Small coarse spaces for overlapping Schwarz algorithms with irregular subdomains

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1 Introduction

Coarse spaces are at the heart of many domain decomposition algorithms. Building on the foundation laid in [10], we have an ongoing interest in the development of coarse spaces based on energy minimization concepts; see [2, 4, 5, 6].

This paper is a short report on a project which substantially extends results in a DD21 conference paper, [7], and which now has resulted in an archival publication [8]. Our work primarily concerns two-level overlapping Schwarz methods and is exclusively for low order, conforming finite element approximations of three–dimensional elliptic problems. What is new in this paper are some variants of the algorithms reported in [8]. The focus of this study is the development of smaller coarse spaces which, to the extent possible, will give us similar rates of convergence as for those developed in the past. Extensive large scale experiments show that this is possible and important; see e.g. [11] in these proceedings.

The domain of a scalar elliptic or elasticity operator is partitioned into non-overlapping subdomains Ω_i each of which is the union of elements. We use *nodal equivalence classes* of finite element nodes on the interface, i.e., the nodes that belong to more than one subdomain boundary, in the construction of our coarse spaces. Two such nodes belong to the same equivalence class if they belong to the same set of subdomain boundaries. The *coarse nodes* are associated with those equivalence classes which are maximal in the sense that they are not subsets of any other. In many cases, the coarse nodes are simply the vertices of the subdomains but there are also other cases which are identified automatically by our algorithm. Each in-

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terface node n is thereby associated with a set of coarse nodes $\mathscr{C}_n$. A coarse node c is included in $\mathscr{C}_n$ if the equivalence class of n is a subset of that of c. For each coarse node, we will construct one coarse basis function for scalar elliptic and six for elasticity problems, which span the coarse space.

2 Elliptic Problems and the Coarse Basis Functions

In our study, we consider scalar elliptic problems defined in terms of a bilinear form

$$\int_\Omega \rho \nabla u \cdot \nabla v \, dx$$

where $\rho(x) > 0$ and constant $= \rho_i$ in each subdomain Ω_i into which Ω has been partitioned. The functions u and v belong to a subspace of $H^1(\Omega)$ subject to a Dirichlet condition on $\partial\Omega$ or a subset thereof. We also consider linear, compressible elasticity defined by a bilinear form

$$2\int_\Omega \mu \ \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) \, dx + \int_\Omega \lambda \ \text{div} \ \boldsymbol{u} \ \text{div} \ \boldsymbol{v} \, dx,$$

where $\mu(x)$ and $\lambda(x)$ are the positive Lamé parameters, $\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$, and $\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) := \sum_{i=1}^3 \sum_{j=1}^3 \varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v})$. The Lamé parameters are also assumed constant, μ_i and $\lambda_i = (2\mu_i\nu_i)/(1-2\nu_i)$, in Ω_i, with $0 < \nu_i < 1/2$. This variational problem is posed in a subspace of $(H^1(\Omega))^3$ determined by a Dirichlet condition. The energy of these systems are defined by these bilinear forms.

Three recipes for the construction of coarse space elements have been developed in [8], each defined in terms of a partition of unity for each interface node. The simplest one, referred to as Option 1, is given by

$$p_{nc} := 1/N_c, \tag{1}$$

where $N_c := |\mathscr{C}_n|$. Of the two other recipes, the one relevant for this paper is the third defined in terms of $d_i(n), i = 1, \ldots, N_c$, the distances between an interface node n and the $c_i \in \mathscr{C}_n$, and given by

$$p_{nc_i} := \frac{1/d_i(n)}{1/d_1(n) + 1/d_2(n) + \ldots + 1/d_{N_c}(n)}. \tag{2}$$

This Option 2 is the only one used in the experiments reported in this paper.

The values of these functions are used as Dirichlet data and extended into the interior of the subdomains, minimizing the energy, and resulting in continuous coarse basis functions for scalar elliptic problems. The support of a coarse basis function associated with the coarse node c is the union of the closure of all Ω_j with c on their boundaries. For elasticity, we multiply the scalar function p_{nc} by a 3×6 matrix with

columns forming a basis for the space of rigid body modes prior to extending the resulting values on the interface into the interiors of the subdomains. We note that the resulting finite element functions will all be continuous given that there are no jumps in the Dirichlet data across the interface.

The choice of minimal energy extensions results in coarse basis functions which sum to 1, in any subdomain that does not touch the Dirichlet boundary, for the scalar elliptic problems and rigid body modes for elasticity. This fact shows that the *null space* condition will be satisfied, a condition necessary to obtain a convergence rate bounded independently of the number of subdomains for any domain decomposition algorithm. This approach works well even for subdomains with irregular boundaries such as those obtained from mesh partitioners. The crucial part of the analysis of an overlapping Schwarz algorithm requires a bound on the sum of the energy of the components of the coarse space and of the local spaces of finite element functions, supported in the overlapping subdomains chosen to define the local problems, in terms of the energy of the sum of these functions; see, e.g., [15, Chapter 2]. The choice of minimal energy extensions therefore makes sense also for this reason.

