Optimized Schwarz Method for Poisson's Equation in Rectangular Domains

José C. Garay, Frédéric Magoulès and Daniel B. Szyld

Abstract An analysis of the convergence properties of Optimized Schwarz methods applied as solvers for Poisson's Equation in a bounded rectangular domain with Dirichlet (physical) boundary conditions and Robin transmission conditions on the artificial boundaries is presented. To our knowledge this is the first time that this is done for multiple subdomains forming a 2D array in a bounded domain.

1 Introduction

Classical Schwarz methods are Domain Decomposition (DD) methods in which the transmission conditions between subdomains are Dirichlet boundary conditions. Optimized Schwarz methods are DD methods in which the transmission conditions are chosen in such a way as to improve the convergence rate with respect to the classical method [2, 3, 5]. These transmission conditions are optimized approximations of the optimal transmission conditions, which are obtained by approximating the global Poincaré-Steklov operator by local differential operators. There is more than one family of transmission conditions that can be used for a given PDE , each of these families consisting of a particular approximation of the optimal transmission conditions. For example, for the problem involving Poisson's equation, we have $OO0$ and $OO2$ family of transmission conditions. The $OO0$ family of transmission

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conditions is obtained by using the zero-th order approximation of the Poincaré-Steklov operator, i.e., it is approximated by a constant α, which leads to have Robin boundary conditions on the artificial boundaries. The $OO2$ family of boundary conditions involves the use of a differential operator that is a linear combination of the normal derivative and tangential second derivatives.

Optimized Schwarz methods (OSM) are fast methods in terms of iteration count when they are used as outer solvers. In [1] it is shown that OSM (as outer solvers) are faster than GMRES preconditioned with a classical Schwarz preconditioner. Also, in parallel computations, OSM requires much less communications between processes in comparison to Krylov methods. Given that communication dominates the execution time of solvers in current supercomputer architectures and will also do so in the upcoming exascale supercomputers, OSM has the potential to be a very good method for solving problems arising from the discretization of PDEs.

In this paper we analyze the convergence properties of OSM applied as solvers for Poisson's Equation in a bounded rectangular domain with Dirichlet (physical) boundary conditions and Robin transmission conditions. To our knowledge, this is the first time an analysis of convergence of Optimized Schwarz applied to a problem defined in a bounded domain and with arbitrary number of subdomains forming a 2D array (i.e., containing cross points) is presented.

2 Equations of OSM for Poisson's in rectangular domain for the $OO0$ case

We want to solve Poisson's equation in a rectangular domain subject to nonhomogeneous Dirichlet boundary conditions, i.e,

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \quad u = g & \text{on } \partial\Omega. \end{cases} \tag{1}$$

where $\Omega = [0,L_1] \times [0,L_2]$.

We divide the physical domain into $p \times q$ overlapping rectangular subdomains. To simplify the presentation, we consider square subdomains where each side is of length H and the same overlap on each side, but the analysis presented here is also valid for arbitrary rectangles and arbitrary ovelaps. Each of these subdomains is represented by a pair of indexes, (s,r), with $s \in \{1,...,p\}$ and $r \in \{1,...,q\}$. Let h be the length of the side of each subdomain as if it were a partition with no overlap. Let us now displace (outward) each of the boundaries of the nonoverlapping subdomains by a γ amount. We have then overlapping square subdomains with side $H = h + 2\gamma$ and can use γ as a parameter to quantify the amount of overlap between subdomains. The Optimized Schwarz iteration process associated with problem (1) and with $OO0$ transmission conditions is defined, for an interior subdomain (i.e., for $1 < s < p$, $1 < r < q$), by

$$
\begin{cases}
\Delta u_{n+1}^{(s,r)} = f & \text{in } \Omega^{(s,r)} \\
-\frac{\partial u_{n+1}^{(s,r)}}{\partial x} + \alpha u_{n+1}^{(s,r)} = -\frac{\partial u_n^{(s-1,r)}}{\partial x} + \alpha u_n^{(s-1,r)} & \text{for } x = (s-1)h - \gamma \\
\frac{\partial u_{n+1}^{(s,r)}}{\partial x} + \alpha u_{n+1}^{(s,r)} = \frac{\partial u_n^{(s+1,r)}}{\partial x} + \alpha u_n^{(s+1,r)} & \text{for } x = sh + \gamma \\
-\frac{\partial u_{n+1}^{(s,r)}}{\partial y} + \alpha u_{n+1}^{(s,r)} = -\frac{\partial u_n^{(s,r-1)}}{\partial y} + \alpha u_n^{(s,r-1)} & \text{for } y = (r-1)h - \gamma \\
\frac{\partial u_{n+1}^{(s,r)}}{\partial y} + \alpha u_{n+1}^{(s,r)} = \frac{\partial u_n^{(s,r+1)}}{\partial y} + \alpha u_n^{(s,r+1)} & \text{for } x = rh + \gamma.
\end{cases} \tag{2}
$$

