Nonoverlapping three grid Additive Schwarz for hp-DGFEM with discontinuous coefficients

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Abstract We discuss a nonoverlapping additive Schwarz method for an *h*-*p* DGFEM discretization of an elliptic PDE with discontinuous coefficients, where the fine grid is decomposed into subdomains of size H and the coarse grid consists of cells size $\mathscr{H}$ such that $h \leq H \leq \mathscr{H}$. We prove the condition number is $O(p^2/q) \cdot O(\mathscr{H}^2/Hh)$ and is independent from the jumps of the coefficient if the discontinuities are aligned with the coarse grid.

1 Introduction

Let us consider a second order elliptic equation

$$-\operatorname{div}(\rho \nabla u) = f, \tag{1}$$

with homogeneous Dirichlet boundary condition. The problem is discretized by an *h*-*p* symmetric weighted interior penalty discontinuous Galerkin finite element method. A nonoverlapping additive Schwarz method (see [3], [1]) is applied to precondition the discrete equations. For $\rho \equiv 1$, Antonietti and Houston [1] conjectured on the basis of numerical experiments that if the coarse space contains piecewise polynomial functions up to degree p, the condition number is $O(p\mathscr{H}/h)$. This conjecture has recently been proved in [5] and independently by Antonietti, Houston and Smears in [2], using slightly different techniques. In the former paper, a general framework for the analysis of problems with discontinuous coefficients and varying polynomial degrees across finite elements has been developed; however, a technical assumption that the basis functions are continuous inside subdomains was made when the coefficient was allowed discontinous in Ω. On the other hand, [2] made use of approximation ideas of [6], allowing for more flexibility in the choice of the finite element spaces. In this note, we extend the analysis to the case when fully

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discontinuous finite elements are employed, under additional assumption that the coefficient is constant inside coarse grid cells and an H^2–regularity assumption on (1) holds.

For more flexibility and enhanced parallelism, we formulate our results addressing the case when the subdomains (where the local problems are solved in parallel) are potentially smaller than the coarse grid cells [4]. By allowing small subdomains of diameter $H \leq \mathscr{H}$, local problems are cheaper to solve and the amount of concurrency of the method is substantially increased, which can be an advantage e.g. on multi-threaded processors. Moreover, small subdomains give more flexibility in assigning them to processors for load balancing in coarse grain parallel processing. In this way, an additional level of domain partitioning gives the user more parameters to fine tune the actual parallel performance, and thus overall efficiency, of the preconditioner for a given hardware architecture.

The paper is organized as follows. In Section 2, the differential problem and its discontinuous Galerkin discretization are formulated. In Section 3, a nonoverlapping two-level, three-grid additive ASM for solving the discrete problem is designed and analyzed under assumption that the coarse mesh resolves the discontinuities of the coefficient, the variation of the mesh size and of the polynomial degree are locally bounded, and the original problem satisfies some regularity assumption. Section 4 presents some numerical experiments.

For nonnegative scalars x, y, we shall write $x \lesssim y$ if there exists a positive constant C, independent of: x, y, the fine, subdomain and coarse mesh parameters $h, H, \mathscr{H}$, the orders of the finite element spaces p, q, and of jumps of the diffusion coefficient ρ as well, such that $x \leq Cy$. If both $x \lesssim y$ and $y \lesssim x$, we shall write $x \simeq y$.

The norm of a function f from the Sobolev space $H^k(S)$ will be denoted by $||f||_{k,S}$, while the seminorm of f will be denoted by $|f|_{k,S}$. For short, the L^2-norm of f will then be denoted by $|f|_{0,S}$.

