Coupling Parareal and Dirichlet-Neumann/Neumann-Neumann Waveform Relaxation Methods for the Heat Equation

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1 Introduction

We introduce two new space-time Waveform Relaxation (WR) methods based on the parareal algorithms and Dirichlet-Neumann waveform relaxation (DNWR) and Neumann-Neumann waveform relaxation (NNWR). The WR method was first introduced by Lelaramee, Ruehli and Sangiovanni-Vincentelli [15], which has been applied to analyze for many different kinds of problems, such as differential algebraic equations[11], fractional differential equations [13] , reaction diffusion equations [17]; for further details, see [12]. Domain decomposition methods for time-dependent partial differential equations (PDEs) can also lead to WR methods, i.e. Schwarz waveform relaxation (SWR) algorithm [8, 10], optimized Schwarz waveform relaxation (OSWR) algorithm [2, 3], and Dirichlet-Neumann and Neumann-Neumann waveform relaxation methods [6, 7, 22].

The parareal algorithm is a time-parallel method that was proposed by Lions, Maday, and Turinici in the context of virtual control to solve evolution problems in parallel [16]. In this algorithm, initial value problems are solved on subintervals in time, and through iterations the initial values on each subinterval are corrected to converge to the correct values of the overall solution [1, 9, 5]. The parareal algorithm has also been combined with waveform relaxation methods [18].

Parallel algorithms based on the decomposition of both time and space domain have been also studied [21, 19]. However, there was no parallel mechanism in the time direction. In [20], it was the first time that the combination of Schwarz waveform relaxation and parareal for PDEs had been introduced. Further, in [4], a new parallel algorithm where there is no order between the Schwarz waveform relaxation algorithm and the parareal algorithm was introduce.

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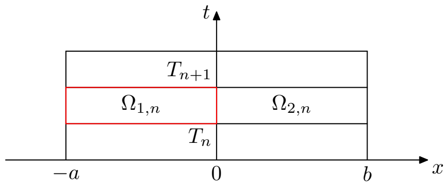

Fig. 1 Space time decomposition on which the proposed algorithms are based.

In this paper, we propose the parareal Dirichlet-Neumann waveform relaxation (PA-DNWR) and the parareal Neumann-Neumann waveform relaxation (PA-NNWR) methods for the time-dependent problem. For ease of presentation for the new algorithms, we derive our results for two subdomains in one spatial dimension.

We consider the following initial-value problem of heat equation on bounded $\Omega \subset \mathbb{R}$

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f(x,t), & x \in \Omega,\ 0 < t < T, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = g(x,t), & x \in \partial\Omega,\ 0 < t < T. \end{cases} \tag{1}$$

2 Parareal Dirichlet-Neumann/Neumann-Neumann waveform relaxation algorithms

We define the new algorithms for the model problem (1) on the space-time domain $\Omega \times (0,T) = (-a,b) \times (0,T)$. We assume that Ω is decomposed into two nonoverlapping subdomains, i.e. $\Omega_1 = (-a,0)$ and $\Omega_2 = (0,b)$, and the time interval $(0,T)$ is decomposed into N equal time subintervals (T_n, T_{n+1}) with $\Delta T = T_{n+1} - T_n = T/N$, $n = 0,1,\ldots,N-1$. We then can define the non-overlapping space-time subdomain $\Omega_{i,n} = \Omega_i \times (T_n, T_{n+1})$, $i = 1,2, n = 0,1,\ldots,N-1$; see Figure 1.

In order to introduce the parareal Dirichlet-Neumann waveform relaxation algorithm for the model problem (1), we first introduce several propagators. We define two propagator $F_{1,n}(U(x),\omega(t))$ and $G_{1,n}(U(x),\omega(t))$ to solve the following Dirichlet problem in $\Omega_{1,n}$

$$\begin{cases} \dfrac{\partial u_{1,n}}{\partial t} = \dfrac{\partial^2 u_{1,n}}{\partial x^2} + f(x,t), & (x,t) \in \Omega_{1,n}, \\ u_{1,n}(-a,t) = g(-a,t), & t \in (T_n, T_{n+1}), \\ u_{1,n}(0,t) = \omega(t), & t \in (T_n, T_{n+1}), \\ u_{1,n}(x,T_n) = U(x), & x \in \Omega_1, \end{cases} \tag{2}$$