where $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are, in this instance, normal derivatives and $u_{n+1}^{(s,r)}$ is the solution of the local problem (2) at the $(n+1)$ iteration in $\Omega^{(s,r)}$. The parameter α is the one which we want to tune to optimize the convergence rate of the method. Note that $\alpha = 0$ would reduce the problem to pure Neuman boundary conditions and therefore this case is not allowed. The subdomains touching the boundary have one or two boundaries that are actually physical (not artificial) boundaries. The equations for the subdomains touching the boundary are similar to (2) with the exception that one or two of the boundary conditions are Dirichlet, namely, the ones associated to the physical boundaries.

3 Recasting equations as an equivalent fixed point iteration

By linearity, we can see that the local error (of interior subdomains) of the iteration process is described by (2) with $f = 0$. Similar equations can be obtained for subdomains touching the boundary. Using separation of variables, Sturm-Liouville theory and superposition principle, we can write the local errors in the form of a series [4]. Then, using the non-homogeneous boundary conditions in each local problem, we obtain a relationship between the error series coefficients at iteration $(n+1)$ and the ones at iteration n.

Fourier Analysis of solution of PDEs defining the local error

We analyze the local error of an interior subdomain, but the same analysis holds for subdomains touching the boundary. Let $\eta_n^{(s,r)}$ be the local error in $\Omega^{(s,r)}$ at the iteration n. By superposition principle, we can write $\eta_n^{(s,r)} = \eta_{n,1}^{(s,r)} + \eta_{n,2}^{(s,r)} + \eta_{n,3}^{(s,r)} + \eta_{n,4}^{(s,r)}$, where $\eta_{n,i}^{(s,r)}$, $i = 1,\ldots,4$, is the solution of (2) with $f = 0$, and with one non-homogeneous boundary condition and the rest homogeneous. Then, each part of the local error $\eta_n^{(s,r)}$ can be written as:

$$\eta_{n,1}^{(s,r)}(x_\ell, y_\ell) = \sum_{m=1}^{\infty} A_{n,m,1}^{(s,r)} \phi_m(x_\ell) \psi_m(H - y_\ell) \tag{3}$$

$$\eta_{n,2}^{(s,r)}(x_\ell, y_\ell) = \sum_{m=1}^{\infty} A_{n,m,2}^{(s,r)} \phi_m(y_\ell) \psi_m(x_\ell) \tag{4}$$

$$\eta_{n,3}^{(s,r)}(x_\ell,y_\ell) = \sum_{m=1}^{\infty} A_{n,m,3}^{(s,r)} \phi_m(x_\ell)\psi_m(y_\ell) \tag{5}$$

$$\eta_{n,4}^{(s,r)}(x_\ell,y_\ell) = \sum_{m=1}^{\infty} A_{n,m,4}^{(s,r)} \phi_m(y_\ell)\psi_m(H-x_\ell), \tag{6}$$

where $\phi_m(x_\ell) = \frac{\bar{\alpha}}{z_m}\sin\left(\frac{z_m x_\ell}{H}\right)+\cos\left(\frac{z_m x_\ell}{H}\right)$ and $\psi_m(x_\ell) = \frac{\bar{\alpha}}{z_m}\sinh\left(\frac{z_m x_\ell}{H}\right)+\cosh\left(\frac{z_m x_\ell}{H}\right)$ with z_m satisfying the transcendental equation

$$\tan(z) = \frac{2z\bar{\alpha}}{\bar{\alpha}^2 - z^2},$$

$\bar{\alpha} = \alpha H$, and x_ℓ and y_ℓ are local coordinates related to the global coordinates x and y by

$$\begin{aligned} x_\ell &= x-(s-1)h+\gamma \\ y_\ell &= y-(r-1)h+\gamma. \end{aligned} \tag{7}$$

