^* corresponds to the point of Π for which the distance between Π and Δ is minimal:

Lemma 2 *The function* $\eta : \Pi \to \mathbb{R}$*, defined by* $\eta(\ell) = |\delta(\ell,\omega_{\min})| = |\delta(\ell,\omega_{\max})|$ *is a strictly increasing function of the signed distance* d*: for any* $(\ell_1,\ell_2) \in \Pi^2$ *such that* $d(\ell_1) < d(\ell_2)$*,* $\eta(\ell_1) < \eta(\ell_2)$*.*

In other words it suffices to study the variations of the distance function d over Π in order to characterize the best parameter ℓ. By a standard investigation of d we prove the following lemma:

Lemma 3 *The function d reaches its minimum over Π for $\ell^* = \sqrt{\omega_{\min}\omega_{\max}}$.*

The proof of Theorem 2 is completed by a standard asymptotic expansion of δ^* for $k_{\max}$ large.

4 Asymptotics of the optimal parameter in the overlapping case

In the overlapping case ($L > 0$), we are not able to obtain an explicit characterization of the best parameter ℓ^*. Nevertheless, we are able to compute its asymptotic behaviour for h small when the overlapping parameter $L = h$ and $k_{\max} = \frac{\pi}{h}$,

Theorem 3 *Assume that $L = h$ and $k_{\max} = \frac{\pi}{h}$.*

- *For h sufficiently small, there exists $k^* \in]k_{\min}, k_{\max}[$ such that*

$$\max_{k\in[k_{\min},k_{\max}]} |\delta(\omega, \ell^*)| = |\delta(\omega_{\min}, \ell^*)| = |\delta(\omega(k^*), \ell^*)|, \tag{15}$$

and, for any $k \in]k_{\min}, k^*[\cup]k^*, k_{\max}]$, $|\delta(\omega(k), \ell^*)| < |\delta(\omega_{\min}, \ell^*)|$.

- *The optimal parameter ℓ^* and the corresponding convergence factor δ^* admit the following asymptotic expansion:*

$$\ell^* = h^{-1/3}\left((c_x - ic_y) + o(1)\right) \quad \text{and} \quad \delta^* = 1 - c_r h^{1/3} + o(h^{1/3}), \tag{16}$$

where, introducing $r_{\min} = \mathrm{Re}(\omega_{\min})$ and $i_{\min} = \mathrm{Im}(\omega_{\min})$,

$$c_x = \left(\frac{r_{\min} + \sqrt{r_{\min}^2 + i_{\min}^2}}{2\sqrt{2}}\right)^{2/3}, \quad c_y = -\frac{i_{\min}}{2\sqrt{2c_x}}, \text{ and } \quad c_r = 2\sqrt{2c_x}. \tag{17}$$

Proof. The proof of Theorem 3 is divided into two main parts. We first construct a formal asymptotic expansion of ℓ^* that we justify *a posteriori*. To start with, we make an 'ansatz' on the asymptotic behaviour of the optimal parameter ℓ^*. We assume that

$$\ell^* \sim ch^{-\alpha} \text{ with } \alpha \in]0, 1[\text{ and } c = c_x - ic_y \quad (c_x > 0, c_y > 0).$$

Then, computing explicitly the derivative of $|\delta(\ell, k)|^2$, we prove that, in this asymptotic regime, the equi-oscillation property (9) holds for exactly two points $k_1 = k_{\min}$ and $k_2 = k_*$, where k_* admits the following asymptotic:

$$k_* = 2^{1/4}(c_x)^{1/4} h^{(-\alpha-1)/4} + o(h^{(-\alpha-1)/4}), \text{ and}$$

$$|\delta(\omega(k_*), \ell^*)|^2 \sim 1 - 4(2c_x)^{1/2} h^{\frac{1-\alpha}{2}} \quad |\delta(\omega_{\min}, \ell^*)|^2 \sim 1 - 4h^{\alpha} \frac{(c_x r_{\min} - c_y i_{\min})}{|c|^2}$$

Identifying the previous two expansions leads to

$$\alpha = \frac{1}{3} \quad \text{and} \quad \sqrt{2c_x}(c_x^2 + c_y^2) - (c_x r_{\min} - c_y i_{\min}) = 0. \tag{18}$$

Thus, in order to minimize the convergence factor (in this asymptotic regime), it suffices to find the couple (c_x, c_y) satisfying (18)(right) and such that c_x is maximal. A direct analysis of equation (18) leads to (17).