using an accurate approximation and a rough approximation, where $U(x)$ and $\omega(t)$ are given data. Furthermore, two propagators $F_{2,n}(U(x),\omega(x,t))$ and $G_{2,n}(U(x),\omega(x,t))$ are defined to solve the following Neumann problem in $\Omega_{2,n}$

$$\begin{cases} \dfrac{\partial u_{2,n}}{\partial t} = \dfrac{\partial^2 u_{2,n}}{\partial x^2} + f(x,t), & (x,t) \in \Omega_{2,n}, \\ \partial_x u_{2,n}(0,t) = \partial_x \omega(0,t), & t \in (T_n, T_{n+1}), \\ u_{2,n}(b,t) = g(b,t), & t \in (T_n, T_{n+1}), \\ u_{2,n}(x,T_n) = U(x), & x \in \Omega_2, \end{cases} \tag{3}$$

using an accurate approximation and a rough approximation. Therefore the parareal Dirichlet-Neumann waveform relaxation algorithm for the model problem (1) consists of the following steps: Given an initial guess $\omega_n^0(t)$ along the interface $\Gamma = \{x=0\} \times (T_n, T_{n+1})$, and an initial guess $U_{i,n}^0(x,t)$, and for $k = 0,1,2,\ldots$, Step I: use the more accurate evolution operator from (2) and (3) to calculate

$$u_{1,n}^{k+1}(x,t) := F_{1,n}(U_{1,n}^k(x), \omega_n^k(t)),$$
$$u_{2,n}^{k+1}(x,t) := F_{2,n}(U_{2,n}^k(x), u_{1,n}^{k+1}(x,t));$$

Step II: update interface information

$$\omega_n^{k+1}(t) = \theta u_{2,n}^{k+1}(0,t) + (1-\theta)\omega_n^k(t);$$

Step III: update new initial conditions using a parareal step both in space and time for $n = 0,1,\ldots,N-1$ by

$$\begin{aligned} U_{1,n+1}^{k+1} &= u_{1,n}^{k+1}(\cdot,T_{n+1}) + G_{1,n}(U_{1,n}^{k+1}(x), \omega_n^{k+1}(t)) - G_{1,n}(U_{1,n}^k(x), \omega_n^k(t)), \\ U_{2,n+1}^{k+1} &= u_{2,n}^{k+1}(\cdot,T_{n+1}) + G_{2,n}(U_{2,n}^{k+1}(x), U_{1,n+1}^{k+1}(x,t)) - G_{2,n}(U_{2,n}^k(x), U_{1,n+1}^k(x,t)). \end{aligned} \tag{4}$$

Next we will introduce the parareal Neumann-Neumann waveform relaxation algorithm. Similar, we first introduce two propagators $FD_{i,n}(U(x),h(t))$ and $GD_{i,n}(U(x),h(t))$ to solve the following Dirichlet problem in $\Omega_{i,n}$

$$\begin{cases} \dfrac{\partial u_{i,n}}{\partial t} = \dfrac{\partial^2 u_{i,n}}{\partial x^2} + f(x,t), & (x,t) \in \Omega_{i,n}, \\ u_{i,n}(x,t) = g(x,t), & x \in \partial\Omega \cap \Omega_i,\ t \in (T_n, T_{n+1}), \\ u_{i,n}(0,t) = h(t), & t \in (T_n, T_{n+1}), \\ u_{i,n}(x,T_n) = U(x), & x \in \Omega_i, \end{cases} \tag{5}$$

and two propagators $FN_{i,n}(u_{1,n}(x,t), u_{2,n}(x,t))$ and $GN_{i,n}(u_{1,n}(x,t), u_{2,n}(x,t))$, $i = 1,2$ to solve the following Neumann problem in $\Omega_{i,n}$

$$\begin{cases} \dfrac{\partial \Psi_{i,n}}{\partial t} = \dfrac{\partial^2 \Psi_{i,n}}{\partial x^2}, & (x,t) \in \Omega_{i,n}, \\ \Psi_{i,n}(x,t) = 0, & x \in \partial\Omega \cap \Omega_i,\ t \in (T_n, T_{n+1}), \\ \partial_{n_i} \Psi_{i,n}(0,t) = \sum_j \partial_{n_j} u_{j,n}(0,t), & x \in \Gamma,\ t \in (T_n, T_{n+1}), \\ \Psi_{i,n}(x,T_n) = 0, & x \in \Omega_i, \end{cases} \tag{6}$$

using an accurate approximation and a rough approximation.

