Domain Decomposition Algorithms for a Generalized Stokes Problem

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1 Introduction

Domain decomposition methods are appropriate to address fluid dynamical problems, especially in complex physical regions and within parallel computational environments. The zonal approach generally consists of partitioning the whole region into subregions of simpler shape, and then reduces the given flow problem to a sequence of subproblems which, to some extent, can be solved simultaneously.

Recently, a mathematical analysis has been carried out in [1] for hyperbolic systems of conservation laws (e.g., Euler equations for compressible inviscid flows). A generalized Stokes problem and an inviscid generalized Stokes problem are addressed in [2] and [3] in the framework of a domain decomposition method, in which the physical computational region Ω is partitioned into two subdomains Ω_1 and Ω_2, and correct transmission conditions across the interface Γ between Ω_1 and Ω_2 are provided. An iterative procedure involving the successive solution of two subproblems is proposed, and the convergence for the iteration-by-subdomain algorithms, which are associated with various domain decomposition approaches, is proved.

Quarteroni et al [3] consider the following generalized Stokes equation

$$\begin{cases} \alpha\sigma + \mathrm{div} u = g \text{ in } \Omega \\ \alpha u - \gamma\Delta u + \beta\nabla\sigma = f \text{ in } \Omega \end{cases} \tag{1.1}$$

where Ω is an open two-dimensional domain, α, β, γ are positive constants, and g and f are given scalar and vector functions, respectively. The unknowns are σ, which is a scalar related to the flow density, and the velocity vector field u.

In this paper, we consider the problem (1.1), and solve it by domain decomposition methods partitioning the computational domain Ω into m subdomains ($m \geq 2$), and propose and analyze some parallel domain decomposition algorithms, and prove that these algorithms are convergent. Furthermore, their convergence rates are nearly optimal if problem (1.1) is discreted by finite elements, which shows that these algorithms are very suitable for solving problem (1.1).

In this paper, c will be a generic constant.

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2 The differential formulation of the problem (1.1)

We first give some notation. Let $C^0(\bar{\Omega})$ be the space of continuous functions on $\bar{\Omega}$, let $H^s(\Omega)$ and $H^s(\Gamma)(s \in \Re, \Gamma$ is a curve) be the usual Sobolev spaces, endowed with the norm $\|\cdot\|_{s,\Omega}$ and $\|\cdot\|_{s,\Gamma}$, respectively. For the seminorms, we use the notations $|\cdot|_{s,\Omega}$ and $|\cdot|_{s,\Gamma}$, for an open curve Γ_0 contained in a closed curve Γ. Let $H_{00}^{1/2}(\Gamma_0)$ be the set of (generalized) functions such that their extension by zero onto Γ belongs to $H^{1/2}(\Gamma)$ and let $(u,v)_\Omega = \int_\Omega uvdxdy$.

We restate the generalized Stokes problem as follows

$$\begin{cases} \alpha\sigma + \text{div}u = g \text{ in } \Omega \\ \alpha u - \gamma\Delta u + \beta\nabla\sigma = f \text{ in } \Omega \end{cases} \tag{2.1}$$

This is an elliptic system. The boundary conditions for problem (2.1) are

$$\begin{cases} u \equiv 0 \text{ on } \Gamma_B \\ u \equiv u_\infty \text{ on } \Gamma_\infty^- \\ S(u,\sigma) \equiv \gamma\dfrac{\partial u}{\partial n} - \beta\sigma n = 0 \text{ on } \Gamma_\infty^+ \end{cases} \tag{2.2}$$

where $\partial\Omega = \Gamma_B \cup \Gamma_\infty^- \cup \Gamma_\infty^+$.

The following result was proved in [2].

Theorem 2.1. *Assume that $f \in H^{-1/2}(\Omega)$, $g \in L^2(\Omega)$, $u_\infty \in H^{1/2}(\Gamma_\infty^-)$. The problem (2.1). (2.2) has a unique solution $(u,\sigma) \in H^1(\Omega) \times L^2(\Omega)$ and*

$$\|u\|_{1,\Omega} + \|\sigma\|_{0,\Omega} \le c(\|f\|_{-1/2,\Omega} + \|g\|_{0,\Omega} + \|u_\infty\|_{1/2,\Gamma_\infty^-}).$$

Let

$$V_0 \equiv \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_B \cup \Gamma_\infty^-\},\ \Sigma_0 \equiv L^2(\Omega).$$