Note that $\{\phi_m\}_{m\in\mathbb{N}}$ is a complete orthogonal set in $[0,H]$. Therefore, equations (3) and (5) can be seen as Generalized Fourier series in x_ℓ and equations (4) and (6) as Generalized Fourier series in y_ℓ. Then, we have that

$$A_{n,m,1}^{(s,r)} = \frac{\int_0^H \eta_{n,1}^{(s,r)}(x_\ell,y_l)\left[\frac{\bar{\alpha}}{z_m}\sin\left(\frac{z_m x_\ell}{H}\right)+\cos\left(\frac{z_m x_\ell}{H}\right)\right]dx_\ell}{\left[\frac{-\bar{\alpha}}{z_m}\sinh\left(\frac{z_m(y_\ell-H)}{H}\right)+\cosh\left(\frac{z_m(y_\ell-H)}{H}\right)\right]\int_0^H\left[\frac{\bar{\alpha}}{z_m}\sin\left(\frac{z_m x_\ell}{H}\right)+\cos\left(\frac{z_m x_\ell}{H}\right)\right]^2 dx_\ell} \tag{8}$$

Let $\beta : \mathbb{N}\times\mathbb{R} \to \{-1\}\cup[0,1]$ such that

$$\beta(m,\bar{\alpha}) = \begin{cases} -1, & \text{if } z_m < 1 \\ \frac{1}{2}, & \text{if } z_m \geq 1 \end{cases}.$$

Then, with $y_\ell = 0$ and using integration by parts in (8) we can write

$$A_{n,m,1}^{(s,r)} = \frac{B_{n,m,1}^{(s,r)}}{z_m^{1+\beta(m,\bar{\alpha})}\left[\frac{\bar{\alpha}}{z_m}\sinh\left(\frac{z_m}{H}\right)+\cosh\left(\frac{z_m}{H}\right)\right]},$$

where $B_{n,m,1}^{(s,r)}$ is uniformly bounded for all $m\in\mathbb{N}$. The same relationship holds between $A_{n,m,i}^{(s,r)}$ and uniformly bounded quantities $B_{n,m,i}^{(s,r)}$ for $i\in\{2,3,4\}$. Plugging these equalities in (3)-(6) and applying the nonhomogeneous boundary conditions, we obtain the expression of the coefficients at iteration $(n+1)$ in terms of those at iteration n. For example, with a normalized overlap $\bar{\gamma} = \gamma/H$, we have for a specific index k,

$$
\begin{aligned}
B^{(s,r)}_{n+1,k,1} = {} & \frac{\left(z_k + \frac{\bar{\alpha}^2}{z_k}\right)\sinh\left(2\bar{\gamma} z_k\right) + 2\bar{\alpha}\cosh\left(2\bar{\gamma} z_k\right)}{\left(z_k + \frac{\bar{\alpha}^2}{z_k}\right)\sinh\left(z_k\right) + 2\bar{\alpha}\cosh\left(z_k\right)} B^{(s,r-1)}_{n,k,1} \\
& + \sum_{m=1}^{\infty} \left\{ \frac{4 z_k^{4+\beta(k,\bar{\alpha})}\left[\frac{\bar{\alpha}}{z_k}\tanh(z_k)+1\right]\left(z_m + \frac{\bar{\alpha}^2}{z_m}\right)\sin\left((1-2\bar{\gamma})z_m\right)}{\left[\left(z_k + \frac{\bar{\alpha}^2}{z_k}\right)\tanh(z_k) + 2\bar{\alpha}\right] z_m^{1+\beta(m,\bar{\alpha})}\left(z_m z_k^3 + z_k z_m^3\right)} \right. \\
& \left. \frac{\left\{\tanh(z_m)\left[\bar{\alpha}(z_k^2+z_m^2)\sin(z_k) - z_k(\bar{\alpha}^2 - z_m^2)\cos(z_k)\right] + z_m(\bar{\alpha}^2+z_k^2)\sin(z_k)\right\}}{\left[\frac{\bar{\alpha}}{z_m}\tanh(z_m)+1\right]\left[(z_k^2-\bar{\alpha})^2\sin(2z_k) + 2z_k(\bar{\alpha}^2+z_k^2+\bar{\alpha}) - 2\bar{\alpha} z_k\cos(2z_k)\right]} B^{(s,r-1)}_{n,m,2} \right\} \\
& + \frac{\left(-z_k + \frac{\bar{\alpha}^2}{z_k}\right)\sinh\left((1-2\bar{\gamma})z_k\right)}{\left(z_k + \frac{\bar{\alpha}^2}{z_k}\right)\sinh\left(z_k\right) + 2\bar{\alpha}\cosh\left(z_k\right)} B^{(s,r-1)}_{n,k,3} \\
& + \sum_{m=1}^{\infty} \left\{ \frac{4 z_k^{4+\beta(k,\bar{\alpha})}\left[\frac{\bar{\alpha}}{z_k}\tanh(z_k)+1\right]\left(z_m + \frac{\bar{\alpha}^2}{z_m}\right)\sin\left((1-2\bar{\gamma})z_m\right)}{\left[\left(z_k + \frac{\bar{\alpha}^2}{z_k}\right)\tanh(z_k) + 2\bar{\alpha}\right] z_m^{1+\beta(m,\bar{\alpha})}\left(z_m z_k^3 + z_k z_m^3\right)} \right. \\
& \left. \frac{\left\{\tanh(z_m) z_k(\bar{\alpha}^2+z_m^2) - z_m\left[-2\bar{\alpha} z_k + \frac{(\bar{\alpha}^2-z_k^2)\sin(z_k)+2\bar{\alpha} z_k\cos(z_k)}{\cosh(z_m)}\right]\right\}}{\left[\frac{\bar{\alpha}}{z_m}\tanh(z_m)+1\right]\left[(z_k^2-\bar{\alpha})^2\sin(2z_k) + 2z_k(\bar{\alpha}^2+z_k^2+\bar{\alpha}) - 2\bar{\alpha} z_k\cos(2z_k)\right]} B^{(s,r-1)}_{n,m,4} \right\}.
\end{aligned}
\tag{9}
$$