Therefore the parareal Neumann-Neumann waveform relaxation algorithm for the model problem (1) consists of the following steps: Given an initial guess $h_n^0(t)$ along the interface $\Gamma = \{x=0\} \times (T_n, T_{n+1})$, and an initial guess $U_{i,n}^0(x,t)$, and for $k = 0,1,2,\ldots$, Step I: use the more accurate evolution operator from (5) to calculate the Dirichlet problem

$$u_{i,n}^{k+1}(x,t) := FD_{i,n}(U_{i,n}^k(x), h_n^k(t)), i = 1,2;$$

Step II: use the more accurate evolution operator from (6) to calculate the Neumann problem

$$\Psi_{i,n}^{k+1}(x,t) := FN_{i,n}(u_{1,n}^{k+1}(x,t), u_{2,n}^{k+1}(x,t))), i = 1,2;$$

Step III: update interface information

$$h_n^{k+1}(t) = h_n^k(t) - \theta(\Psi_{1,n}^{k+1}(0,t) + \Psi_{2,n}^{k+1}(0,t));$$

Step IV: update the new initial conditions using a parareal step both in space and time for $n = 0,1,\ldots,N-1$ by

$$\begin{aligned} U_{1,n+1}^{k+1} &= u_{1,n}^{k+1}(\cdot, T_{n+1}) + GD_{1,n}(U_{1,n}^{k+1}(x), h_n^{k+1}(t)) - GD_{1,n}(U_{1,n}^k(x), h_n^k(t)), \\ U_{2,n+1}^{k+1} &= u_{2,n}^{k+1}(\cdot, T_{n+1}) + GD_{2,n}(U_{2,n}^{k+1}(x), h_n^{k+1}(t)) - GD_{2,n}(U_{2,n}^k(x), h_n^k(t)). \end{aligned} \tag{7}$$

Different from regular DNWR/NNWR and using parareal to solve the subproblems, our new methods are in parallel both in space and time, and there is no order between DNWR/NNWR and parareal. Meanwhile, we don't need to using parareal to achieve the convergence for each subproblem for each DNWR/NNWR iteration.

Theorem 1 (Convergence for parareal DNWR). *Assuming that the F-propagator is an exact solver and G-propagator is chosen as backward Euler method, if $a = b$, then $\theta = 1/2$ is the optimal parameter and fixed $T > 0$, and the parareal DNWR algorithm is convergent in finite steps; if $a \neq b$, for $\theta = 1/2$ and fixed $T > 0$, the parareal DNWR algorithm is convergent.*

Theorem 2 (Convergence for parareal NNWR). *Assuming that the F-propagator is an exact solver and G-propagator is chosen as backward Euler method, if $a = b$, then $\theta = 1/4$ is the optimal parameter and fixed $T > 0$, and the parareal DNWR algorithm is convergent in finite steps; if $a \neq b$, for $\theta = 1/4$ and and fixed $T > 0$, the parareal DNWR algorithm is convergent*

Proof. The first parts of both theorems can be directly obtained by the convergence results of parareal in [9], and DNWR and NNWR in [6]; and the proves of the second parts are technical and will in [14], a detailed numerical study of how the algorithm depends on the various parameters in Section 3.

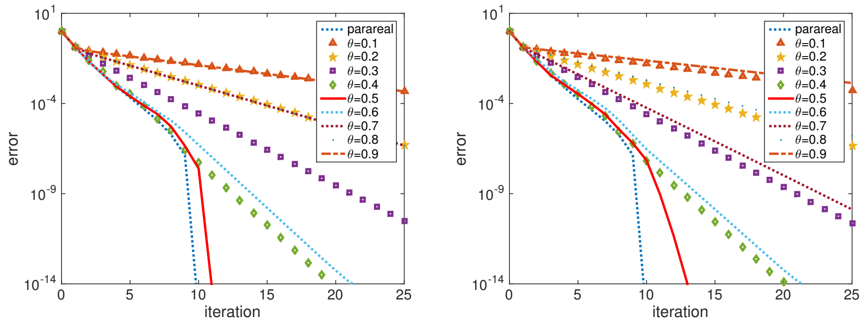

Fig. 2 Convergence of parareal DNWR for various values of the parameter θ with $T = 2$ and $\Delta T = 1/5$ for $a = b = 3$ on the left and $a = 2$, $b = 3$ on the right.

