Combining space-time multigrid techniques with multilevel Monte Carlo methods for SDEs

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Abstract *In this work we combine multilevel Monte Carlo methods for time-dependent stochastic differential equations with a space-time multigrid method. The idea is to use the space-time hierarchy from the multilevel Monte Carlo method also for the solution process of the arising linear systems. This symbiosis leads to a robust and parallel method with respect to space, time and probability. We show the performance of this approach by several numerical experiments which demonstrate the advantages of this approach.*

1 Introduction

Stochastic differential equations (SDEs) have become an invaluable tool for modelling time-dependent problems that are perturbed by random influences. Since the importance of such models increases constantly, there is a high demand on improving the efficiency of numerical algorithms for SDEs, especially, if one is interested in the approximation of $\mathbb{E}[\varphi(X(T))]$, where $X(T)$ denotes the (mild) solution of an SDE evaluated at time T and $\mathbb{E}$ denotes the expectation, where φ is a mapping determining the statistical quantity of interest.

In this work we focus on approximating $\mathbb{E}[\varphi(X(T))]$ for the solution process of linear SDEs driven by additive noise. For this we combine space-time multigrid methods for approximating solutions of time-dependent deterministic differential equations, see [4] and the references therein, and multilevel Monte Carlo (MLMC) methods, see e.g. [5, 6]. Both methods as such are well-known to be parallelizable,

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however, the combination of both methods is a completely new approach that enables the full parallelization of the problem in space, time and probability.

The outline of this article is as follows: In the remainder of this section, we introduce two model problems (the Ornstein-Uhlenbeck process and the stochastic heat equation) together with discretization techniques for these model problems with respect to space and time. Afterwards, we consider the multilevel Monte Carlo (MLMC) method for approximating the expectation in Section 2 and we discuss parallelizable space-time multigrid methods based on the inherited space-time hierarchy of the MLMC estimator in Section 3. Finally, we conclude by presenting numerical experiments in Section 4.

1.1 Model problems

Let $T > 0$ and let $(\Omega, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathscr{F}, \mathbb{P})$ be a complete filtered probability space. At first, we consider a one-dimensional model problem given by the stochastic ordinary differential equation (SODE)

$$\begin{aligned} \mathrm{d}u(t) + \lambda u(t)\,\mathrm{d}t &= \sigma\,\mathrm{d}\beta(t) \qquad \text{for } t \in (0,T], \\ u(0) &= u_0, \end{aligned} \tag{1}$$

where $\lambda \in \mathbb{R}_0^+, \sigma, u_0 \in \mathbb{R}$ and $\beta = (\beta(t), t \in [0,T])$ is a standard Brownian motion. The solution of this SODE is a *Ornstein-Uhlenbeck* process defined by

$$u(t) = u_0 e^{-\lambda t} + \sigma \int_0^t e^{-\lambda(t-s)}\,\mathrm{d}\beta(s), \qquad t \in [0,T]. \tag{2}$$

As second model problem we consider the *stochastic heat equation* on a bounded and convex domain $D \subset \mathbb{R}^d, d = 1,2,3$, with homogeneous Dirichlet boundary conditions. We rewrite the corresponding stochastic partial differential equation (SPDE) as a stochastic evolution equation on the Hilbert space $H = L^2(D)$

$$\begin{aligned} \mathrm{d}U(t) &= \Delta U(t)\,\mathrm{d}t + \mathrm{d}W(t) \qquad \text{for } t \in (0,T], \\ U(0) &= U_0 \in H^2(D) \cap H_0^1(D). \end{aligned} \tag{3}$$

Subsequently, we denote by $(e_j, j \in \mathbb{N})$ the set of eigenfunctions of the Laplace operator $-\Delta$, which forms an orthonormal basis of H. Furthermore, let $W = (W(t), t \in [0,T])$ be an H-valued Q-Wiener process with a linear, positive definite, symmetric, trace class covariance operator Q. Then W can be represented as (see e.g. [3, 7])

$$W(t) = \sum_{j=1}^{\infty} \sqrt{\mu_j} e_j \beta_j(t), \tag{4}$$

where $(\mu_j, j \in \mathbb{N})$ denotes the set of eigenvalues of Q satisfying $Qe_j = \mu_j e_j$ and $(\beta_j, j \in \mathbb{N})$ is a sequence of independent standard Brownian motions.