2 Differential problem and its *h-p* discontinuous Galerkin discretization

Let Ω be a bounded open convex polyhedral domain in R^d, $d \in \{2,3\}$, with Lipschitz boundary $\partial\Omega$. We consider the following problem for given $f \in L^2(\Omega)$ and $\rho \in L^\infty(\Omega)$:

Find $U^* \in H_0^1(\Omega)$ such that

$$a(U^*, v) = (f, v)_\Omega, \qquad \forall v \in H_0^1(\Omega), \tag{2}$$

where

$$a(u,v) = \int_\Omega \rho\, \nabla u \cdot \nabla v\, dx, \qquad (f,v)_\Omega = \int_\Omega f v\, dx.$$

We assume that there exist constants α_0 and α_1 such that $0 < \alpha_0 \leq \rho \leq \alpha_1$ a.e. in Ω so that (2) is well–posed. Without loss of generality we shall additionally suppose that $\alpha_0 \geq 1$ and $\operatorname{diam}(\Omega) = 1$, which can always be guaranteed by simple scaling. We also assume that ρ is piecewise constant, i.e. Ω can be partitioned into nonoverlapping polyhedral subregions with the property that ρ restricted to any of these subregions is some positive constant.

Let $\mathscr{T}_h = \{K_1, \ldots, K_{N_h}\}$ denote an affine nonconforming partition of Ω, where K_i are either triangles in 2-D or tetrahedrons in 3-D. For $K \in \mathscr{T}_h$ we set $h_K = \operatorname{diam}(K)$. By $\mathscr{E}_h^{\text{in}}$ we denote the set of all common (internal) faces (edges in 2-D) of elements in $\mathscr{T}_h$, so that $e \in \mathscr{E}_h^{\text{in}}$ iff $e = \partial K_i \cap \partial K_j$ is of positive measure. We will use symbol $\mathscr{E}_h$ to denote the set of all faces (edges in 2-D) of fine mesh $\mathscr{T}_h$, that is those either in $\mathscr{E}_h^{\text{in}}$ or on the boundary $\partial\Omega$. For $e \in \mathscr{E}_h$ we set $h_e = \operatorname{diam}(e)$. We assume that $\mathscr{T}_h$ is shape- and contact–regular, that is, it admits a matching submesh $\mathscr{T}_{\hat{h}}$ which is shape–regular and such that for any $K \in \mathscr{T}_h$ the ratios of h_K to diameters of simplices in $\mathscr{T}_{\hat{h}}$ covering K are uniformly bounded by an absolute constant. In consequence, if $e = \partial K_i \cap \partial K_j$ is of positive measure, then $h_e \simeq h_{K_i} \simeq h_{K_j}$. We shall refer to $\mathscr{T}_h$ as the "fine mesh". Throughout the paper we will assume that the fine mesh is chosen in such a way that $\rho_{|K}$ is already constant for all $K \in \mathscr{T}_h$.

We define the finite element space V_h^p in which problem (2) is approximated,

$$V_h^p = \{v \in L^2(\Omega) : v_{|K} \in \mathbb{P}_{p_K} \text{ for } K \in \mathscr{T}_h\} \tag{3}$$

where $\mathbb{P}_{p_K}$ denotes the set of polynomials of degree not greater than p_K. We shall assume that $1 \leq p_K$ and that polynomial degrees have bounded local variation, that is, if $e = \partial K_i \cap \partial K_j \in \mathscr{E}_h^{\text{in}}$, then $p_{K_i} \simeq p_{K_j}$.

Next, we discretize (2) by the symmetric weighted interior penalty discontinuous Galerkin method, see for example [3], [1]:

Find $u^* \in V_h^p$ such that

$$\mathscr{A}_h^p(u^*, v) = (f, v)_\Omega, \qquad \forall v \in V_h^p, \tag{4}$$

where

$$\mathscr{A}_h^p(u, v) = A_h^p(u, v) - F_h^p(u, v) - F_h^p(v, u)$$

and

$$A_h^p(u, v) = \sum_{K \in \mathscr{T}_h} (\rho \nabla u, \nabla v)_K + \sum_{e \in \mathscr{E}_h} \langle \gamma [u], [v] \rangle_e, \qquad F_h^p(u, v) = \sum_{e \in \mathscr{E}_h} \langle \{\rho \nabla u\}, [v] \rangle_e.$$

