Auxiliary space preconditioners for a DG discretization of $H(\mathbf{curl};\Omega)$- elliptic problem on hexahedral meshes

B. Ayuso de Dios, R. Hiptmair, and C. Pagliantini

Abstract We present a family of preconditioners based on the *auxiliary space method* for a discontinuous Galerkin discretization on cubical meshes of $H(\mathbf{curl};\Omega)$- elliptic problems with possibly discontinuous coefficients. We address the influence of possible discontinuities in the coefficients on the asymptotic performance of the proposed solvers and present numerical results in two dimensions.

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a simply connected bounded domain with Lipschitz boundary and let $\boldsymbol{f} \in L^2(\Omega)^3$. We consider the following $H(\mathbf{curl};\Omega)$-elliptic problem

$$\begin{cases} \nabla \times (\nu \nabla \times \boldsymbol{u}) + \beta \boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{u} \times \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

where $\nu = \nu(\mathbf{x}) \geq \nu_0 > 0$ and $\beta = \beta(\mathbf{x}) \geq \beta_0 > 0$ are assumed to be in $L^\infty(\Omega)$ but possibly discontinuous, and represent properties of the medium or material: ν is typically the inverse of the magnetic permeability and β is proportional to the ratio of electrical conductivity and the time step. Problem (1) arises in the modelling of magnetic diffusion and also after implicit time discretization of

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resistive magneto-hydrodynamics (MHD). In connection with the MHD application the use of hexahedral meshes is typically preferred over to family partitions made of simplices [Pagliantini(2016)].
Finite element discretizations using edge elements of the first family [Nédélec(1980)] are probably the most satisfactory methods to approximate (1) from a theoretical point of view. Only recently, a new *compatible* element (corresponding to an edge element of the second family) has been introduced in [Arnold and Awanou(2014)]. Discontinuous Galerkin (DG) methods offer an attractive alternative to conforming FE edge elements [Houston et al.(2005)] and allow for great flexibility in incorporating the discontinuities of the medium. For both methods, the condition number of the resulting linear systems degrades with mesh refinement and the size of the variations of the coefficients. Hence, designing a preconditioner able to cope with the combined effect of the mesh width and of highly varying coefficients turns out to be essential. For constant coefficients, efficient solvers for FE edge discretizations have been successfully developed using domain decomposition (DD) and the Auxiliary Space (AS) method [Hiptmair and Xu(2007)]. For discontinuous coefficients, a non-overlapping BDDC algorithm has been proposed and analyzed in [Dohrmann and Widlund(2016)], improving previous results in the DD literature, see e.g. [Toselli(2006)]. Recently, in [Ayuso de Dios et al.(2017)], we have developed a family of AS preconditioners for DG discretizations of (1), providing the analysis for simplicial meshes and in the case of cubical meshes when edge elements of the first kind are used as local spaces. In this paper, we report on the construction of the AS preconditioners focusing on the case of cubical meshes, discussing also their performance in the case of jumping coefficients. The proposed preconditioners rely on $H(\mathbf{curl};\Omega)$-conforming auxiliary spaces (as *auxiliary space*) and hence is presumed the availability of a (direct) solver for standard $H(\mathbf{curl};\Omega)$-conforming Galerkin discretizations.

2 SIPG Discretization on Hexahedral Meshes

Let $\mathscr{T}_h$ be a family of shape-regular partitions of Ω into cubes T. For each $T \in \mathscr{T}_h$, let $h_T = \mathrm{diam}(T)$ and set $h = \max_{T\in\mathscr{T}_h} h_T$. We assume that $\mathscr{T}_h$ is conforming and resolves the piece-wise constant coefficients β and ν. (i.e., $\nu_T, \beta_T \in \mathbb{P}^0(T)$ for all $T \in \mathscr{T}_h$).We denote by $\mathscr{F}_h$ the set of all faces of the partition; $\mathscr{F}_h^o$ and $\mathscr{F}_h^\partial$ refer respectively, to the collection of all interior and boundary faces. Similarly, $\mathscr{E}_h = \mathscr{E}_h^o \cup \mathscr{E}_h^\partial$ denote the set of all edges of the skeleton of $\mathscr{T}_h$; with $\mathscr{E}_h^o$ and $\mathscr{E}_h^\partial$ referring to interior and boundary edges, respectively. We define the sets:

$$\mathscr{T}(e) := \{T \in \mathscr{T}_h : e \subset \partial T\}; \qquad \mathscr{E}(T) := \{e \in \mathscr{E}_h : e \subset \partial T\};$$
$$\mathscr{F}(T) := \{f \in \mathscr{F}_h : f \subset \partial T\}; \qquad \mathscr{F}(e) := \{f \in \mathscr{F}_h : e \subset \partial f\}.$$

