

Complete, Optimal and Optimized Coarse Spaces for Additive Schwarz

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1 Introduction and Model Problem

Coarse spaces are needed to achieve scalability in domain decomposition methods, see [16] and references therein. More recently, new coarse corrections were also designed to improve convergence, for example in high contrast problems. Such enriched coarse spaces were first proposed in [4, 5], where volume eigenfunctions were combined with different types of partition of unity functions, and further developed in [3]. A coarse space using the eigenfunctions of the Dirichlet-to-Neumann maps on the boundary of each subdomain has been proposed and analyzed in [13, 2], and further development led to solving a generalized eigenvalue problem in the overlap (GenEO), see [14, 15].

A new, different idea is to first define an optimal coarse space, which leads to the best possible convergence and makes the method nilpotent [6]¹ and then to approximate it [8, 9, 10, 11]. Following this principle, we design here for the first time an optimal coarse space for the additive Schwarz (AS) method with arbitrary sized overlaps, and then define an optimized approximation using a specific function in the overlap, combined with harmonic extensions of interface eigenfunctions. We compare our new coarse space to the GenEO coarse space [14, 15] and the local spectral multiscale coarse space (also with reduced energy) in [4].

Determining an optimal coarse space and then approximating it is a very general idea, see for example the SLEM coarse space [10, 11], but for simplicity we consider here

$$\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1)$$

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¹ Classical one level domain decomposition methods can even be nilpotent in certain situations, see [1].

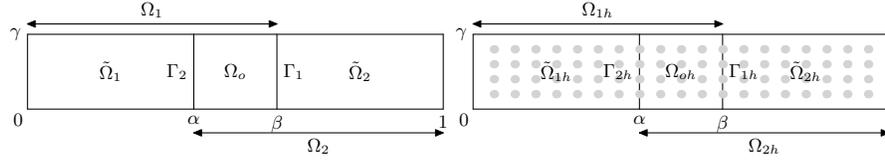


Fig. 1 Continuous (left) and discrete (right) partition of Ω into two overlapping subdomains.

where $\Omega := (0, 1) \times (0, \gamma)$ is decomposed into two overlapping subdomains $\Omega_1 := (0, \beta) \times (0, \gamma)$ and $\Omega_2 := (\alpha, 1) \times (0, \gamma)$, with overlap $\Omega_o := \Omega_1 \cap \Omega_2$, and interfaces $\Gamma_1 := \{(x, y) | x = \beta, 0 < y < \gamma\}$ and $\Gamma_2 := \{(x, y) | x = \alpha, 0 < y < \gamma\}$, which leads to the partition of the domain $\tilde{\Omega} = \tilde{\Omega}_1 \cup \tilde{\Omega}_o \cup \tilde{\Omega}_2$; see Figure 1.

Discretizing (1) by the classical five-point finite difference scheme, we obtain the linear system $\mathbf{A}\mathbf{u} = \mathbf{f}$. Starting with an initial guess \mathbf{u}^0 , the iterative two level additive Schwarz method with multiplicative (hybrid) coarse grid correction computes

$$\begin{aligned} \mathbf{u}^{n-1/2} &= \mathbf{u}^{n-1} + (R_1^T A_1^{-1} R_1 + R_2^T A_2^{-1} R_2)(\mathbf{f} - \mathbf{A}\mathbf{u}^{n-1}), \\ \mathbf{u}^n &= \mathbf{u}^{n-1/2} + R_c^T A_c^{-1} R_c(\mathbf{f} - \mathbf{A}\mathbf{u}^{n-1/2}), \end{aligned} \quad (2)$$

where R_i are rectangular restriction matrices corresponding to Ω_i , $A_i = R_i A R_i^T$, $i = 1, 2$, and R_c is a restriction matrix to a coarse space, $A_c = R_c A R_c^T$.

2 Complete, Optimal and Optimized Coarse Spaces

Definition 1 (Complete coarse space). A complete coarse space for the additive Schwarz method (2) is given by R_c such that (2) converges after one iteration for an arbitrary initial guess \mathbf{u}^0 , i.e. the method is nilpotent and becomes a direct solver.