Here for $K \in \mathscr{T}_h$ and $e \in \mathscr{E}_h$ we use standard notation: $(u, v)_K = \int_K u\, v\, dx$ and $\langle u, v \rangle_e = \int_e u\, v\, d\sigma$. On $e \in \mathscr{E}_h^{\text{in}}$ such that $e = \partial K_i \cap \partial K_j$ we set

$$\{\rho \nabla u\} = \overline{\rho}(\nabla u_{|K_i} + \nabla u_{|K_j}), \qquad [u] = u_{|K_i} n_{|K_i} + u_{|K_j} n_{|K_j},$$

with

$$\overline{\rho} = \frac{\rho_{|K_i} \rho_{|K_j}}{\rho_{|K_i} + \rho_{|K_j}}, \qquad \underline{h} = \min\{h_{K_i}, h_{K_j}\}, \qquad \overline{p} = \max\{p_{K_i}, p_{K_j}\}, \qquad \gamma = \frac{\overline{\rho}\, \overline{p}^2}{\underline{h}}\, \delta,$$

where $\delta > 0$ is a prescribed constant. The unit normal vector pointing outward K_i is denoted by $n_{|K_i}$. On e which lies on the boundary of Ω and belongs to a face of K_i, we set $\{\rho\nabla u\} = \rho_{|K_i}\nabla u_{|K_i}$, $[u] = u_{|K_i} n_{|K_i}$ and $\gamma = \rho_{|K_i} p_{K_i}\delta/h_{K_i}$.

For sufficiently large penalty constant δ the discrete problem (4) is well–defined, therefore we can define a norm $|||u|||_\Omega$ by the identity $|||u|||_\Omega^2 = A_h^p(u, u)$.

3 Nonoverlapping two-level, three-grid additive Schwarz method

Let us introduce the subdomain grid $\mathscr{T}_H$ as a partition of Ω into N_H disjoint open polygons (polyhedrons in 3-D) Ω_i, $i = 1,\ldots,N_H$, such that $\bar{\Omega} = \bigcup_{i=1,\ldots,N_H} \bar{\Omega}_i$ and that each Ω_i is a union of certain elements from the fine mesh $\mathscr{T}_h$. We shall retain the common notion of "subdomains" while referring to elements of $\mathscr{T}_H$. We set $H_i = \mathrm{diam}(\Omega_i)$ and $H = (H_1,\ldots,H_{N_H})$. We assume that there exists a reference simply-connected polygonal (polyhedral in 3-D) domain $\hat{\Omega} \subset R^d$ with Lipschitz boundary, such that every Ω_i is affinely homeomorphic to $\hat{\Omega}$ and the aspect ratios of Ω_i are bounded independently of h and H. Moreover, we assume that the number of neighboring regions in $\mathscr{T}_H$ is uniformly bounded by an absolute constant $\mathscr{N}$.

Next, let $\mathscr{T}_{\mathscr{H}}$ be a shape-regular affine triangulation by triangles in 2-D or tetrahedrons in 3-D, with diameter $\mathscr{H}$. We denote the elements of $\mathscr{T}_{\mathscr{H}}$ by D_n, $n = 1,\ldots,N_{\mathscr{H}}$. We shall call this partition the "coarse grid" and assume:

$$\rho_{|D_n} = \rho_n \text{ is a constant for each } D_n \in \mathscr{T}_{\mathscr{H}}.$$

We clearly have $N_{\mathscr{H}} \le N_H \le N_h$ and $\mathscr{T}_{\mathscr{H}} \subseteq \mathscr{T}_H \subseteq \mathscr{T}_h$ (inclusions understood in the sense of subsequent refinements of the coarsest partitioning), and $\max h \le \max H \le \mathscr{H}$. We define the additive Schwarz method following [1] and [4], by introducing the following decomposition of V_h^p:

$$V_h^p = V_0 + \sum_{i=1}^{N_H} V_i, \tag{5}$$

where the coarse space consists of functions which are polynomials inside each element of the coarse grid:

$$V_0 = \{v \in V_h^p : v_{|D_n} \in \mathbb{P}_q \text{ for all } n = 1,\ldots,N_{\mathscr{H}}\} \tag{6}$$

where $1 \le q \le \min\{p_K : K \in \mathscr{T}_h\}$. Next, for $i = 1,\ldots,N_H$ we define