Then (2.1), (2.2) can be formulated as

$$\begin{cases} (u - U_\infty) \in V_0,\ \sigma \in \Sigma_0 \\ a[(u,\sigma),(v,\varphi)] = (f,v)_\Omega + \beta(g,\varphi)_\Omega,\ \forall(v,\varphi) \in V_0 \times \Sigma_0 \end{cases} \tag{2.3}$$

where U_∞ is a suitable extension of u_∞ to Ω satisfying $U_\infty \in H^1(\Omega)$, $U_\infty = 0$ on Γ_B and $\|U_\infty\|_{1,\Omega} \le c\|u_\infty\|_{1/2,\Gamma_\infty^-}$, $a[(u,\sigma),(v,\varphi)] = \int_\Omega(\alpha\beta\sigma\varphi + \beta\varphi\text{div}u + \alpha uv + \gamma\nabla u\nabla v - \beta\sigma\text{div}v)dxdy$

Remark 2.1. $a[(u,\sigma),(v,\varphi)]$ is continuous and coercive in $V_0 \times \Sigma_0$.

Now, we solve problem (2.3) by domain decomposition methods. Ω is decomposed into two nonoverlapping subdomains, and we assume for simplicity that $u_\infty = 0$, $\Gamma_\infty^- \cup \Gamma_B = \partial\Omega$. If this is not the case, we can argue in a similar way.

Define

$V_i \equiv \{v \in H^1(\Omega_i) : v = 0 \text{ on } \partial\Omega_i \cap (\Gamma_\infty^- \cup \Gamma_B)\}$

$V_{0,i} \equiv \{v \in V_i : v = 0 \text{ on } \Gamma\},\ i = 1,2$

and the bilinear forms

$$a_i[(u_i,\sigma_i),(v_i,\varphi_i)] \equiv \int_{\Omega_i}(\alpha\beta\sigma_i\varphi_i + \beta\varphi_i\text{div}u_i + \alpha u_i v_i + \gamma\nabla u_i\nabla v_i - \beta\sigma_i\text{div}v_i)dxdy$$

which are continuous and coercive in $V_i \times L^2(\Omega_i)$.

The first domain decomposition algorithm studied in this paper can be described as

Algorithm 1.

Given $\lambda^0 \in H_{00}^{1/2}(\Gamma)$

Step 1. Compute $(u_i^n, \sigma_i^n) \in V_i \times L^2(\Omega_i)$ by solving the equations ($i = 1, 2$)

$$\begin{cases} a_i[(u_i^n, \sigma_i^n), (v, \varphi)] = (f, v)_{\Omega_i} + \beta(g, \varphi)_{\Omega_i} (v, \varphi) \in V_{0,i} \times L^2(\Omega_i) \\ u_i^n = \lambda^n \text{on } \Gamma \end{cases}$$

Step 2. Compute $(\omega_i^n, \psi_i^n) \in V_i \times L^2(\Omega_i)$ by solving the equation (i=1, 2)

$$a_i[(\omega_i^n, \psi_i^n), (v, \varphi)] = \frac{1}{2} \sum_{j=1}^{2} \{a_j[(u_j^n, \sigma_j^n), (v_j, \varphi_j)] - (f, v_j)_{\Omega_j} - \beta(g, \varphi_j)_{\Omega_j}\},$$

$\forall (v, \varphi) \in V_i \times L^2(\Omega_i), (v_j, \varphi_j) \in V_j \times L^2(\Omega_j), v_j|_\Gamma = v|_\Gamma, j = 1, 2.$

Step 3. $\lambda^{n+1} = \lambda^n - \rho(\omega_1^n + \omega_2^n)$ on Γ, where ρ is a positive constant.

Remark 2.2. Obviously, it is true that

$$\begin{aligned} &a_j[(u_j^n, \sigma_j^n), (v_1, \varphi_1)] - (f, v_1)_{\Omega_j} - \beta(g, \varphi_1)_{\Omega_j} \\ &= a_j[(u_j^n, \sigma_j^n), (v_2, \varphi_2)] - (f, v_2)_{\Omega_j} - \beta(g, \varphi_2)_{\Omega_j}, \end{aligned}$$

$\forall (v_1, \varphi_1),\ (v_2, \varphi_2) \in V_j \times L^2(\Omega_j), v_1|_\Gamma = v_2|_\Gamma, j = 1, 2$. This implies that (ω_i^n, ψ_i^n), given in Step 2, is unique.