Then, by [3], there exists a unique, square-integrable mild solution to SPDE (3)

$$U(t) = S(t)U_0 + \int_0^t S(t-s)\,\mathrm{d}W(s) \qquad \text{for } t \in [0,T], \tag{5}$$

where $S(t), t \in [0,T]$, denotes the semigroup generated by the Laplace operator.

1.2 Discretization of model problems

In this section, we present fully discrete schemes for approximating the solution processes from Eq. (2) and Eq. (5). For this we fix an equidistant partition Θ_K of the time interval $[0,T]$ given by $\Theta_K = \{0 = t_0 < t_1 < \cdots < t_K = T\}$, where for $0 \le j \le K$ we choose $t_j = j\Delta t$ with time step size $\Delta t = T/K$.

For approximating the solution of the Ornstein-Uhlenbeck process (2), we consider the backward Euler–Maruyama scheme given by the recursion

$$(1+\lambda\Delta t)\mathbf{u}_j = \mathbf{u}_{j-1} + \sigma\Delta\beta^j, \qquad \text{for } 1 \le j \le K, \tag{6}$$

where $\mathbf{u}_0 = u_0$ and $\Delta\beta^j = \beta(t_j) - \beta(t_{j-1})$. Rewriting the recursion (6) in a matrix-vector representation yields

$$\begin{pmatrix} (1+\lambda\Delta t) & & & \\ -1 & (1+\lambda\Delta t) & & \\ & \ddots & \ddots & \\ & & -1 & (1+\lambda\Delta t) \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_K \end{pmatrix} = \begin{pmatrix} \sigma\Delta\beta^1(\omega) + \mathbf{u}_0 \\ \sigma\Delta\beta^2(\omega) \\ \vdots \\ \sigma\Delta\beta^K(\omega) \end{pmatrix}. \tag{7}$$

In this article, we abbreviate this linear system by $\mathscr{L}_\tau \mathbf{u} = \mathbf{f}(\omega)$, where we use the ω-dependency in $\mathbf{f}(\omega)$ to indicate that the right hand side is a random vector.

For the stochastic heat equation we want to obtain a fully discrete approximation U_h^j of the mild solution $U(t_j), t_j \in \Theta_K$, where U_h^j attains values in a finite-dimensional subspace $V_h \subset H_0^1(D)$. Besides an appropriate time integration method, we apply a discretization scheme in space. For this we consider a standard Galerkin finite element (FE) discretization based on a regular family $(\mathscr{T}_h, h \in (0,1])$ of triangulations of D with maximal mesh size h. Then V_h denotes the space of globally continuous and on $\mathscr{T}_h$ piecewise linear functions. Furthermore, we denote by N_h the dimension of V_h. By using the nodal basis functions $(\phi_i, 1 \le i \le N_h) \subset H_0^1(D)$, the fully discrete approximation scheme based on Galerkin finite elements in space and on the backward Euler–Maruyama scheme in time is given by (see e.g. [2])

$$(M_h + \Delta t K_h)\mathbf{U}_j = M_h\mathbf{U}_{j-1} + \Delta\mathbf{W}^j \qquad \text{for } 1 \le j \le K, \tag{8}$$

where $\Delta\mathbf{W}^j$ denotes the vector representation of the FE approximation of the Q-Wiener increments $\Delta W^j(\mathbf{x}) = W(t_j,\mathbf{x}) - W(t_{j-1},\mathbf{x}), \mathbf{x} \in D$, and for $j = 0,\ldots,K$,