$$V_i = \{v \in V_h^p : v_{|\Omega_j} = 0 \text{ for all } j \ne i\}.$$

One can view V_0 as a rough approximation to V_h^p (using coarser grid and lower order polynomials), cf. condition (9), while V_i can be thought of as V_h^p restricted to Ω_i, extended by zero elsewhere. Note that V_h^p already is a direct sum of spaces $V_1,\ldots,N_H$

and when $\mathscr{T}_{\mathscr{H}} = \mathscr{T}_H$, this decomposition coincides with [1]. Using decomposition (5) we define, for $i = 1, \ldots, N_H$, subdomain solvers $T_i : V_h^p \to V_i$, by

$$A_h^p(T_i u, v) = \mathscr{A}_h^p(u, v) \qquad \forall v \in V_i,$$

so that on each subdomain one has to solve only a relatively small system of linear equations (a "local problem") for $u_i = T_i u|_{\Omega_i}$. These problems are independent one from another, so can be solved in parallel. The coarse solve operator is $T_0 : V_h^p \to V_0$ defined analogously as $A_h^p(T_0 u, v_0) = \mathscr{A}_h^p(u, v_0)$ for all $v_0 \in V_0$. The preconditioned operator is

$$T = T_0 + \sum_{i=1}^{N_H} T_i. \tag{7}$$

Obviously, T is symmetric with respect to $\mathscr{A}_h^p(\cdot, \cdot)$. For D_n in $\mathscr{T}_{\mathscr{H}}$ let us define an auxiliary seminorm

$$|||u|||_{D_n,\mathrm{in}}^2 = \sum_{K \in \mathscr{T}_h(D_n)} \rho |\nabla u|_{0,K}^2 + \sum_{e \in \mathscr{E}_h^{\mathrm{in}}(D_n)} \gamma |[u]|_{0,e}^2, \tag{8}$$

where $\mathscr{E}_h^{\mathrm{in}}(D_n) = \{e \in \mathscr{E}_h : e \subset \bar{D}_n \setminus \partial D_n\}$.

Lemma 1 (see [5]). *Assume that V_0 has the following approximation property:*

$$\forall u \in V_h^p \quad \exists u^{(0)} \in V_0 : \quad \sum_{n=1}^{N_{\mathscr{H}}} \left(\frac{\rho_n q^2}{\mathscr{H}^2} |u - u^{(0)}|_{0,D_n}^2 + |||u - u^{(0)}|||_{D_n,\mathrm{in}}^2 \right) \lesssim \mathscr{A}_h^p(u, u). \tag{9}$$

Then the operator T defined in (7) *satisfies the inequalities*

$$\beta^{-1} \mathscr{A}_h^p(u, u) \lesssim \mathscr{A}_h^p(Tu, u) \lesssim \mathscr{A}_h^p(u, u) \qquad \forall u \in V_h^p,$$

where

$$\beta = \frac{\mathscr{H}^2}{q} \max_{n=1,\ldots,N_H} \left\{ \frac{\overline{p}_i^2}{\underline{h}_i H_i} \right\} \tag{10}$$

with $\underline{h}_i = \min\{h_K : K \in \mathscr{T}_h(\Omega_i)\}$ and $\overline{p}_i = \max\{p_K : K \in \mathscr{T}_h(\Omega_i)\}$.

Theorem 1. *Let us assume that there holds the following H^2–stability property: for every $g \in L^2$ the solution $z \in H_0^1(\Omega)$ of the problem*

$$-\mathrm{div}(\rho \nabla z) = \rho g \tag{11}$$

belongs to $H^2(\Omega)$ and $\sum_{n=1}^{N_{\mathscr{H}}} \rho_n ||z||_{2,D_n}^2 \lesssim \sum_{n=1}^{N_{\mathscr{H}}} \rho_n |g|_{0,D_n}^2$ with constant independent of g. Then $\mathrm{cond}(T) = O(\beta)$ where β is as in (10).