We introduce the (family of) DG finite element spaces

$$\mathbf{V}_h^{DG} = \{\boldsymbol{v} \in L^2(\Omega)^3 : \boldsymbol{v} \in \mathscr{M}(T), T \in \mathscr{T}_h\}, \qquad \mathscr{M}(T) \subseteq \mathbb{Q}_k(T)^3$$

where the local space $\mathscr{M}(T)$ of vector-valued polynomials can be of three types:

1. Nédélec elements of first family on cubical meshes [Nédélec(1980)]

$$\mathscr{M}(T) = \mathscr{N}^I(T) := \mathbb{Q}_{k-1,k,k}(T) \times \mathbb{Q}_{k,k-1,k}(T) \times \mathbb{Q}_{k,k,k-1}(T), \qquad k \geq 1,$$

where $\mathbb{Q}_{\ell,m,n}(T)$ is the space of polynomials of degree at most ℓ, m, n in each vector variable.

2. Compatible elements (of second kind) [Arnold and Awanou(2014)]:

$$\mathscr{M}(T) = \mathscr{S}_k(T) := (\mathbb{P}_k(T))^3 + \text{span}\{[yz(w_2(x,z) - w_3(x,y)), zx(w_3(x,y) - w_1(y,z)), xy(w_1(y,z) - w_2(x,z))] + \nabla s(x,y,z)\},$$

where each $w_i \in \mathbb{P}_k$ and $s \in \mathbb{P}_k(T)$ has superlinear degree (ordinary degree ignoring variables which appear linearly) at most $k+1$, with $k \geq 1$.

3. Full polynomials: We set the local space $\mathscr{M}(T) = (\mathbb{Q}_k(T))^3$, and $k \geq 1$.

For each choice of the resulting $\mathbf{V}_h^{DG}$ space, the corresponding $\mathrm{H}_0(\mathbf{curl},\Omega)$-conforming finite element spaces are defined as:

$$\mathbf{V}_h^c := \mathbf{V}_h^{DG} \cap \mathrm{H}_0(\mathbf{curl},\Omega) = \{\boldsymbol{v} \in \mathrm{H}_0(\mathbf{curl},\Omega) : \boldsymbol{v} \in \mathscr{M}(T), T \in \mathscr{T}_h\}. \tag{2}$$

For a piecewise smooth vector-valued function $\boldsymbol{v}$, we denote by $\boldsymbol{v}^{\pm}$ the traces of $\boldsymbol{v}$ taken from within $T^{\pm}$. The tangential jump, indicated by $[\![\cdot]\!]_\tau$, is defined by

$$[\![\boldsymbol{v}]\!]_\tau := \mathbf{n}^+ \times \boldsymbol{v}^+ + \mathbf{n}^- \times \boldsymbol{v}^- \quad \text{on} \quad f \in \mathscr{F}_h^o, \qquad [\![\boldsymbol{v}]\!]_\tau := \mathbf{n} \times \boldsymbol{v} \text{ on } f \in \mathscr{F}_h^\partial$$

where $\mathbf{n}^+$ and $\mathbf{n}^-$ denote the unit normal vectors on $f = \partial T^+ \cap \in \partial T^-$ pointing outwards from T^+ and T^-, respectively. We will also use the notation

$$(\theta \boldsymbol{u}, \boldsymbol{v})_{\mathscr{T}_h} = \sum_{T \in \mathscr{T}_h} \int_T \theta_T \boldsymbol{u}\boldsymbol{v} d\mathbf{x}, \qquad \langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\mathscr{F}_h} = \sum_{f \in \mathscr{F}_h} \int_f \boldsymbol{u}\boldsymbol{v} ds \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbf{V}_h^{DG}$$

where $\theta \in \mathbb{P}^0(\mathscr{T}_h)$ will be either $\theta = \nu$ or $\theta = \beta$.