It remains to justify the obtained formal asymptotic. For $h \in (0,1)$ and $\varepsilon > 0$ sufficiently small, let

$$\mathscr{L}_h = \left\{ \ell \in \mathbb{C}, \text{ s. t. } h^{1/3}(\ell_x, \ell_y) \in \ [c_x - \varepsilon\,, c_x + \varepsilon] \times [-c_y - \varepsilon, -c_y + \varepsilon] \right\},$$

where c_x and c_y are defined by (17). Then, for h sufficiently small (in order to be able to define k^*), let $\Gamma_h = \{\ell \in \mathscr{L}_h, |\delta(\omega_{\min}, \ell)| = |\delta(\omega(k^*, \ell)|\}$. Because Γ_h is closed and non empty, there exists ℓ_*^h such that

$$|\delta(\omega_{\min}, \ell_*^h)| = \inf_{\ell \in \Gamma_h} |\delta(\omega_{\min}, \ell)|. \tag{19}$$

It is not difficult to prove that ℓ_*^h admits the asymptotic expansion (16). The end the proof of Theorem 3 consists in showing that $\ell_*^h = \ell^*$. This is done by proving the following lemma:

Lemma 4 *ℓ_*^h is a strict local minimum for $\ell \mapsto \|R(\omega(k), \ell)\|_{L^\infty(k_{\min}, k_{\max})}$.*

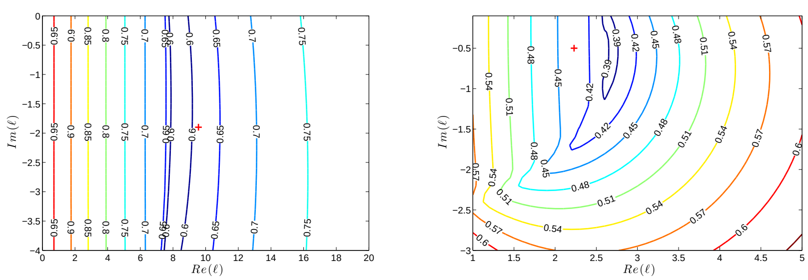
Indeed, Corollary 2.16 in [3] guarantees that any strict local minimum of the function $\ell \mapsto \|R(\omega(k), \ell)\|_{L^\infty(k_{\min}, k_{\max})}$ is the global minimum. Consequently $\ell_*^h = \ell^*$ and the proof is complete. The proof of Lemma (4) is an adaptation of the proof of [3, Theorem 4.2].

5 Numerical illustration

Let $\Omega =]0, \pi[^2$, $\nu = 1$ and $f = z = 0$ (hence $g = 0$), so that the exact solution is 0. The discretization is done using a standard second order finite difference scheme. We choose a similar discretization in the x and y directions ($h_x = h_y = h$) and we set $k_{\min} = 1$ and $k_{\max} = \frac{\pi}{h}$. In the non-overlapping case, we split the domain Ω into two domains Ω_1 and Ω_2 of equal size: $\Omega_1 =]0, \pi/2[\times]0, \pi[$ and $\Omega_2 =]\pi/2, \pi[\times]0, \pi[$. In the overlapping case, we take $\Omega_1 =]0, \pi/2[\times]0, \pi[$ and $\Omega_2 =]\pi/2 - h, \pi[\times]0, \pi[$ (i.e. $L = h$). The domain decomposition algorithm is initialized with a uniform (over $]0,1[$) random data λ_1^0. In the next experiments, we evaluate the numerical (or observed) convergence rate $\delta_{\mathrm{num}}(\ell, N)$ defined by

$$\delta_{\mathrm{num}}(\ell, N) = \left(\frac{e_N}{e_{N-1}}\right)^{1/2}, \quad e_n = \sqrt{\|u_{h,1}^n\|^2 + \|u_{h,2}^n\|^2} \tag{20}$$