Proof. We will show that the assumptions of Lemma 1 are satisfied. The proof will extend the tools from [2] to the case of discontinuous coefficient; see also [6]. Let us define the lifting operator $R : L^2(\mathscr{E}_h) \to V_h^p$ by

$$(\rho R(\phi), w) = \sum_{e\in\mathscr{E}_h} \langle \{\rho w\}, \phi\rangle_e \qquad \forall w \in V_h^p$$

and the discrete gradient of $u \in V_h^p$ as $G(u) = \nabla_h u - R([u])$. Note that

$$(\rho R([u]), R([u])) = \sum_{e\in\mathscr{E}_h} \langle \{\rho R([u])\}, [u]\rangle_e \lesssim \sum_{e\in\mathscr{E}_h} \frac{\underline{h}^{1/2}}{\overline{p}} |\rho^{1/2} R([u])|_{0,e} \cdot \frac{\overline{p}}{\underline{h}^{1/2}} |\overline{\rho}^{1/2}[u]|_{0,e},$$

so by trace inequality $(\rho R([u]), R([u])) \lesssim |\rho^{1/2} R([u])|_{0,\Omega} \cdot \sum_{e\in\mathscr{E}_h} \langle \gamma[u], [u]\rangle_e$, from which we conclude stability estimate

$$|\rho^{1/2} R([u])|^2_{0,\Omega} \lesssim \sum_{e\in\mathscr{E}_h} \langle \gamma[u], [u]\rangle_e \qquad \forall u \in V_h^p. \tag{12}$$

Let $U \in H_0^1(\Omega)$ solve the problem

$$(\rho\nabla U, \nabla w)_\Omega = (\rho G(u), \nabla w)_\Omega \qquad \forall w \in H_0^1(\Omega).$$

From the definition of U and mentioned above property of the lifting operator R it directly follows that

$$|\rho^{1/2}\nabla U|_{0,\Omega} \lesssim |||u|||. \tag{13}$$

In order to prove (9) we estimate separately

$$\sum_{n=1}^{N_{\mathscr{H}}} |||u-u^{(0)}|||^2_{D_n,\mathrm{in}} \lesssim \sum_{n=1}^{N_{\mathscr{H}}} |||u-U|||^2_{D_n,\mathrm{in}} + \sum_{n=1}^{N_{\mathscr{H}}} |||U-u^{(0)}|||^2_{D_n,\mathrm{in}} = I_1 + I_2$$

and

$$\sum_{n=1}^{N_{\mathscr{H}}} \rho_n |u-u^{(0)}|^2_{0,D_n} \lesssim \sum_{n=1}^{N_{\mathscr{H}}} \rho_n |u-U|^2_{0,D_n} + \sum_{n=1}^{N_{\mathscr{H}}} \rho_n |U-u^{(0)}|^2_{0,D_n} = I_3 + I_4.$$

Clearly, $I_1 \lesssim |||u|||^2 + |||U|||^2 = |||u|||^2 + |\rho^{1/2}\nabla U|^2_{0,\Omega} \lesssim |||u|||^2$ by (13). In order to bound I_3, we use a variant of Aubin–Nitsche trick [2], which is the reason for our H^2–stability assumption. Let us define $z \in H_0^1(\Omega)$ as in (11) with $g = u - U$. After multiplying (11) by $(u-U)$ and integrating by parts on each fine grid element K, we sum over all $K \in \mathscr{T}_h$; using the definition of R we arrive after some calculations at

$$I_3 = |\rho^{1/2}(u-U)|^2_{0,\Omega} = \sum_{e\in\mathscr{E}_h} \langle \{\rho\nabla(z_h - z)\}, [u]\rangle_e + (\rho\nabla(z - z_h), R([u]))_\Omega = I_5 + I_6$$

for any $z_h \in V_h^p$. Applying Schwarz inequality first and then choosing z_h as the approximation to z in V_h^p we have, by the approximation property of V_h^p (cf. e.g. [2, eq. (13)],

$$I_6 \lesssim |\rho^{1/2} R([u])|_{0,\Omega} \cdot |\rho^{1/2} \nabla(z - z_h)|_{0,\Omega}$$
$$\lesssim |||u|||(\sum_{K \in \mathscr{T}_h} \rho \frac{h_K^2}{p_K^2} ||z||_{2,K}^2)^{1/2} \lesssim |||u||| \frac{\mathscr{H}}{q} (\sum_{n=1}^{N_{\mathscr{H}}} \rho_n ||z||_{2,D_n}^2)^{1/2},$$

so from H^2–stability assumption we conclude that $I_6 \lesssim |||u||| \cdot \frac{\mathscr{H}}{q} |\rho^{1/2}(u - U)|_{0,\Omega}$. In a similar way we obtain $I_5 \lesssim |||u||| \cdot \frac{\mathscr{H}}{q} |\rho^{1/2}(u - U)|_{0,\Omega}$, whence $I_3 \lesssim \frac{\mathscr{H}}{q} |||u|||$.