Let B_n be the infinite vector containing all the error series coefficients at iteration n, i.e., $B_n = (b_{n_1}, b_{n_2}, \ldots)$ with $b_{n_j} \in \left\{B^{(s,r)}_{n,k,i} : s \in \{1,\ldots,p\}, r \in \{1,\ldots,q\}, k \in \mathbb{N}, i \in \{1,\ldots,4\}\right\}$. Then the relation between coefficients can be written as $B_{n+1} = \hat{T} B_n$, where $\hat{T} : \mathbb{R}^\infty \to \mathbb{R}^\infty$ is an infinite matrix. Note that $\hat{T} = (\hat{T}^{1,1}, \ldots, \hat{T}^{p,q})$, where $\hat{T}^{(s,r)}$ is a local operator such that $B^{(s,r)}_{n+1} = \hat{T}^{(s,r)} B_n$ with $B^{(s,r)}_{n+1}$ being a vector containing all the error coefficients of the local problem (s,r) at iteration $(n+1)$.

Our main result is the following.

Theorem 1. *For any positive value of the normalized overlap $\bar{\gamma}$ there exist a computable range of values of the normalized boundary parameter $\bar{\alpha}$ for which the OSM iteration given by (2) converges.*

For its proof it suffices to show that each of the series in (3)-(6) converge uniformly and that the error series coefficients tend to zero as the number of iteration goes to infinity.

4 Approximation of the infinite operator $\hat{T}$ by a matrix of finite dimensions

Note that the following statements hold

1. In the r.h.s. of (9), the terms containing the coefficients $B^{(s,r-1)}_{n,k,i}$, $i = 1,3$, decrease with k.

2. For a given $n \in \mathbb{N}_0$, $B_{n,m,i}^{(s,r-1)}$ is uniformly bounded in $m \in \mathbb{N}$ and $i = 1,...,4$. Moreover, $B_{n,m,i}^{(s,r-1)} \leq M/z_m$ for all $m \in \mathbb{N}$ and some $M > 0$.
3. For any number $\delta > 0$ there exists a number k_δ, such that for $k > k_\delta$, the sum of the absolute values of the terms in the r.h.s. of (9) is less than δ.
4. For any number $\delta > 0$ there exists a number m_δ, such that for every $k \in \mathbb{N}$ the sum of the absolute values of the terms in the r.h.s. of (9) corresponding to for $m > m_\delta$ is less than δ.