The development of domain decomposition theory has often focused on the effect of large discontinuities of the coefficient. Thus, for iterative substructuring algorithms, based on non-overlapping subdomains, a number of strong results have been developed for elliptic problems where the coefficients are constant or vary slowly inside the subdomains but without any restrictions on their variation across the interface between the subdomains; see, e.g., [15, Chapters 4–6] and [14]. Many of these algorithms are well-defined for arbitrary subdomains although the theory has been fully developed mostly for subdomains that are tetrahedral or unions of a few large tetrahedra; we note that some of the standard tools now have been extended to Lipschitz subdomains, see [8]. In contrast, the theory for two–level additive Schwarz methods is developed only for constant coefficients in [15, Section 3.2]. However, the classical coarse spaces for these Schwarz algorithms have been shown to be stable for *quasi–monotone* coefficients in [9]; for a related condition, see Assumption 1 of this paper. The results in [9] considerably expanded the class of subdomain coefficients for which results quite similar to those for constant coefficients became possible.

To derive the final bounds for our overlapping Schwarz algorithms, we also need to consider the components associated with local problems on overlapping subdomains, which are often constructed by extending the nonoverlapping subdomains, Ω_i, into which the given domain Ω has been decomposed, by adding one or a few layers of elements. Observing that we need solvers for Dirichlet problems on the original subdomains Ω_i to construct the coarse basis functions, we will in some of our numerical experiments instead use the Ω_i as part of the covering. In addition, to cover all of the domain Ω, we can then use *boundary layers*, which are unions of elements which include all points within a minimal distance δ_i to the boundary of an individual Ω_i. As an alternative, we also use sets created by adding one or more element layers to the closure of the individual subdomain faces.

We note that no new ideas are required to complete the part of the analysis related to these locally supported subspaces; cf. [15, Subsection 3.2] and the discus-

sion in [6, Section 3]. Therefore, we have been able to focus on developing the coarse spaces and bounds for the coarse component, which are always required in the analysis of any Schwarz algorithm; see [15, Subsection 2.3].

The subdomains, Ω_i, are unions of elements, that form quasi–uniform meshes for each subdomain, and often have irregular boundaries, in particular, if they have been generated by a mesh partitioner. Some of the tools used in our analysis, such as a trace theorem, will require that the subdomains are Lipschitz. We note that in our previous studies of two-dimensional problems, [3, 6], we have been able to extend our analysis even to subdomains with fractal boundaries assuming only that they are uniform in the sense of Jones [12].

We note from formula (1) that p_{nc} is the same for all n in a nodal equivalence class and for a particular $c \in \mathscr{C}_n$; in the case of tetrahedral subdomains, the basis functions constructed will be built from the face and edge functions, $\theta_{\mathscr{F}}$ and $\theta_{\mathscr{E}}$, used extensively in the development of iterative substructuring algorithms as in [15, Chapters 4-6]. The fact that these functions are piecewise constant causes large changes in the coarse basis functions across boundaries between equivalence classes, resulting in logarithmic factors, $(1+\log(H_i/h_i))$, in our estimate of the energy of the coarse basis functions; cf. [15, Lemma 4.25] for a bound on the energy of the classical face function $\theta_{\mathscr{F}}$. Here, H_i is the diameter of Ω_i and h_i the diameter of its smallest element. In [8], we have obtained the same quality bound for Lipschitz subdomains by generalizing bounds for the face and edge functions for subdomains to the Lipschitz case. By using the alternative (2), we obtain smoother coarse basis functions and improved bounds.

3 Assumptions and Major Results

We will now consider two different assumptions on the coefficient μ of the elasticity problem. The same assumptions are also used for the coefficient, ρ, of the scalar elliptic problems.

Assumption 1 *(Quasi-monotone face-connected paths) Let c be any coarse node of Ω_i and $\mathscr{S}_c$ be the index set of all subdomains containing c on their boundaries. Select $j_c \in \mathscr{S}_c$ such that $\mu_{j_c} \geq \mu_j$ for all $j \in \mathscr{S}_c$. Assume that there exists a constant C and for any $i \in \mathscr{S}_c$ a sequence $\{i = j_c^0, j_c^1, ..., j_c^p = j_c\}$, all in $\mathscr{S}_c$, such that $\mu_i \leq C\mu_{j_c^\ell}$ and that $\Omega_{j_c^{\ell-1}}$ and $\Omega_{j_c^\ell}$ have a subdomain face $\mathscr{F}_{j_c^{\ell-1},j_c^\ell}$ in common for all $\ell = 1,\ldots,p$ and $i = 1,\ldots,N$. In the case that $c \in \partial\Omega$, we also assume that $\partial\Omega_{j_c} \cap \partial\Omega$ contains at least one subdomain face.*

In other words, Assumption 1 means that there is a face connected path between Ω_i and Ω_{j_c} such that the Lamé parameter μ_i is no greater than a constant times the Lamé parameter of any subdomain along the path. This assumption is similar to the quasi-monotonicity assumption of [9]. We will also work with an additional assumption.