3 Numerical experiments

The numerical experiments in this section were performed for the model problem (1) on the domain $(-a,b)\times(0,T)$ with $f = 0$, $u_0(x) = x(x+1)(x+3)(x-2)\exp(-x)$, $g(-a,t) = t$ and $g(b,t) = t\exp(t)$. The diffusion problem is discretized using a centered finite differences with mesh size $h = \Delta x = 2\times 10^{-2}$ in space and backward Euler with $\Delta t = 4\times 10^{-3}$ in time. The domain is decomposed into the space-time subdomains $\Omega_{i,n}$ as described in Section 2. We test the algorithms by choosing $h_n^0(t) = t^2$, $t\in(T_n, T_{n+1})$ as an initial guess.

We first test the parareal DNWR algorithm. Figure 2 shows the convergence behavior for different values of θ with $T = 2$ and $\Delta T = 1/5$ for the case $a = b = 3$ on the left, and for the case $a = 2$, $b = 3$ on the right. Note that $\theta = 1/2$ is the best parameter in both cases as sated in Theorem 1, and the performance of the parareal DNWR algorithm is similar when compared to the parareal algorithm, especially when chose the parameter $\theta = 1/2$. Then we show the convergence behavior for the best parameters $\theta = 1/2$ for different numbers of the time subintervals N with $T = 2$ for both cases in Figure 3, and for different time window length T with $\Delta T = 1/5$ in Figure 4. We observe that the convergence of the parareal DNWR slows down when the number of time intervals N is increased and time interval T is increased, which is similar to the performance of the parareal algorithm; see [9].

For the parareal DNWR algorithm, Figure 5 shows the convergence behavior for different values of θ with $T = 2$ and $\Delta T = 1/5$ for the case $a = b = 3$ on the left, and for the case $a = 2$, $b = 3$ on the right. Note that $\theta = 1/4$ is the best parameter in both cases. Then we show the convergence behavior for the best parameters $\theta = 1/4$ for different numbers of the time subintervals N with $T = 2$ for both cases in Figure 6, and for different time window length T with $\Delta T = 1/5$ in Figure 7. We observe that parareal NNWR also has the similar perfomance as that of the parareal algorithm and parareal DNWR. However, compared to parareal DNWR, the parareal NNWR needs almost double numbers of iterations to achieve convergence in the same cases.

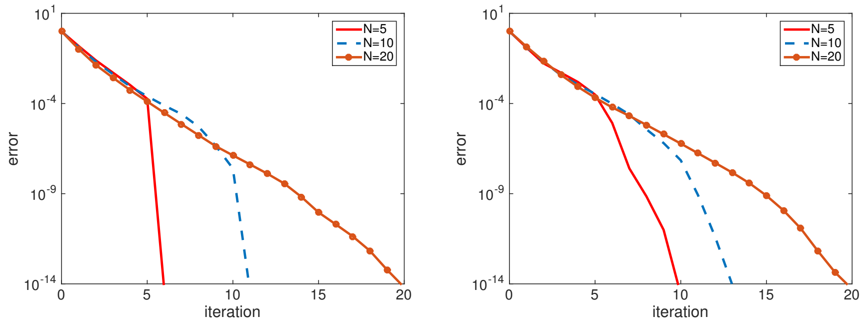

Fig. 3 Convergence of parareal DNWR for various values of the number of time subintervals N with $T = 2$ and $\theta = 1/2$ for $a = b = 3$ on the left and $a = 2, b = 3$ on the right.

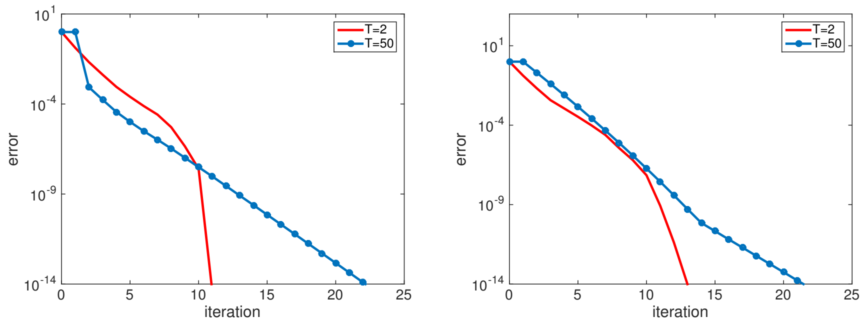

Fig. 4 Convergence of parareal DNWR for various values of the time window length T with $\Delta T = 1/5$ and $\theta = 1/2$ for $a = b = 3$ on the left and $a = 2, b = 3$ on the right.