Let $(B_n)_{|k \leq k_\delta}$ denote the vector resulting after discarding all the entries of B_n corresponding to $k > k_\delta$. Then, based on the above three facts, we can write

$$(B_{n+1})_{|k\leq k_\delta} = \left(\hat{T}\,(B_n)\right)_{|k\leq k_\delta} = \hat{T}_\delta\left((B_n)_{|k\leq k_\delta}\right) + \xi_{n+1,k_\delta}((B_n)_{|k>k_\delta}), \qquad (10)$$

where $\hat{T}_\delta$ is a finite matrix obtained by discarding the rows and columns of $\hat{T}$ related to the coefficients pertaining to $k > k_\delta$, and $\xi_{n+1,k_\delta}((B_n)_{|k>k_\delta})$ is the error obtained by approximating $(B_{n+1})_{|k\leq k_\delta}$ by $\hat{T}_\delta\left((B_n)_{|k\leq k_\delta}\right)$.

We will discuss in the next section situations in which $\rho(\hat{T}_\delta) < 1$, i.e., the spectral radius of $\hat{T}_\delta$ is less than one. In the rest of this section we show that in addition the error $\xi_{n+1,k_\delta}((B_n)_{|k>k_\delta})$ tends to zero as $n \to \infty$, and consequently $B_n \to 0$ as $n \to \infty$.

A necessary condition for convergence of Optimized Schwarz is that $B_n \to 0$ as $n \to \infty$. Note that each entry of $\xi_{n+1,k_\delta}((B_n)_{|k>k_\delta})$ is the truncation error that results after truncating the series in the formulas of the coefficients $B_{n+1,k,i}^{(s,r)}$, by keeping only the terms corresponding to $k \leq k_\delta$. Thus, as it can be seen in (9), each entry of $\xi_{n+1,k_\delta}((B_n)_{|k>k_\delta})$ is just a linear combination of the entries of $(B_n)_{|k>k_\delta}$. Note also that the entries of $(B_n)_{|k>k_\delta}$ are linear combinations of the entries of B_{n-1}. Hence, based on the four facts from above, we can choose a large enough k_δ so that the entries of $(B_{n+1})_{|k>k_\delta}$ and $\xi_{n+1,k_\delta}((B_n)_{|k>k_\delta})$ are as small as desired.

Using equation (10) recursively, we obtain the following equation

$$(B_{n+1})_{|k\leq k_\delta} = \hat{T}_\delta^{n+1}((B_0)_{|k\leq k_\delta}) + \sum_{j=1}^{n+1} \hat{T}_\delta^{n+1-j}(\xi_{j,k_\delta}((B_{j-1})_{|k>k_\delta})). \qquad (11)$$

Using (11), the four facts from above, and assuming that the spectral radius of $\hat{T}_\delta$ is less than one and that remains practically constant for large values of k_δ, it can be shown that given a $0 < \varepsilon < 1$ there exists a n_ε such that $||B_n||_\infty \leq \varepsilon ||B_0||_\infty$ for all $n \geq n_\varepsilon$. Repeating this argument, we can then show that $\lim_{n\to\infty} B_n = 0$. Hence in order to prove that $B_n \to 0$ as $n \to \infty$, it suffices to show that $\rho(\hat{T}_\delta) < 1$ and that it remains practically constant for large values of k_δ. We show this in the next section.

It can be shown that the series describing the local errors converge uniformly in $\Omega^{(s,r)}$. This implies that if each term of the error series goes to zero as n goes to infinity, so will do the series. Thus, given that $B_n \to 0$ as $n \to \infty$, i.e., the coefficients of the error series go to zero as n goes to infinity, the error of the iterative process converges to zero as n goes to infinity, which means that Optimized Schwarz converges for the given Poisson's problem for any initial error.

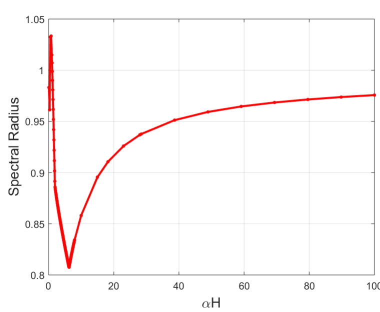
5 Spectral Radius of $\hat{T}_\delta$

The spectral radius of $\hat{T}_\delta$ describes the convergence rate of the Optimized Schwarz method. Thus, we define the optimal normalized boundary parameter $\bar{\alpha} = \alpha H$ as the one which minimizes the spectral radius of $\hat{T}_\delta$ and thus gives the optimal asymptotic convergence rate.