Assumption 2 *(Quasi-monotone edge-connected paths) Using the same notation as in Assumption 1, assume that there exists a sequence* $\{i = j_c^0, j_c^1, ..., j_c^p = j_c\}$, *all in* $\mathscr{S}_c$, *such that* $\rho_i \leq C\rho_{j_c^\ell}$ *and* $\Omega_{j_c^{\ell-1}}$ *and* $\Omega_{j_c^\ell}$ *have at least a subdomain edge in common for all* $\ell = 1,\ldots,p$ *and* $i = 1,\ldots,N$. *In the case that* $c \in \partial\Omega_i$ *also assume that* $\partial\Omega_{j_c} \cap \partial\Omega$ *contains at least one subdomain edge.*

We note that Assumption 2 is weaker than Assumption 1 since we have more options of continuing at every step in the construction of a path. We note that in our proof for linear elasticity, we have had to use the more restricted Assumption 1. The need for this has also been demonstrated by experiments reported in [8].

Our analysis can closely follow the theory as developed in [15, Section 2.3]; a main effort is directed to constructing a coarse component u_0, for any u, with a good bound on the energy $E(u_0)$ in terms of $E(u)$, the energy of the function u..

With estimates for our coarse interpolants in hand, we can then perform a local analysis for an overlapping additive Schwarz algorithm using basically the same approach as in [3] or [6]. This involves a set of partition of unity functions $\{\vartheta_j\}_{j=1}^N$ with $0 \leq \vartheta_j \leq 1$, $|\nabla\vartheta_j| \leq C/\delta_i$, and with ϑ_j supported in the closure of a subdomain which is part of the covering of Ω. Here, δ_j is the thickness of the part of subdomain which is common to its neighbors. Given an estimate of the form

$$E(u_0) \leq C\Theta(H/h)E(u),$$

where $H/h := \max_i H_i/h_i$, the resulting condition number estimate for the preconditioned operator is given by

$$\kappa(M^{-1}A) \leq C\Theta(H/h)(1+H/\delta), \tag{3}$$

where $H/\delta := \max_i H_i/\delta_i$. For Option 2, we can prove a uniform bound of $\Theta(H/h)$ if Assumption 1 is satisfied. In addition, we have a bound $\Theta(H/h) \leq (1+\log(H/h))$ for the scalar case if Assumption 2 holds.

We note that our coarse spaces could alternatively be combined with local spaces previously developed for iterative substructuring algorithms such as those of [10]; see also [15, Chapter 5].

4 Numerical Results

Numerical results are presented in this section to help confirm the theory and to demonstrate some advantages of the face-based local spaces. We note that large-scale experiments with closely related algorithms are also reported in [11]. Our results are for a unit cube domain with homogeneous essential boundary conditions applied to one of its faces. Condition numbers (cond) of the preconditioned operator and the number of iterations (iter) needed to achieve a relative residual tolerance of 10^{-8} for the solution of the linear system of equations, $Ax = b$, with random right-hand-side vectors b are reported. The domain is decomposed into smaller cubic

subdomains, and formula (2) is used to construct the coarse space. We note that the interface preconditioner is of a hybrid type which employs overlapping Schwarz local spaces as in [5]. We also note that at the end of each step of the iteration, the residual will vanish at all interior nodes of the subdomains. We use the lowest order hexahedral nodal elements and Matlab.

Three different local spaces are considered. The *standard* one starts with all the nodes of a non-overlapping subdomain and adds to them nodes from an integer number of layers of elements adjacent to the original nodes. The *boundary layer* local spaces are identical to the standard ones with the exception that the starting nodes only include those on the subdomain interfaces. We note these local spaces were considered previously in [5]. Finally, the *face* local spaces of this study start with nodes in the closure of each subdomain face and add layers of elements just as for the other two local spaces. These spaces locally precondition an interface problem.

Our example has the overlap parameter $H/\delta = 3$ fixed while the number of elements (H/h) in each subdomain direction increases. In addition to condition numbers and iteration counts, we also report in Table 1 the number of non-zeros in the sparse Cholesky factorizations for the local spaces. Specifically, r_{nnz} denotes these numbers normalized by the number for the standard local space. Further, we report estimates of the maximum eigenvalue $\lambda_{\max}$ of the preconditioned operator.