4 Conclusions

We introduced the parareal DNWR and parareal NNWR algorithms for the heat equation, and provide their convergence properties for the two subdomain decomposition in one spatial dimension case. We showed that the convergence can be achieved in a finite number of iterations when choosing a proper relaxation parameter as chose for the DNWR and NNWR algorithms. Numerical results illustrate our analysis, which also indicate that the performance of parareal DNWR is better than that of parareal NNWR. We will further find the possible way to improve the performance parareal NNWR.

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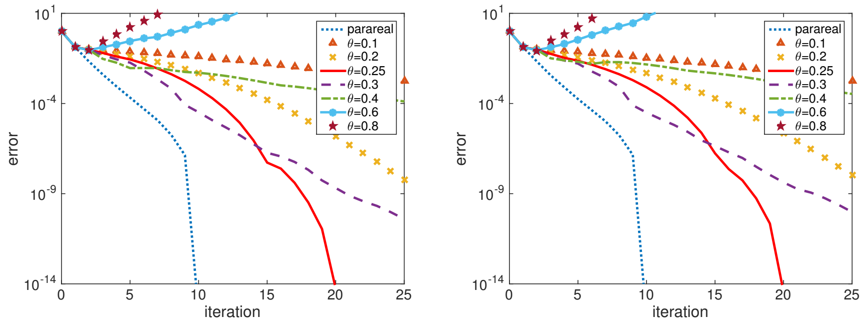

Fig. 5 Convergence of parareal NNWR for various values of the parameter θ with $T = 2$ and $\Delta T = 1/5$ for $a = b = 3$ on the left and $a = 2$, $b = 3$ on the right.

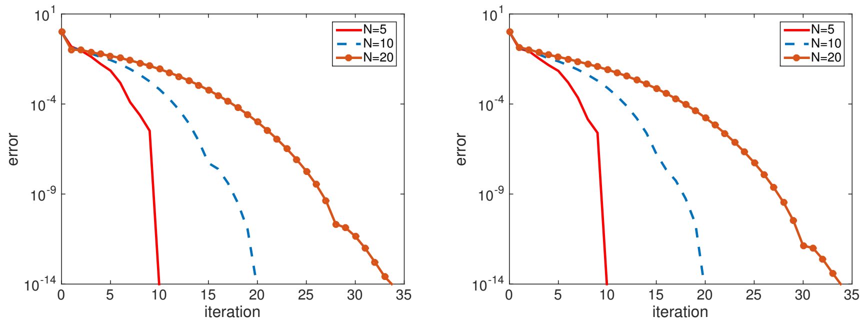

Fig. 6 Convergence of parareal NNWR for various values of the number of time subintervals N with $T = 2$ and $\theta = 1/4$ for $a = b = 3$ on the left and $a = 2$, $b = 3$ on the right.

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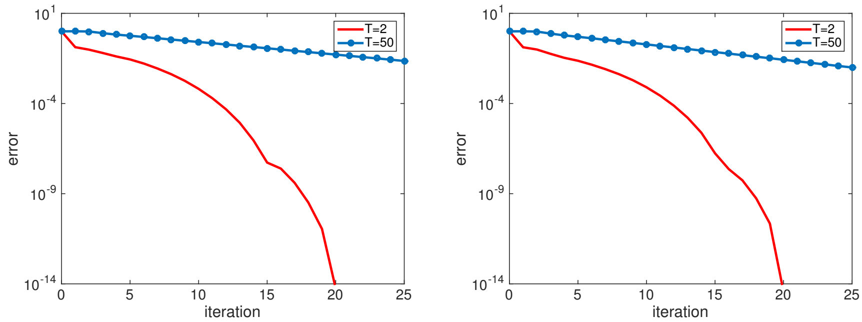

Fig. 7 Convergence of parareal NNWR for various values of the time window length T with $\Delta T = 1/5$ and $\theta = 1/4$ for $a = b = 3$ on the left and $a = 2$, $b = 3$ on the right.

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