On Figure 1, we evaluate $\delta_{\mathrm{num}}(\ell, N)$ for different values of ℓ taking $N = 60$ and $h = \pi/80$. The red cross corresponds to theoretical optimal parameter ℓ^*: in the non-overlapping case, $\ell^* = \sqrt{\omega_{\min}\omega_{\max}}$ while in the overlapping case, ℓ^* is numerically computed. Although the theoretical analysis is done for a two dimensional unbounded domain, we remark that the theoretical optimal parameter ℓ^* and the observed optimal parameter are relatively closed. Moreover, for $L = 0$, the convergence factor slowly varies with respect to the imaginary part of ℓ (cf. [5]). Then, Figure 2a presents the evolution of the error e_n with respect to the number n of iterations of the domain decomposition algorithm for two different values of ℓ: $\ell = \ell^*$ and $\ell = \ell^*_{\mathrm{num}}$, where ℓ^*_{num} denotes the numerical optimized coefficient obtained by optimizing $\delta_{\mathrm{num}}(\ell, N)$. Finally, Figure 2b shows the evolution of $1 - \delta_{\mathrm{num}}(\ell, N)$ with respect to the discretization parameter h. The introduction of the overlap perceptibly improves the observed convergence rate (although the asymptotic regime is not entirely reached in this case).

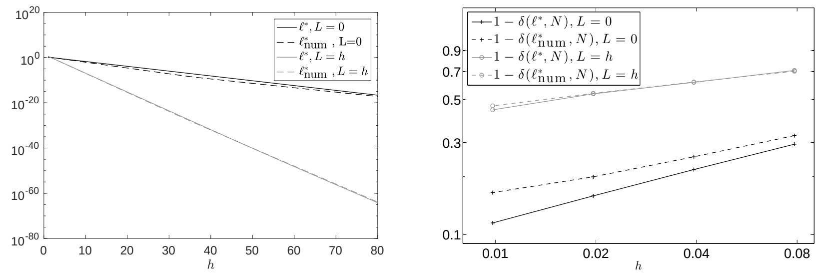


(a) $L = 0$. (b) $L = h$

Fig. 1: Contour plot of $\delta_{\mathrm{num}}(\ell, N)$ for $h = \pi/80$, $N = 60$

References

[1] Jean-David Benamou. A domain decomposition method with coupled transmission conditions for the optimal control of systems governed by



(a) Error w.r.t the num. of iter. (b) $\delta_{\text{num}}(\ell, N)$ w.r.t h

Fig. 2: Error and convergence factor in the overlapping and non-overlapping cases

elliptic partial differential equations. *SIAM J. Numer. Anal.*, 33(6):2401–2416, 1996.

[2] Jean-David Benamou and Bruno Després. A domain decomposition method for the Helmholtz equation and related optimal control problems. *J. Comput. Phys.*, 136(1):68–82, 1997.

[3] D. Bennequin, M. J. Gander, and L. Halpern. A homographic best approximation problem with application to optimized Schwarz waveform relaxation. *Math. Comp.*, 78(265):185–223, 2009.

[4] Daniel Bennequin, Martin J. Gander, Loic Gouarin, and Laurence Halpern. Optimized Schwarz waveform relaxation for advection reaction diffusion equations in two dimensions. *Numer. Math.*, 134(3):513–567, 2016.

[5] B. Delourme and L. Halpern. Optimized Schwarz method for control problems. In preparation.

[6] M. J. Gander, L. Halpern, and F. Magoulès. An optimized Schwarz method with two-sided Robin transmission conditions for the Helmholtz equation. *Internat. J. Numer. Methods Fluids*, 55(2):163–175, 2007.

[7] Martin J. Gander. Optimized Schwarz methods. *SIAM J. Numer. Anal.*, 44(2):699–731 (electronic), 2006.

[8] Martin J. Gander, Frédéric Magoulès, and Frédéric Nataf. Optimized Schwarz methods without overlap for the Helmholtz equation. *SIAM J. Sci. Comput.*, 24(1):38–60, 2002.

[9] J.-L. Lions. *Optimal control of systems governed by partial differential equations.* Translated from the French by S. K. Mitter. Die Grundlehren der mathematischen Wissenschaften, Band 170. Springer-Verlag, New York-Berlin, 1971.