Finally, we bound the terms I_2 and I_4 in a standard way, by choosing $u^{(0)}$ on each D_n as the q–th order polynomial interpolant of $U_{|D_n}$. See [5, Corollary 2] for details.

4 Numerical experiments

The H^2–stability requirement in Theorem 1 is quite limiting. As the following experimental results indicate, the preconditioner works well for checkerboard distribution of the coefficient, so there is room to relax assumptions Theorem 1.

Let us choose $\Omega = (0,1)^2$. We divide Ω into $N_{\mathscr{H}} = 2^{\mathscr{M}} \times 2^{\mathscr{M}}$ squares D_n ($n = 1, \dots, N_{\mathscr{H}}$) of equal size. Let ρ be constant on a 2×2 grid with checkerboard distribution: $\rho = 1$ in "white" squares and $\rho = \rho_R$ (specified later) in "red" squares. For simplicity we choose $\mathscr{T}_H = \mathscr{T}_{\mathscr{H}}$, refined into a uniform fine triangulation $\mathscr{T}_h$ based on a square $2^m \times 2^m$ grid, with each square split into two triangles of identical shape. We discretize problem (2) on the fine mesh $\mathscr{T}_h$ using (4) with equal polynomial degree p across all elements in $\mathscr{T}_h$ and with $\delta = 7$. For the coarse problem, we use polynomials of degree q.

We report the number of Preconditioned Conjugate Gradient iterations (with zero as the initial guess) for operator T, required to reduce the initial norm of the preconditioned residual by a factor of 10^8 and (in parentheses) the condition number of T estimated from the PCG convergence history. We set the coefficients of the discrete solution u^* as random numbers from uniform distribution and construct f such that (4) holds.

$q \rightarrow$ $\rho_R \downarrow$	1	2	3	4	5
10^0	90 (166)	72 (96)	64 (70)	57 (56)	54 (47)
10^8	89 (155)	69 (94)	63 (71)	57 (55)	53 (48)

Table 1 Dependence of the number of iterations and the condition number (in parentheses) on the contrast ratio ρ_R and the coarse space polynomial degree q. Fixed $p = 6$, $\mathscr{M} = 2$, $m = 4$.

From Table 1 it is clear the converegence rate is independent from the jump of the coefficient and the improvement of the condition number due to increase of q is diminishing roughly like $O(1/q)$. Table 2 confirms that the condition number

$p \rightarrow$ / $m \downarrow$	2	3	4	5	6
3	26 (11)	37 (22)	47 (37)	58 (57)	67 (78)
4	36 (20)	50 (42)	62 (72)	75 (112)	83 (149)
5	48 (38)	65 (79)	81 (140)	98 (219)	113 (303)

Table 2 Dependence of the number of iterations and the condition number (in parentheses) on the fine mesh size $h = 2^{-m}$ and polynomial degree p. Fixed $q = 1$, $\mathscr{M} = 2$ and $\rho_R = 10^4$.

$\mathscr{M} \rightarrow$ / $m \downarrow$	2	3	4	5
3	47 (37)	38 (20)		
4	62 (72)	49 (39)	38 (20)	
5	81 (140)	65 (75)	50 (39)	38 (20)

Table 3 Dependence of the number of iterations and the condition number (in parentheses) on $\mathscr{H} = H = 2^{-\mathscr{M}}$ and $h = 2^{-m}$. Fixed $p = 4$, $q = 1$, $\rho_R = 10^4$.

dependence on p and h behaves approximately like $O(p^2/h)$. For varying h and $\mathscr{H} = H$, an $O(\mathscr{H}/h)$ dependence of the condition number is verified in Table 3. See [5] for more experimental results.

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