$$U_h^j = \sum_{i=1}^{N_h} \mathbf{U}_j[i]\phi_i,$$

where $\mathbf{U}_j[i]$ denotes the ith component of the vector $\mathbf{U}_j \in \mathbb{R}^{N_h}$. Here, we denote by M_h the standard *mass* matrix and K_h the standard *stiffness* matrix defined by

$$M_h[i,j] := \int_D \phi_j(\mathbf{x})\phi_i(\mathbf{x})\,\mathrm{d}\mathbf{x}, \quad K_h[i,j] := \int_D \nabla\phi_j(\mathbf{x})\cdot\nabla\phi_i(\mathbf{x})\,\mathrm{d}\mathbf{x}, \quad \text{for } i,j = 1,\dots,N_h.$$

Finally, by rewriting the numerical scheme (8) in terms of a matrix-vector formulation we obtain the large linear system

$$\begin{pmatrix} M_h + \Delta t K_h & & & \\ -M_h & M_h + \Delta t K_h & & \\ & \ddots & \ddots & \\ & & -M_h & M_h + \Delta t K_h \end{pmatrix} \begin{pmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \vdots \\ \mathbf{U}_K \end{pmatrix} = \begin{pmatrix} \Delta\mathbf{W}^1(\omega) + M_h\mathbf{U}_0 \\ \Delta\mathbf{W}^2(\omega) \\ \vdots \\ \Delta\mathbf{W}^K(\omega) \end{pmatrix} \tag{9}$$

that is subsequently abbreviated by $\mathscr{L}_{h,\tau}\mathbf{U} = \mathbf{F}(\omega)$.

2 Multilevel Monte Carlo methods

The goal is to approximate $\mathbb{E}[\varphi(u(T))]$ or $\mathbb{E}[\varphi(U(T))]$ for a sufficiently smooth mapping $\varphi : H \to B$, where B is a separable Hilbert space, by using suitable estimators. For $Y \in L^2(\Omega;B)$ a common way to approximate $\mathbb{E}[Y]$ is to use a standard *Monte Carlo* (MC) estimator defined by

$$E_M[Y] := \frac{1}{M}\sum_{i=1}^{M} Y^{(i)},$$

where $(Y^{(i)}, i = 1,\dots,M)$ are independent realizations of Y. Here, $L^2(\Omega;B)$ denotes the space of strongly measurable random variables Y that satisfy

$$\|Y\|^2_{L^2(\Omega;B)} := \mathbb{E}[\|Y\|_B^2] < \infty.$$

Due to the rather slow convergence of the MC estimator of order $M^{-1/2}$ in the $L^2(\Omega;B)$-sense, the efficient *multilevel Monte Carlo* (MLMC) estimator has been proposed in [5]. For its definition we consider a sequence $(Y_\ell, \ell \in \mathbb{N}_0)$ of approximations of $Y \in L^2(\Omega;B)$ based on different refinement levels $\ell \in \mathbb{N}_0$. The MLMC estimator is then given by

$$E^L[Y_L] := \sum_{\ell=0}^{L} E_{M_\ell}[Y_\ell - Y_{\ell-1}],$$

where $Y_{-1} = 0$. The $L^2(\Omega;B)$-error of the MLMC estimator satisfies (see [6])

$$\|\mathbb{E}[Y] - E^L[Y_L]\|_{L^2(\Omega;B)} \leq \|\mathbb{E}[Y - Y_L]\|_B + \left(\sum_{\ell=0}^{L} M_\ell^{-1} \operatorname{Var}[Y_\ell - Y_{\ell-1}] \right)^{1/2} \tag{10}$$

where $\operatorname{Var}[Y] = \mathbb{E}[\|Y - \mathbb{E}[Y]\|_B^2]$ for $Y \in L^2(\Omega;B)$.

In the following two subsections, we discuss how to choose the number of samples $(M_\ell, \ell \in \mathbb{N}_0)$ and the refinement parameter h and Δt in order to guarantee the convergence of the MLMC estimator.