The values of the entries of the matrix $\hat{T}_\delta$ depend on the normalized overlap $\bar{\gamma}$, $\bar{\alpha}$ and the truncation parameter k_δ. The structure of the matrix depends on k_δ, p, q and the way we order the entries of B_n, i.e., the way we order each coefficient $B^{(s,r)}_{n,k,i}$ based on its values of s, r, k and i. For the ordering we have chosen, we computed the spectral radius of the resulting matrix $\hat{T}_\delta$, for $\bar{\gamma} \in \{0, 0.001, 0.01, 0.04, 0.08\}$, a set of values of $\bar{\alpha}$ in the range $[0.1, 500]$, $k_\delta \in \{3, 5, 10, 20, 50, 100\}$, and $p, q \in \{4, 5, 10, 20, 30\}$. In these computations we have observed the following.

1. There exist values of $\bar{\alpha}$ for which the spectral radius of $\hat{T}_\delta$ is less than one.
2. For a given $\bar{\gamma}$ and the range of $\bar{\alpha}$ considered in the experiments, $\rho(\hat{T}_\delta)$ has a local minimum, and it approaches a constant less than one for large values of $\bar{\alpha}$.
3. Given $\bar{\gamma}$, $\bar{\alpha}$, p and q, the value of $\rho(\hat{T}_\delta)$ remains practically constant for large enough k_δ (see Figure 2).
4. For a given $\bar{\gamma}$, the optimal spectral radius of $\hat{T}_\delta$ remains practically constant as p and q increase.

In Figure 1, the results for the cases $\bar{\gamma} = 0.001$ and $\bar{\gamma} = 0.01$, with $p, q = 10$, $k_\delta = 20$, $\bar{\alpha} \in [1, 100]$, are shown.



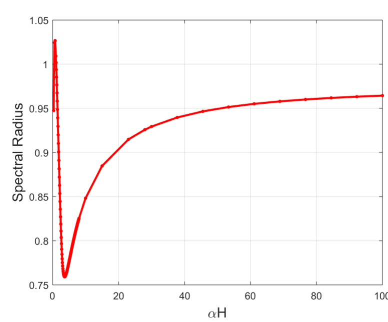


Fig. 1 (a) Spectral Radius of $\hat{T}_\delta$ vs. $\bar{\alpha}$ for $\bar{\gamma} = 0.001$, $p, q = 10$, $k_\delta = 20$ and $\bar{\alpha} \in [0.1, 100]$. (b) Spectral radius of $\hat{T}_\delta$ vs. $\bar{\alpha}$ for $p, q = 10$, $k_\delta = 20$, $\bar{\gamma} = 0.01$ and $\bar{\alpha} \in [0.1, 100]$

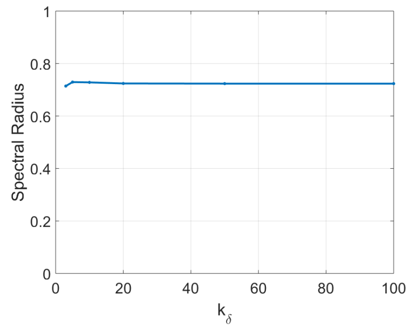


Fig. 2 Spectral radius of $\hat{T}_\delta$ vs. k_δ for $p,q = 10$, $\bar{\gamma} = 0.01$ and $\bar{\alpha} = 3.9697$

6 Further comments and conclusion

In the case of elliptic problems with varying coefficients, the same procedure can be applied to obtain an operator $\hat{T}$ such that $B_{n+1} = \hat{T}B_n$ as long as the coefficients are separable as products of one-variable functions. In this case, as well as in the constant coefficients case, the entries of the operator $\hat{T}$ depend on values and first derivatives of ϕ_m and ψ_m with $m \in \mathbb{N}$ at specific points. Note that in the constant coefficient case an explicit formula can be obtained for ϕ_m and ψ_m. In the varying coefficients case, an explicit formula for ϕ_m and ψ_m may not always be available. However, we can still compute values of ϕ_m and ψ_m and their first derivatives at specific points using numerical methods and then use these values to compute $\rho(\hat{T}_\delta)$.

In conclusion, we analyzed the convergence of the Optimized Schwarz method applied to Poisson's equation in a bounded rectangular domain subject to nonhomogeneous Dirichlet boundary conditions and transmission conditions of the family $OO0$. The spectral radius of $\hat{T}_\delta$ can be less than one for any positive amount of overlap. One can obtain the optimal boundary parameter that minimizes this spectral radius. We outlined a proof showing that this bound on the spectral radius, together with other results, can guarantee convergence of OSM for the problem studied.

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