The SIPG-DG method. We consider a symmetric Interior Penalty method (SIPG) introduced recently in [Ayuso de Dios et al.(2017)] for approximating (1) robustly (w.r.t the discontinuous coefficients). The method reads:

$$\text{Find} \quad \boldsymbol{u}_h \in \mathbf{V}_h^{DG} \quad \text{such that} \quad \mathrm{a}_{\mathrm{DG}}(\boldsymbol{u}_h, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_{\mathscr{T}_h} \quad \forall \boldsymbol{v} \in \mathbf{V}_h^{DG}, \tag{3}$$

with $\mathrm{a}_{\mathrm{DG}}(\cdot,\cdot)$ defined by

$$\begin{aligned}\mathrm{a}_{\mathrm{DG}}(\boldsymbol{u}, \boldsymbol{v}) &:= (\nu \nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v})_{\mathscr{T}_h} + (\beta \boldsymbol{u}, \boldsymbol{v})_{\mathscr{T}_h} - \langle \{\!\{\nu \nabla \times \boldsymbol{u}\}\!\}_\gamma, [\![\boldsymbol{v}]\!]_\tau \rangle_{\mathscr{F}_h} \\ &- \langle [\![\boldsymbol{u}]\!]_\tau, \{\!\{\nu \nabla \times \boldsymbol{v}\}\!\}_\gamma \rangle_{\mathscr{F}_h} + \sum_{T \in \mathscr{T}_h} \alpha_T(\nu) \sum_{e \in \mathscr{E}(T)} \sum_{f \in \mathscr{F}(e)} (s_f [\![\boldsymbol{u}]\!]_\tau, [\![\boldsymbol{v}]\!]_\tau)_{0,f}.\end{aligned} \tag{4}$$

In (4), the *weighted average* $\{\{\cdot\}\}_\gamma$ is defined as the plain trace for a boundary face, whereas for $\partial T^+ \cap \partial T^- = f \in \mathscr{F}_h^o$, is given by

$$\{\{\boldsymbol{u}\}\}_\gamma := \gamma_f^+ \boldsymbol{u}^+ + \gamma_f^- \boldsymbol{u}^- \quad \text{with} \quad \gamma_f^\pm = \frac{\nu^\mp}{\nu^+ + \nu^-}, \qquad \nu^\pm := \nu_{|_{T^\pm}} .$$

The penalization is defined by $s_f := c h_f^{-1}$ on all $f \in \mathscr{F}_h$ with some $c > 0$ and the mesh function $h_f = \min\{h_{T^+}, h_{T^-}\}$ on $f \in \mathscr{F}_h^o$ and $h_f = h_T$ on $f = \partial T \cap \partial\Omega$.
The coefficient function $(\alpha_T(\nu))_{T\in\mathscr{T}_h} \in \mathbb{P}_0(\mathscr{T}_h)$ is defined by

$$\alpha_T(\nu) := \max_{f\in\mathscr{F}(T)} \{\{\nu\}\}_{*,f} \quad \text{with} \quad \{\{\nu\}\}_{*,f} := \begin{cases} \max\limits_{\substack{T\in\mathscr{T}(e)\\ e\subset\partial f}} \nu_T & f \in \mathscr{F}_h^o, \\ \nu_T & f \in \mathscr{F}_h^\partial . \end{cases}$$

Notice that $\alpha_T(\nu)$ picks the maximum conductivity coefficient over a patch of elements surrounding T. In Figure 1 a 2D sketch of such patch is given.

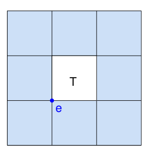

Fig. 1: 2D sketch of the patch involved in definition of $\alpha_T(\nu)$.