To give an example of a complete coarse space, and being able to write discrete problems using the same notation as continuous ones, we denote by Δ_h the discretized Laplacian, and by Ω_h , $\tilde{\Omega}_{ih}$, Ω_{oh} , Γ_{ih} the corresponding discretized spaces, $i = 1, 2$. Let N_{Γ_i} be the number of degrees of freedom (DOFs) on the interface Γ_{ih} , $i = 1, 2$, and let $\phi_{i,cs}^j$ be defined for each DOF on Γ_{ih} by harmonic extension,

$$\begin{aligned} \Delta_h \phi_{i,cs}^j &= 0 && \text{in } \tilde{\Omega}_{ih}, \\ \phi_{i,cs}^j &= 1 && \text{at DOF } j \text{ of } \Gamma_{3-i,h}, \quad j = 1, \dots, N_{\Gamma_{3-i}}, \\ \phi_{i,cs}^j &= 0 && \text{elsewhere in } \Omega_h. \end{aligned} \quad (3)$$

Denoting by N_o the number of DOFs in the overlap Ω_{oh} , we define for each of them the further basis function $\phi_{o,cs}^j = 1$, extended by zero to the rest of Ω_h , $j = 1, \dots, N_o$, and

$$V_{0,cs} := \text{span}\left\{ \left\{ \phi_{i,cs}^j \right\}_{j=1}^{N_{\Gamma_{3-i}}} \right\}_{i=1}^2 \cup \left\{ \phi_{o,cs}^j \right\}_{j=1}^{N_o} \}. \quad (4)$$

Theorem 1. *A complete coarse space for the iterative two level additive Schwarz algorithm (2) is given by R_c containing in its columns the vectors of $V_{0,cs}$ from (4).*

Proof. The proof is technical, see [12], but the property is illustrated in Section 4.

The dimension of this complete coarse space depends on the number of DOFs in the overlap for AS which was designed to be symmetric [7], but it can be reduced².

Definition 2 (Optimal coarse space). An optimal coarse space for (2) is a complete coarse space such that its associated R_c has the smallest number of columns possible.

For an optimal coarse space, we define the restriction matrix in the overlap, $R_o := [0 \ I_{\Omega_{oh}} \ 0]$, where I_{Ω_o} is the identity matrix whose dimension equals the number of unknowns in Ω_{oh} , and the associated local solver in the overlap, $A_o := R_o A R_o^T$. We then construct *just one* specific basis function ϕ_o in the overlap Ω_{oh} for (2), based on the initial guess \mathbf{u}^0 by solving

$$A_o \phi_o = R_o (\mathbf{f} - A \mathbf{u}^0),$$

and then extending ϕ_o with zero to the rest of Ω_h . We also need the basis functions $\phi_{i,opt}^j$, $j = 1, \dots, N_{\Gamma_{3-i}}$, $i = 1, 2$, based on harmonic extensions,

$$\begin{aligned} \Delta_h \phi_{i,opt}^j &= 0 & \text{in } \tilde{\Omega}_{ih}, & & \Delta_h \phi_{i,opt}^j &= 0 & \text{in } \Omega_{oh}, \\ \phi_{i,opt}^j &= 1 & \text{at DOF } j \text{ of } \Gamma_{3-i,h}, & & \phi_{i,opt}^j &= 1 & \text{at DOF } j \text{ of } \Gamma_{3-i,h}, \\ \phi_{i,opt}^j &= 0 & \text{elsewhere in } \Omega_h, & & \phi_{i,opt}^j &= 0 & \text{elsewhere in } \Omega_h, \end{aligned}$$

and define

$$V_{0,opt} := \text{span} \left\{ \left\{ \left\{ \phi_{i,opt}^j \right\}_{j=1}^{N_{\Gamma_{3-i}}} \right\}_{i=1}^2 \cup \{ \phi_o \} \right\}. \quad (5)$$

Theorem 2. *An optimal coarse space for the iterative two level additive Schwarz algorithm (2) is given by R_c containing in its columns the vectors of $V_{0,opt}$ from (5).*

Proof. The proof is given in [12], but the property is again illustrated in Section 4.