2.1 Ornstein-Uhlenbeck process

Let u be given in Eq. (2) and for $\ell \in \mathbb{N}_0$ let $\mathbf{u}_{K_\ell}$ be the numerical approximation of $u(T)$ based on the backward Euler–Maruyama scheme (6) with respect to the partition Θ_{K_ℓ} with time step size Δt_ℓ. Furthermore, let $\varphi \in C_b^2(\mathbb{R},\mathbb{R})$, i.e., $\varphi : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable with bounded first and second derivatives. Due to the additive noise structure of SDE (1) we obtain by results from [8] that

$$|\mathbb{E}[\varphi(u(T)) - \varphi(\mathbf{u}_{K_L})]| \leq C\Delta t_L, \quad \operatorname{Var}[\varphi(\mathbf{u}_{K_\ell}) - \varphi(\mathbf{u}_{K_{\ell-1}})]^{1/2} \leq C\Delta t_\ell.$$

Thus, by similar arguments as in [6], if we choose for any $\varepsilon, C_M > 0$,

$$M_0 = \lceil C_M \Delta t_L^{-2} \rceil, \qquad M_\ell = \lceil C_M \Delta t_\ell^2 \Delta t_L^{-2} \ell^{1+\varepsilon} \rceil \quad \text{for } \ell = 1, \dots, L, \tag{11}$$

then $\|\mathbb{E}[\varphi(u(T))] - E^L[\varphi(\mathbf{u}_{K_L})]\|_{L^2(\Omega;\mathbb{R})} = \mathcal{O}(\Delta t_L)$.

2.2 Stochastic heat equation

Let U be given in Eq. (5) and for $\ell \in \mathbb{N}_0$ let $U_{h_\ell}^{K_\ell}$ be an approximation of $U(T)$ based on the FE backward Euler–Maruyama scheme (8) with respect to the partition Θ_{K_ℓ} and the FE space V_{h_ℓ}. Furthermore, let $\varphi \in \mathscr{C}_b^2(H,B)$, i.e., $\varphi : H \to B$ is twice Fréchet differentiable with bounded first and second Fréchet derivatives. Then by using the results from [1], we get by choosing $\Delta t_\ell = h_\ell^2$ for any $\gamma \in [0,1)$

$$\|\mathbb{E}[\varphi(U(T)) - \varphi(U_{h_L}^{K_L})]\|_B \leq C h_L^{2\gamma}, \ \operatorname{Var}[\varphi(U_{h_\ell}^{K_\ell}) - \varphi(U_{h_{\ell-1}}^{K_{\ell-1}})] \leq C h_\ell^{2\gamma}.$$

Thus, by [6], if we choose $\Delta t_\ell = h_\ell^2$ and for any

$$M_0 = \lceil C_M h_L^{-2\gamma} \rceil, \qquad M_\ell = \lceil C_M h_\ell^{2\gamma} h_L^{-2\gamma} \ell^{1+\varepsilon} \rceil \quad \text{for } \ell = 1, \dots, L. \tag{12}$$

then $\|\mathbb{E}[\varphi(U(T))] - E^L[\varphi(U_{h_L}^{K_L})]\|_{L^2(\Omega;B)} = \mathcal{O}(h_L^\gamma)$.

3 Space-time multigrid methods

The idea is to use the space-time hierarchy from the MLMC methods discussed in Sections 2.1 and 2.2 also for a space-time multigrid approach. In detail we use the space-time multigrid method presented in [4] to solve the linear system (7) and (9) at once. The advantage is that we can also add parallelization in time direction and also with respect to the space dimension. So using the space-time hierarchy coming from the MLMC method for the linear solver we obtain an algorithm which can be applied in parallel with respect to space, time and probability. For the space-time multigrid method we use a (inexact) damped block Jacobi smoother, see also [4], i.e. for the problem (7) we use