We stress that the weighted average $\{\{\cdot\}\}_\gamma$ together with $\{\{\cdot\}\}_{*,f}$ and the definition of $\alpha_T(\nu)$ ensure robustness (with respect to the coefficients) of both the approximation (3) in the energy norm (see [Ayuso de Dios et al.(2017), Proposition 2.1], and [Pagliantini(2016), Proposition 5.1.1]) and the preconditioners, see Theorem 1 and [Ayuso de Dios et al.(2017), Pagliantini(2016)] for details in the analysis. Observe that when the variational formulation (3) is restricted to $\mathbf{V}_h^c$ in (2), the corresponding $\mathrm{H}_0(\mathbf{curl},\Omega)$-conforming discretization of (1) is obtained. In fact,

$$a_{\mathscr{W}}(\boldsymbol{u},\boldsymbol{v}) := (\nu\nabla\times\boldsymbol{u}, \nabla\times\boldsymbol{v})_\Omega + (\beta\boldsymbol{u},\boldsymbol{v})_\Omega = a_{DG}(\boldsymbol{u},\boldsymbol{v}) \quad \forall\, \boldsymbol{u},\boldsymbol{v} \in \mathbf{V}_h^c. \tag{5}$$

We denote by $\mathscr{A} : \mathbf{V}_h^{DG} \longrightarrow (\mathbf{V}_h^{DG})'$ the discrete operator $(\mathscr{A}\boldsymbol{u},\boldsymbol{w}) = a_{DG}(\boldsymbol{u},\boldsymbol{w})$ and by $\mathbb{A}$ the matrix representation of $\mathscr{A}$ with respect to a localized "nodal" basis of $\mathbf{V}_h^{DG}$ (using any of the choices for $\mathscr{M}(T)$). It can be verified that the spectral condition number $\kappa(\mathbb{A})$ is proportional to

$$h^{-2}\frac{\max_T \alpha_T(\nu)}{\min_T \nu_T} + \frac{\max_T \beta_T}{\min_T \beta_T} .$$

3 Auxiliary Space Preconditioning

The Auxiliary Space Method (ASM) was introduced in [Xu(1996), Oswald(1996)] as an expansion of the Fictitious Space Method [Nepomnyaschikh(1991)] providing a neat methodology for developing and analysing preconditioners. To describe the preconditioners we propose, based on the AS methodology, we first review the basic ingredients behind the Fictitious Space Method:

(1) the *fictitious space*: a real finite dimensional Hilbert space $\overline{\mathscr{V}}$, endowed with an inner product $\bar{a}(\cdot,\cdot)$, induced operator $\overline{\mathscr{A}}:\overline{\mathscr{V}}\to\overline{\mathscr{V}}'$ and norm $\|\cdot\|_{\overline{\mathscr{A}}}$.
(2) A continuous, linear and surjective transfer operator $\Pi:\overline{\mathscr{V}}\to\mathbf{V}_h^{DG}$.

By virtue of [Nepomnyaschikh(1991)], an optimal preconditioner for $\overline{\mathscr{A}}$ would result in an optimal preconditioner for $\mathscr{A}$. The distinguishing feature of ASM is the particular choice of $\overline{\mathscr{V}}$ as a product space, including the original space as one of the components. Here, we set $\overline{\mathscr{V}}=\mathbf{V}_h^{DG}\times\mathscr{W}$, endowed with the inner product

$$\bar{a}(\bar{\boldsymbol{v}},\bar{\boldsymbol{v}})=s(\boldsymbol{v}_0,\boldsymbol{v}_0)+a_{\mathscr{W}}(\boldsymbol{w},\boldsymbol{w}),\qquad \forall\bar{\boldsymbol{v}}=(\boldsymbol{v}_0,\boldsymbol{w}),\ \boldsymbol{v}_0\in\mathbf{V}_h^{DG},\ \boldsymbol{w}\in\mathscr{W},\tag{6}$$