At the continuous level, even the optimal coarse space would still be infinite dimensional, and we thus introduce now an approximation of the optimal coarse space based on SHEMA (Spectral Harmonically Enriched Multiscale coarse space [10]) using an interface eigenvalue problem:

Definition 3 (Interface eigenvalue problem). Denoting by D_{yy} an approximation of the second derivative along the interface Γ_i , the interface eigenvalue problem is

$$-D_{yy} \psi_i = \lambda \psi_i \quad \text{on } \Gamma_{ih}, \quad (6)$$

with zero Dirichlet boundary conditions $\psi_i(0) = \psi_i(\gamma) = 0$, $i = 1, 2$.

² This problem does not arise with overlap of one or two mesh sizes [11], or RAS [10].

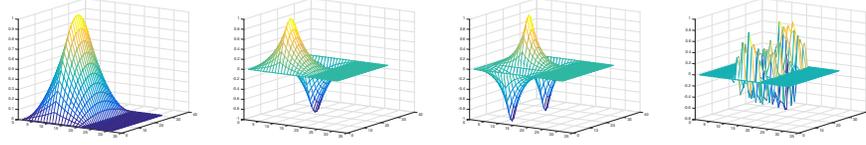


Fig. 2 First 3 basis functions used to approximate the optimal coarse space based on the interface eigenfunctions, and the single mode in the overlap for a random initial guess on the right.

In our example, the eigenvectors of the interface eigenvalue problem (6) are $\psi_i^j = \sin((j\pi/\gamma)y_m)$, $y_m = mh$. We can thus construct basis functions $\phi_{i,\text{app}}^j$ by the harmonic extensions of the sine functions for $i = 1, 2$,

$$\begin{aligned} \Delta_h \phi_{i,\text{app}}^j &= 0 & \text{in } \tilde{\Omega}_{ih}, & & \Delta_h \phi_{i,\text{app}}^j &= 0 & \text{in } \Omega_{oh}, \\ \phi_{i,\text{app}}^j &= \psi_{3-i}^j & \text{on } \Gamma_{3-i,h}, & & \phi_{i,\text{app}}^j &= \psi_{3-i}^j & \text{on } \Gamma_{3-i,h} \\ \phi_{i,\text{app}}^j &= 0 & \text{elsewhere in } \Omega_h, & & \phi_{i,\text{app}}^j &= 0 & \text{elsewhere in } \Omega_h, \end{aligned} \quad (7)$$

$j = 1, \dots, \ell$, where ℓ is the number of the eigenvectors of the interface eigenvalue problem (6) selected; see Figure 2 for an illustration. We then define an optimized approximation of the optimal coarse space

$$V_{0,\text{cs-1}} = \text{span}\{\{\{\phi_{i,\text{app}}^j\}_{j=1}^\ell\}_{i=1}^2 \cup \{\phi_o\}\}. \quad (8)$$

Theorem 3. *The iterative two level additive Schwarz algorithm (2) with R_c containing in its columns the vectors of $V_{0,\text{cs-1}}$ in (8) satisfies the error estimate*

$$\|\mathbf{u}^n - \mathbf{u}\|_{\infty,2} \leq \left(\frac{\cosh(\frac{(\ell+1)\pi}{\gamma}(\alpha + \beta - 1)) - \cosh(\frac{(\ell+1)\pi}{\gamma}(\beta - \alpha - 1))}{\cosh(\frac{(\ell+1)\pi}{\gamma}(\alpha + \beta - 1)) - \cosh(\frac{(\ell+1)\pi}{\gamma}(\beta - \alpha + 1))} \right)^{n/2} \|\mathbf{u}^0 - \mathbf{u}\|_{\infty,2},$$

and there is no other coarse space of this dimension that leads to faster convergence.

Proof. The proof can be obtained by a direct calculation using separation of variables for the residual after one additive Schwarz iteration, and will be given in [12].