Consistent with the theory, condition numbers for the face local spaces exhibit sub-linear growth with respect to H/h. Although the number of iterations and condition numbers are noticeably larger compared with the standard and boundary layer spaces, the number of non-zeros in the local factorizations are considerably smaller for the face spaces. One reason for the larger condition numbers of the face spaces are the larger values of $\lambda_{\max}$ shown in Table 1. The larger values can be explained using a coloring argument. For instance, there are 12 different faces which include each subdomain vertex in their closures. In contrast, each subdomain vertex is included in only 8 of the standard or boundary layer spaces.

Table 1 Results for a unit cube decomposed into 64 smaller cubic subdomains with overlap $H/\delta = 3$ for three different local spaces. The material properties are constant with $\rho = 1$ for scalar problems and $\mu = .385$, $\lambda = 1.54$ for elasticity problems.

	standard			boundary layer				face			
H/h	iter	cond	$\lambda_{\max}$	iter	cond	r_{nnz}	$\lambda_{\max}$	iter	cond	r_{nnz}	$\lambda_{\max}$
scalar problem results											
3	26	14.1	8.2	27	14.8	0.47	8.2	39	32.7	0.18	12.1
6	28	17.7	8.2	31	18.9	0.77	8.2	50	40.9	0.25	12.0
9	30	19.7	8.2	33	21.1	0.80	8.2	55	45.9	0.28	12.0
12	30	30.0	8.2	33	22.5	0.87	8.2	58	49.6	0.30	12.0
elasticity problem results											
3	33	13.6	8.2	34	14.4	0.47	8.2	47	30.9	0.17	12.1
6	36	15.8	8.2	38	16.8	0.69	8.2	59	35.3	0.23	12.0
9	37	17.1	8.2	40	18.2	0.78	8.2	62	38.7	0.26	11.9
12	38	18.0	8.2	41	19.1	0.78	8.2	64	41.2	0.27	11.9

Normalized solution times for the preconditioned conjugate gradient algorithm applied to an elasticity problem (see bottom half of Table 1) are shown in Figure 1 for the boundary layer and face local spaces. Notice for all values of H/h that the normalized times are less than 1 for the boundary layer local spaces. Remarkably, the smallest times are achieved using the face local spaces for $H/h > 5$ even though the number of iterations are larger than those for the other two local spaces. The improved performance here can be attributed to the much smaller factorization sizes for the face spaces.

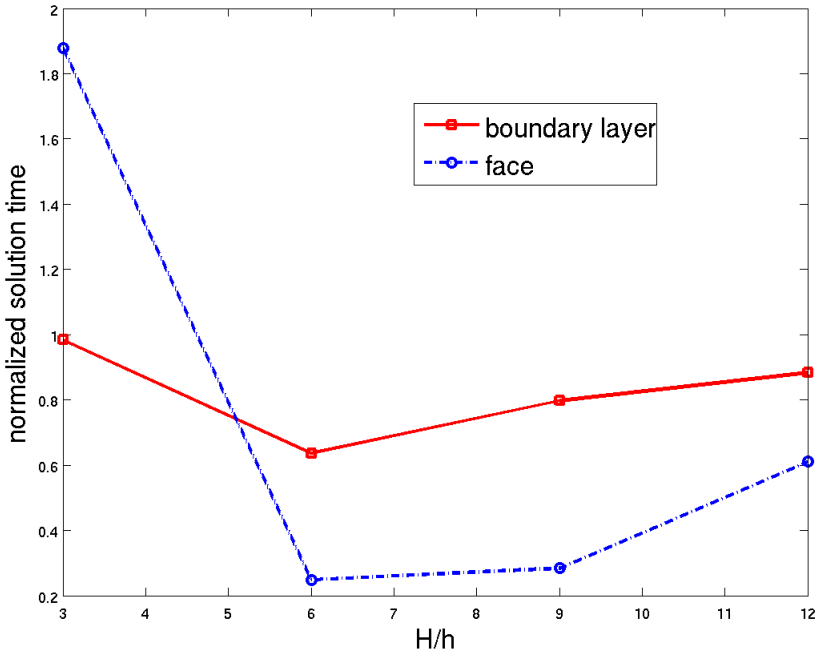

Fig. 1 Elasticity problem solution times for the preconditioned conjugate gradient algorithm normalized with respect to solution times for the standard local space.

As found in [5], the number of iterations can be reduced significantly, for all three local spaces, by dividing each element in the right-hand-side vectors for the local solvers by the number of local spaces which share this element. Although this results in a non-symmetric preconditioner, reduced solution times can be achieved as for restricted additive Schwarz preconditioners [1]. As a final note, for parallel computations, it makes sense to assign the work for each face to just one of the two subdomains, i.e. processors, which contain it. To achieve good load balance, an assignment algorithm can be used to approximately minimize the maximum work for any one processor.

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