where $\mathscr{W}$ is the (truly) so-called auxiliary space and $a_{\mathscr{W}}(\cdot,\cdot)$ is the auxiliary bilinear form. We will always take as $\mathscr{W}$ an $\mathrm{H}_0(\mathbf{curl},\Omega)$-conforming space $\mathbf{V}_h^c$. In (6), $s(\cdot,\cdot)$ is the bilinear form associated with a relaxation operator $\mathscr{S}$ on $\mathbf{V}_h^{DG}$. Denoting by $\mathscr{A}_{\mathscr{W}}$ the operator associated with $a_{\mathscr{W}}(\cdot,\cdot)$, the auxiliary space preconditioner operator is $\mathscr{B}=\mathscr{S}^{-1}+\Pi_{\mathscr{W}}\circ\mathscr{A}_{\mathscr{W}}^{-1}\circ\Pi_{\mathscr{W}}^{*}$ where the linear transfer operator $\Pi_{\mathscr{W}}:\mathscr{W}\to\mathbf{V}_h^{DG}$ is the standard inclusion and its adjoint $\Pi_{\mathscr{W}}^{*}:\mathbf{V}_h^{DG}\to\mathscr{W}$ is defined by $a_{\mathscr{W}}(\Pi_{\mathscr{W}}^{*}\boldsymbol{v},\boldsymbol{w})=a(\boldsymbol{v},\Pi_{\mathscr{W}}\boldsymbol{w})$, $\boldsymbol{v}\in\mathbf{V}_h^{DG}$, $\boldsymbol{w}\in\mathscr{W}$. If $\mathbb{S}\in\mathbb{R}^{N\times N}$ with $N:=\dim\mathbf{V}_h^{DG}$ and $\mathbb{A}_W\in\mathbb{R}^{N_W\times N_W}$, $N_W:=\dim\mathscr{W}$, then the preconditioner in algebraic form reads

$$\mathbb{B}=\mathbb{S}^{-1}+\mathbb{P}\mathbb{A}_W^{-1}\mathbb{P}^{\mathsf{T}},\tag{7}$$

where $\mathbb{P}\in\mathbb{R}^{N\times N_W}$ is the matrix representation of the transfer operator $\Pi_{\mathscr{W}}$.

We now specify the precise components for the two preconditioners we propose:

1. Natural Preconditioner: We set $\mathscr{W}=\mathbf{V}_h^c=\mathbf{V}_h^{DG}\cap\mathrm{H}_0(\mathbf{curl},\Omega)$ for any choice of the local space $\mathscr{M}(T)$ and $a_{\mathscr{W}}(\cdot,\cdot)$ is as in (5). Hence, $\mathscr{A}_{\mathscr{W}}:\mathbf{V}_h^c\to(\mathbf{V}_h^c)'$ is self-adjoint and positive definite. As relaxation operator $\mathscr{S}$ it is sufficient to use a simple Jacobi or block Jacobi smoother.

2. Coarser or Economical Preconditioner: When the local space is either $\mathscr{M}(T)=\mathscr{S}_k(T)$ or $\mathscr{M}(T)=(\mathbb{Q}_k(T))^3$ in the construction of the $\mathbf{V}_h^{DG}$-space, we consider a second possibility for the AS preconditioner. We take $\mathscr{W}$ as

$$\mathscr{W}:=\mathscr{W}_h^c=\{\boldsymbol{w}\in\mathrm{H}_0(\mathbf{curl},\Omega):\boldsymbol{w}|_T\in\mathscr{N}^I(T),T\in\mathscr{T}_h\}\subset\mathbf{V}_h^c\subset\mathbf{V}_h^{DG}.$$

As to the relaxation operator, we demonstrate numerically that a non-overlapping Schwarz smoother is not able to resolve the components in the kernel of $\mathbf{curl}(\mathscr{W})$ and as a consequence an overlapping smoother is necessary. We will show numerically that in the case $\mathscr{M}(T)=(\mathbb{Q}_k(T))^3$, the resulting AS preconditioner is not effective, independently of the choice of the smoother and the amount of domain overlaps involved in its construction. We suspect that this is connected to the fact that the DG method using $\mathscr{M}(T)=(\mathbb{Q}_k(T))^3$ is not spectrally correct, while $\mathscr{W}_h^c$ is.

Next result provides the convergence of the *Natural Preconditioner*.