3 Comparison to Two Other Coarse Spaces

We now compare our optimized coarse space to the GenEO coarse space from [14, 15], and the local spectral multiscale coarse space with reduced energy from [4]. The GenEO coarse space was designed for high contrast problems and is based on generalized eigenproblems “in the overlap”: in our example it solves in Ω_i , $i = 1, 2$,

$$\hat{A}_i \mathbf{p}_i^j = \lambda_i^j X_i \hat{A}_i^o X_i \mathbf{p}_i^j \quad (9)$$

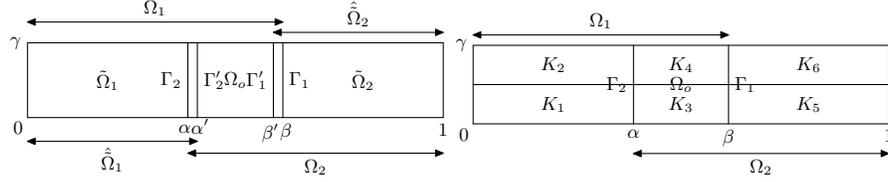


Fig. 3 Left: partition of the domain for the GenEO coarse space. Right: partition of the domain for the local multiscale coarse space with reduced energy.

for eigenvectors $\mathbf{p}_i^j \in \mathbb{R}^{\#\text{dof}(\Omega_i)}$ associated with small eigenvalues $\lambda_i^j \in \mathbb{R} \cup \{+\infty\}$. In (9), X_j is a diagonal matrix indicating the partition of unity used to combine subdomain solutions, and \hat{A}_i, \hat{A}_i^o are Neumann matrices for each subdomain. Selecting the ℓ eigenfunctions corresponding to the smallest eigenvalues then leads to the GenEO coarse space

$$V_{0,\text{GenEO}} = \text{span}\{\mathcal{R}_i^T X_i \mathbf{p}_i^j, j = 1, \dots, \ell, i = 1, 2\}.$$

To understand how GenEO is related to our optimized coarse space, we first rewrite the eigenvalue problem (9) at the continuous level for $\lambda_i \neq 0$ and $\lambda_i \neq 4$,

$$\begin{aligned} \Delta \hat{p}_i(x, y) &= 0 & \text{in } \hat{\Omega}_i, & \quad \Delta p_{io}(x, y) = 0 & \text{in } \Omega_o, \\ \hat{p}_i &= p_{io} & \text{on } \hat{\Gamma}_{3-i}, & \quad p_{io} = \frac{4}{4 - \lambda_i} \hat{p}_i & \text{on } \Gamma_{3-i}, \end{aligned} \quad (10)$$

with boundary conditions $\hat{p}_i = 0$ on $\partial\Omega \cap \hat{\Omega}_i$, $p_{io} = 0$ on $(\partial\Omega \cap \hat{\Omega}_o) \setminus \Gamma_i$ and $\partial_n p_{io} = 0$ on Γ_i , and then define $p_i := \hat{p}_i$ in $\hat{\Omega}_i$, $p_i := p_{io}$ in Ω_o , and $p_i := 0$ in the rest of Ω , $i = 1, 2$. Here, Γ'_i are within one mesh size from the corresponding boundary Γ_i , $i = 1, 2$, see Figure 3 on the left. Solving (10) with separation of variables for Ω_1 , we find for our model problem

$$p_1^j(x, y) = \begin{cases} \frac{4}{4 - \lambda_1^j} \frac{\sinh(\frac{j\pi}{\gamma} x) \cosh(\frac{j\pi}{\gamma} (\beta - \alpha'))}{\sinh(\frac{j\pi}{\gamma} \alpha') \cosh(\frac{j\pi}{\gamma} (\beta - \alpha))} \sin(\frac{j\pi}{\gamma} y), & (x, y) \in (0, \alpha) \times (0, \gamma), \\ \frac{4}{4 - \lambda_1^j} \frac{\cosh(\frac{j\pi}{\gamma} (\beta - x))}{\cosh(\frac{j\pi}{\gamma} (\beta - \alpha))} \sin(\frac{j\pi}{\gamma} y), & (x, y) \in (\alpha, \beta) \times (0, \gamma), \end{cases}$$

$$\lambda_1^j = 4 - 4 \frac{\sinh(\frac{j\pi}{\gamma} \alpha) \cosh(\frac{j\pi}{\gamma} (\beta - \alpha'))}{\sinh(\frac{j\pi}{\gamma} \alpha') \cosh(\frac{j\pi}{\gamma} (\beta - \alpha))}.$$