$$\mathbf{u}^{(n+1)} = \mathbf{u}^{(n)} + \alpha \mathscr{D}_{\tau}^{-1} \left[\mathbf{f}(\omega) - \mathscr{L}_{\tau} \mathbf{u}^{(n)} \right] \qquad \text{for } n = 0, 1, \dots,$$

with the diagonal matrix $\mathscr{D}_{\tau} := \operatorname{diag}(1 + \lambda \Delta t)$. Whereas, for the problem (9) we use the smoothing iteration

$$\mathbf{U}^{(n+1)} = \mathbf{U}^{(n)} + \alpha \mathscr{D}_{h,\tau}^{-1} \left[\mathbf{F}(\omega) - \mathscr{L}_{h,\tau} \mathbf{U}^{(n)} \right] \qquad \text{for } n = 0, 1, \dots,$$

with the block diagonal matrix $\mathscr{D}_{h,\tau} := \operatorname{diag}(M_h + \Delta t K_h)$. To speed up the application of the smoothing procedure we replace the exact inverse of $\mathscr{D}_{h,\tau}$ by applying one iteration of a multigrid V-cycle with respect to the matrix $M_h + \Delta t K_h$. Moreover we always set the damping parameter to $\alpha = \frac{1}{2}$, see [4] for more details. Choosing $\Delta t \approx h^2$ leads – in combination with the space-time hierarchy coming from the MLMC method – to a robust solver which is independent of the number of time steps K the time step size Δt and the randomness ω.

4 Numerical experiments

4.1 Ornstein-Uhlenbeck process

We consider the SODE (1) with $\lambda = 1, \sigma = 1$ and $u_0 = 1$. By choosing $T = 1$ and $\varphi(x) = x$ for all $x \in \mathbb{R}$ we are interested in approximating $\mathbb{E}[u(T)] = e^{-T}$.

For the numerical approximation we consider the backward Euler–Maruyama scheme from Eq. (6) in the matrix-vector representation $\mathscr{L}_{\tau} \mathbf{u} = \mathbf{f}(\omega)$, which is solved by the time multigrid method described in Section 3. For the approximation of the expectation we consider a multilevel Monte Carlo estimator based on the sample size selection from Eq. (11) with $\varepsilon = \frac{1}{2}$ and $C_M = 10$.

In Table 1, $\|\mathbb{E}[u(T)] - E^L[\mathbf{u}_{K_L}]\|_{L^2(\Omega;\mathbb{R})}$ is approximated by a standard Monte Carlo estimator given by

$$\text{MS-err} = \left(\frac{1}{M} \sum_{i=1}^{M} \left| e^{-T} - E^L[\mathbf{u}_{K_L}]^{(i)} \right|^2 \right)^{1/2},$$

where $(E^L[\mathbf{u}_{K_L}]^{(i)}, 1 \le i \le M)$ are independent realizations of the MLMC estimator $E^L[\mathbf{u}_{K_L}]$. For this we choose $M = 100$ in the numerical experiments from Table 1 and we observe the right convergence rates as predicted by the theory.

Table 1 Numerical test for SODE (1) (Ornstein-Uhlenbeck process).

L	time steps	realizations level 0	realizations level L	MS-err	EOC
0	1	10	10	2.61915E-01	-
1	2	40	20	1.39399E-01	0.91
2	4	160	50	6.73215E-02	1.05
3	8	640	80	3.92162E-02	0.78
4	16	2560	110	2.02307E-02	0.95
5	32	10240	140	1.00032E-02	1.02
6	64	40960	180	4.80065E-03	1.06
7	128	163840	220	2.31171E-03	1.05
8	256	655360	270	1.13875E-03	1.02
9	512	2621440	310	5.29684E-04	1.10
10	1024	10485760	360	2.62618E-04	1.01

4.2 Stochastic heat equation

For the stochastic heat equation (3) we consider the one-dimensional case $D = (0,1)$ with initial value $U_0(\mathbf{x}) = \sin(\pi x)$. By choosing $T = 0.2$ and $\varphi(v) = v$ for all $v \in L^2(D)$, we are interested in approximating $\mathbb{E}[U(T,\mathbf{x})] = \exp(-\pi^2 T)\sin(\pi\mathbf{x}), \mathbf{x} \in D$.