Theorem 1. *Let $\mathbb{B}$ be the auxiliary space preconditioner in* (7)*, with $\mathscr{W} = \mathbf{V}_h^c$ and simple Jacobi smoother on $\mathbf{V}_h^{DG}$. Let Δ_h and Δ_h' denote the set of elements in the* ***curl****-dominated regime and* **reaction***-dominated region, respectively:*

$$\Delta_h := \{T \in \mathscr{T}_h : h_T^2 \beta_T < \alpha_T(\nu)\}, \quad \Delta_h' := \{T \in \mathscr{T}_h : h_T^2 \beta_T \geq \alpha_T(\nu)\}.$$

Then, the spectral condition number of the resulting preconditioned system satisfies

$$\kappa(\mathbb{BA}) \lesssim \max\{1, \Theta(\nu,\beta)\},$$

$$\text{with } \Theta(\nu,\beta) := \min\left\{ \max_{T\in\mathscr{T}_h} \frac{h_T^2\beta_T}{\nu_T}, \max_{\substack{T,T'\in\mathscr{T}_h \\ \partial T\cap\partial T'\neq\emptyset}} \frac{\beta_T}{\beta_{T'}}, \max_{\substack{T\in\Delta_h, T'\in\Delta_h' \\ \partial T\cap\partial T'\neq\emptyset}} \frac{\alpha_T(\nu)}{\alpha_{T'}(\nu)} \right\}.$$

The proof can be found in [Ayuso de Dios et al.(2017), Pagliantini(2016)] as well as the analysis of the *Coarser AS Preconditioner* on simplicial meshes. The analysis of a *Coarser AS Preconditioner* on hexahedral meshes is still an open problem.

4 Numerical Results

In the following numerical simulations we will restrict to the two dimensional problem (1) on a square. We set the constant entering in the penalty parameter s_f in (4) to $c = 10$. The tolerance for the CG and PCG is set to 10^{-7}. In the tables we always report the number of iterations required for convergence. We refer to the AS preconditioners by $\mathbf{V}_h^{DG} - \mathscr{W}$, or more precisely by the local spaces $\mathscr{M}(T)$ in the construction of each $\mathbf{V}_h^{DG}$ and $\mathscr{W}$. Since the experiments are in 2D we use the rotated Nédélec elements of the first family $\mathscr{N}^I(T) = \mathscr{RT}_0$; the rotated version of the space $\mathscr{S}_1 := \mathscr{RT}_0 + \{\mathbf{curl}(x^2y), \mathbf{curl}(xy^2), \mathbf{curl}(x^2), \mathbf{curl}(y^2)\}$, and the 2D full polynomials space $\mathbb{Q}_1(T)^2$. For the *Natural AS Preconditioner* a simple Jacobi smoother is always used. For the *Coarser or Economical AS Preconditioner* we will specify the smoother used at each time.

Test Cases with Continuous Coefficients. We consider first the constant coefficient case $\beta = \nu = 1$. As shown in Table 1, the natural AS preconditioner is indeed optimal in all the cases, as predicted by Theorem 1. In contrast, the coarser AS preconditioner performs optimally for $\mathscr{S}_1 - \mathscr{RT}_0$ only if an overlapping smoother is included. However, the coarser AS preconditioner $\mathbb{Q}_1 - \mathscr{RT}_0$ is not efficacious regardless the smoother involved in the construction.

To get some insight on the failure of the coarser AS preconditioner for $\mathbb{Q}_1$, we explore the spectral approximation of the considered DG methods to (1) on $\Omega = [0,\pi]^2$ with $\nu = 1$ and $\beta = 0$. The exact eigenvalues are given by $n^2 + m^2$ for n and m positive integers. In Figure 2 is given the lower part of the spectrum using a DG discretization based on the three possible choices of local spaces $\mathscr{M}(T)$. As it can be observed in in Figure 2, the DG discretization based on the full polynomial space $(\mathbb{Q}_1)^2$, is not spectrally correct. Therefore, a preconditioner built on an auxiliary

$\sharp\mathscr{T}_h$	16×16	32×32	64×64	128×128	256×256
$\mathscr{RT}_0$ Unpreconditioned	128	204	376	753	1504
$(Q_1)^2$ Unpreconditioned	410	815	1454	2796	4554
$\mathscr{S}_1$ Unpreconditioned	543	1083	2031	4056	7316
$\mathscr{RT}_0$-$\mathscr{RT}_0$ Jacobi	9	9	9	9	9
Q_1-Q_1 Jacobi	22	21	20	19	19
Q_1-$\mathscr{RT}_0$: Jacobi | overlapping	259 | 61	471 | 113	844 | 202	1622 | 337	2936 | 618
$\mathscr{S}_1$-$\mathscr{RT}_0$: Jacobi | overlapping	88 | 18	72 | 19	49 | 20	34 | 20	36 | 19

Table 1: Number of iterations for test case with constant coefficients.

space where the $H_0(\mathbf{curl},\Omega)$-conforming discretization is spectrally correct (e.g. Nédélec elements of the first family) is not effective.