We show in Figure 4 the three types of GenEO eigenfunctions. The eigenfunctions corresponding to the smallest eigenvalues are like the ones in our optimized coarse space within the subdomains, but in the overlap they differ. Since GenEO uses an eigenvalue problem in the entire subdomain volume, it also contains many more eigenfunctions (which one avoids to compute in GenEO), like the overlap ones for

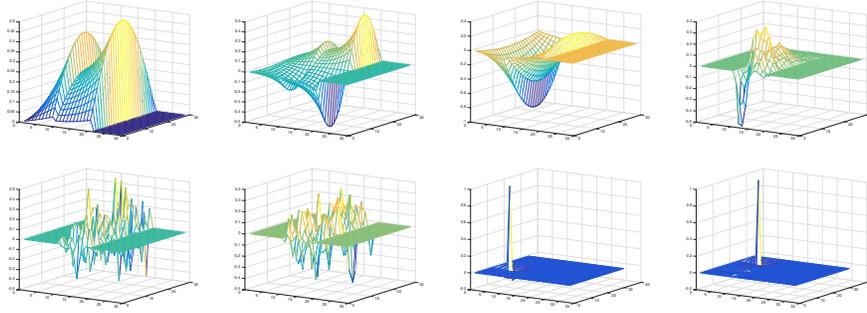


Fig. 4 First 3 basis functions on Ω_1 of the eigenvalue problem from GenEO for the smallest λ , followed by 3 of the randomly looking modes for $\lambda = 4$ and 2 of the modes for $\lambda = \infty$.

$\lambda = 4$ corresponding to $\phi_{o,cs}^j$ in our complete coarse space, plus the ones for $\lambda = \infty$ which do not contain relevant information for the coarse space.

We next compare our optimized coarse space with the local spectral multiscale coarse space (also with reduced energy) in [4]. The domain Ω is still decomposed into two overlapping subdomains Ω_1 and Ω_2 , with six coarse blocks K_i , $i = 1, \dots, 6$, see Figure 3 on the right. Let \mathbf{q}_i^j denote the j th eigenvector of the volume eigenvalue problem in subdomain Ω_{ih} , $i = 1, 2$,

$$\begin{aligned} \Delta_h \mathbf{q}_i &= \lambda_i \mathbf{q}_i && \text{in } \Omega_{ih}, \\ \partial_n \mathbf{q}_i &= 0 && \text{on } \Gamma_{ih}, \\ \mathbf{q}_i &= 0 && \text{on } \partial\Omega_{ih} \setminus \Gamma_{ih}. \end{aligned} \quad (11)$$

With the partition of unity χ_i , $i = 1, 2$, the local spectral multiscale coarse space of [4] using ℓ functions is defined by

$$V_{0,\text{mul}} = \text{span}\{R_i^T \chi_i \mathbf{q}_i^j, 1 \leq j \leq \ell, i = 1, 2\}. \quad (12)$$

The local spectral multiscale coarse space with reduced energy of [4] is defined by

$$\tilde{V}_{0,\text{mul}} = \text{span}\{R_i^T \tilde{\mathbf{q}}_i^j, 1 \leq j \leq \ell, i = 1, 2\},$$

where for each block $K_h \in \Omega_{ih}$, $i = 1, 2$, $1 \leq j \leq \ell$, one still needs to solve $\Delta_h \tilde{\mathbf{q}}_i^j = 0$ in K_h , $\tilde{\mathbf{q}}_i^j = \chi_i \mathbf{q}_i^j$ on ∂K_h . Solving (11) using separation of variables, we find in Ω_1

$$\mathbf{q}_1^{jk}(x, y) = \sin\left(\frac{k\pi - \pi/2}{\alpha}x\right) \sin\left(\frac{j\pi}{\gamma}y\right), \quad \lambda_1^{jk} = \left(\frac{k\pi - \pi/2}{\alpha}\right)^2 + \left(\frac{j\pi}{\gamma}\right)^2.$$