The eigenvalues of the Q-Wiener process are $\mu_j = j^{-(2r+1+\varepsilon)}$ with $r = 2$ and any $\varepsilon > 0$, see e.g. [7] for details. For approximating paths of the Q-Wiener process we truncate the series representation (4) after the first $J_h = N_h$ summands, see e.g. [2].

For the numerical approximation in space and time, we consider the FE Euler–Maruyama scheme from Eq. (8) on an equidistant mesh with grid width $h_\ell = 2^{-\ell-1}$ in the matrix-vector formulation $\mathscr{L}_{h,\tau}\mathbf{U} = \mathbf{F}(\omega)$, which is again solved by the space-time multigrid method described in Section 3. For the approximation of the expectation we consider the MLMC method based on the sample size selection (12) with $\varepsilon = 0.5$ and $C_M = 10$.

In numerical experiments $\|\mathbb{E}[U(T)] - E^L[U_{h_L}^{K_L}]\|_{L^2(\Omega;B)}$ is approximated by a standard Monte Carlo estimator, i.e., we consider

$$\text{MS-err} = \left(\frac{1}{M} \sum_{i=1}^{M} \left\| \mathbb{E}[U(T)] - E^L[U_{h_L}^{K_L}]^{(i)} \right\|_{L^2(D)}^2 \right)^{1/2},$$

where $(E^L[U_{h_L}^{K_L}]^{(i)}, 1 \le i \le M)$ are independent realizations of the estimator $E^L[U_{h_L}^{K_L}]$ and

$$\|\mathbb{E}[U(T)] - E^L[U_{h_L}^{K_L}]^{(i)}\|_{L^2(D)}^2 = \int_0^1 \left(\exp(-\pi^2 T)\sin(\pi \mathbf{x}) - E^L[U_{h_L}^{K_L}(\mathbf{x})]^{(i)}\right)^2 \mathrm{d}\mathbf{x}.$$

In Table 2 we use $M = 100$ independent realizations of the MLMC estimator and we observe the optimal convergence rates as predicted by the theory. Moreover we give in Table 3 the averaged solving times for one signle MLMC run for different levels and different distributions of 512 cores. Here we observe that the best possible setting is given by a balanced distribution of cores between parallelization in time and parallelization of the Monte Carlo estimators. For example for level $L = 7$ the best possible setting is given by 8 cores for time parallelization and 64 cores for the Monte Carlo parallelization.

Table 2 Numerical test for SPDE (1) (stochastic heat equation) – convergence.

L	time steps	# elements	realizations level 0	realizations level L	MS-err	EOC
0	1	2	10	10	7.83487E-02	-
1	4	4	40	20	3.39860E-02	1.20
2	16	8	160	30	1.29145E-02	1.40
3	64	16	640	60	5.99035E-03	1.11
4	256	32	2560	90	2.71909E-03	1.14
5	1024	64	10240	120	1.39772E-03	0.96
6	4096	128	40960	150	6.89668E-04	1.02
7	16384	256	163840	190	3.41996E-04	1.01

Table 3 Numerical test for SPDE (1) (stochastic heat equation) – computation time for one MLMC run with respect to different distributions of 512 cores (in sec).

	cores time / cores Monte Carlo							
L	1 / 512	2 / 256	4 / 128	8 / 64	16 / 32	32 / 16	64 / 8	128 / 4
3	0.04	0.02	0.02	0.02	0.03	0.06	0.1	0.14
4	0.27	0.17	0.12	0.13	0.16	0.26	0.47	0.93
5	2.64	1.51	0.95	1.01	1.17	1.64	2.47	4.41
6	24.12	13.92	13.64	11.47	10.76	12.53	15.88	23.5
7	282.46	157.97	153.41	125.56	127.84	133.6	146.81	178.76

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