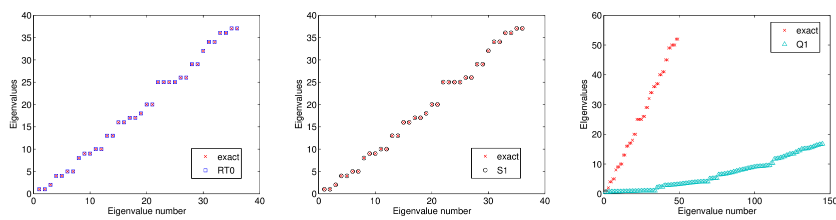

Fig. 2: Lower part of the spectrum for different DG discretizations: rotated Nédélec elements of the first family $\mathscr{RT}_0$ (left), rotated $\mathscr{S}_1$ (center), and the full polynomial space $(Q_1)^2$ (right).

Test Case with Discontinuous Coefficients. We consider now the more challenging case of β and ν both discontinuous following a checkerboard distribution according to the partition $\Omega_1 := [0,0.5]^2 \cup [0.5,1]^2 \subset \Omega = [0,1]^2$. We define

$$\nu(\mathbf{x}) = \begin{cases} 10^2 & \text{if } \mathbf{x}\in\Omega_1\,, \\ 1 & \text{otherwise}\,, \end{cases} \qquad \text{and} \qquad \beta(\mathbf{x}) = \begin{cases} 10^{-3} & \text{if } \mathbf{x}\in\Omega_1\,, \\ 10 & \text{otherwise}\,. \end{cases}$$

In Table 2 we report the iteration counts of the different preconditioners and in Figure 3 are given graphically the estimated condition numbers of the preconditioned systems. As it can be observed in Figure 3 and Table 2, the natural AS preconditioner performs optimally in the presence of discontinuous coefficients, as predicted by Theorem 1. The coarser AS preconditioner $\mathscr{S}_1$-$\mathscr{RT}_0$ is also efficacious in this case, when using an overlapping relaxation. As regards the $(\mathbb{Q}_1)^2$ DG discretization, the coarser AS preconditioner is totally ineffective.

$\sharp\mathscr{T}_h$	16×16	32×32	64×64	128×128	256×256
$\mathscr{RT}_0$-$\mathscr{RT}_0$ Jacobi	11	10	10	10	10
Q_1-Q_1 Jacobi	23	22	21	21	20
$\mathscr{S}_1$-$\mathscr{RT}_0$: overlapping	24	24	24	25	24
Q_1-$\mathscr{RT}_0$: overlapping	69	129	248	425	–

Table 2: Number of iterations for test case with discontinuous coefficients.

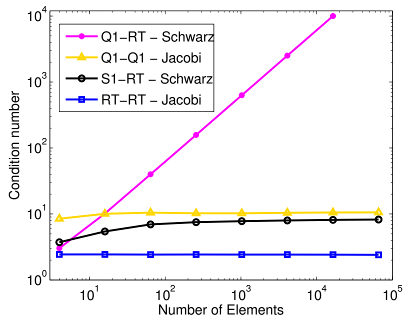

Fig. 3: Test case with discontinuous coefficients. Condition number vs. number of elements: $\mathscr{S}_1$ DG discretization with ASM based on rotated $\mathscr{RT}_0$ elements with overlapping additive Schwarz smoother (black); DG discretization with rotated $\mathscr{RT}_0$ discontinuous elements and rotated $\mathscr{RT}_0$ as auxiliary space with pointwise Jacobi smoother (blue); discontinuous bilinear Lagrangian elements with $\mathrm{H}(\mathbf{curl},\Omega)$-conforming full polynomial auxiliary space and Jacobi smoother (orange).

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