We show in Figure 5 the first few of those modes. Note that these modes are different from the modes in our optimized coarse space and GenEO, and again one needs to

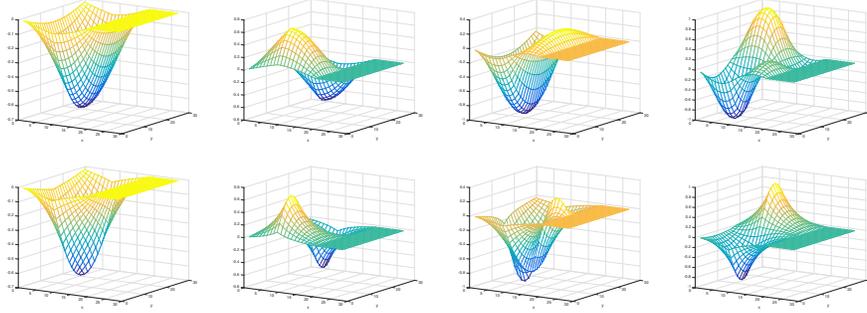


Fig. 5 Top: First 4 basis functions of the local spectral multiscale coarse space $V_{0,mul}$. Bottom: corresponding modes for the reduced energy case.

solve volume eigenvalue problems to construct the coarse spaces $V_{0,mul}$ and $\tilde{V}_{0,mul}$, which now also contain many redundant modes.

4 Numerical experiments

We solve (1) with $f = -3$ on $\Omega = (0, 1) \times (0, 1)$ discretized by centered finite differences and using an overlap of $4h$, h being the mesh parameter. We start with a random initial guess, and stop the iteration when the error in the iterative method or the residual in PCG reaches the tolerance $1e-8$. In Table 1 we show the dependence of the number of iterations on h for the complete coarse space (CS), the optimal coarse space (CS-opt), our optimized coarse space SHEM(ℓ), and GenEO(ℓ), GalvisI(ℓ) and GalvisII(ℓ) (reduced energy), using $\ell = 3$ enrichment functions for each subdomain. We see that CS and CS-opt are direct solvers, and only SHEM(ℓ) leads to a convergent stationary iteration; and SHEM(ℓ) also performs best with PCG.

In Table 2, we show the iteration numbers for 2×2 , 4×4 , and 8×8 subdomains using $h = 1/32, 1/64, 1/128$, i.e. keeping H/h fixed. We choose again $\ell = 3$ for

Table 1 Iteration number comparison in the two subdomain case for different mesh parameters h .

	h	AS	CS	CS-opt	SHEM(3)	GenEO(3)	GalvisI(3)	GalvisII(3)
Iterative	$\frac{1}{16}$	na	1	1	6	na	na	na
	$\frac{1}{32}$	na	1	1	11	na	na	na
	$\frac{1}{64}$	na	1	1	20	na	na	na
	$\frac{1}{128}$	na	1	1	38	na	na	na
PCG	$\frac{1}{16}$	9	1	1	4	8	8	7
	$\frac{1}{32}$	11	1	1	6	9	9	9
	$\frac{1}{64}$	14	1	1	9	11	12	11
	$\frac{1}{128}$	17	1	1	12	14	14	14

Table 2 Iteration number comparison for many subdomains.

	Subdomains	AS	CS	CS-opt	SHEM	GenEO	GalvisI	GalvisII
Iterative	2×2	na	1	1	8	na	na	na
	4×4	na	1	1	8	na	na	na
	8×8	na	1	1	8	na	na	na
PCG	2×2	15	1	1	5	13	12	10
	4×4	25	1	1	5	13	14	10
	8×8	42	1	1	5	13	14	10

SHEM, and approximately the same total number of coarse functions for the other coarse spaces. We see again that CS and CS-opt are direct solvers, and only SHEM leads to a convergent stationary method. When used with PCG, the methods are all scalable, but SHEM needs only half the number of iterations compared to the other methods.

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