It is easy to see that Algorithm 1 can be implemented in parallel. In the following, we analyze its convergence.

Theorem 2.2. *There exists a positive constant c such that if $0 < \rho < c$, then $\{(u_1^n, \sigma_1^n), (u_2^n, \sigma_2^n)\}$ generated by Algorithm 1 converge to the solution of equation (2.3).*

3 Domain decomposition algorithms for the finite element problem

Let us first briefly recall a finite element approximation for the problem (2.3); see [3]. Let T_h be a family of decompositions of Ω (which will be assumed hereafter to be a polygon) into triangles Δ. Assume that T_h is regular, i.e. there exists $\tau_0 > 0$ independent of h such that

$$h_\Delta / \rho_\Delta \leq \tau_0, \quad \forall \Delta \in T_h$$

where $h_\Delta \equiv \text{diam}\Delta,\ \rho_\Delta \equiv \sup\{2\rho | \exists x_0 \in \Delta : B(x_0, \rho) \subset \Delta\},\ h \equiv \max h_\Delta$.

Define the following finite element spaces for $r \geq 1$

$$\begin{cases} V_{0,h} = \{v \in C^0(\bar{\Omega}) : v|_\Delta \in P_r, \forall \Delta \in T_h, v = 0 \text{ on } \Gamma_\infty^- \cup \Gamma_B\} \\ \Sigma_{0,h} = \{\varphi \in L^2(\Omega) : \varphi|_\Delta \in P_{r-1},\ \forall \Delta \in T_h\} \end{cases} \tag{3.1}$$

where P_r denotes the space of polynomials of degree $\leq r$ on Δ, and consider the following finite dimensional approximation to (2.3)

$$\begin{cases} (u_h, \sigma_h) \in V_{0,h} \times \Sigma_{0,h} \\ a[(u_h, \sigma_h), (v_h, \varphi_h)] = (f, v_h)_\Omega + \beta(g, \varphi_h)_\Omega, \quad \forall (v_h, \varphi_h) \in V_{0,h} \times \Sigma_{0,h}. \end{cases} \tag{3.2}$$

Since $a[(u_h,\sigma_h),(v_h,\varphi_h)]$ is coercive on $V_{0,h}\times\Sigma_{0,h}$, problem (3.2) has a unique solution, and the following error estimate holds (see [16])

$$\|u-u_h\|_{1,\Omega}+\|\sigma-\sigma_h\|_{0,\Omega}\le ch^s(|u|_{s+1,\Omega}+|\sigma|_{s,\Omega}),\quad 0<s\le r$$

where (u,σ) is the solution of problem (2.1). By assuming $f\in H^{-1/2}(\Omega)$, $g\in H^s(\Omega)$, we obtain $u\in H^{1+s}(\Omega)$, $\sigma\in H^s(\Omega)$ for each $0\le s\le 1/2$. Higher regularity cannot be expected for the solution of problem (2.1) endowed with the mixed Dirichlet-Neumann boundary conditions (2.2). Hence it is safe to choose $r=1$ in (3.1) and (3.2). So, in the following, we set $r=1$.

Decompose Ω into nonoverlapping subdomains $\Omega_1,\ldots,\Omega_m$, and let

$$\Gamma\equiv\cup_{i=1}^m\partial\Omega_i\backslash\partial\Omega,\quad \Phi_h\equiv\{v|_\Gamma: v\in V_{0,h}\}$$

$$V_{i,h}\equiv\{v|_{\Omega_i}: v\in V_{0,h}\},\quad \Sigma_{i,h}\equiv\{\varphi|_{\Omega_i}:\varphi\in\Sigma_{0,h}\},\ i=1,\ldots,m$$

$$V_{i,h}^0\equiv\{v|_{\Omega_i}: v=0 \text{ on } \partial\Omega_i\cap\Gamma, v\in V_{0,h}\},\ i=1,\ldots,m.$$

Assume that the nonoverlapping subdomains $\{\Omega_i\}_{i=1}^m$ satisfy

A1. $\bar\Omega=\cup_{i=1}^m\bar\Omega_i$, the sides of $\Omega_i(i=1,\ldots,m)$ follow the finite element mesh lines of Ω;

A2. $\{\Omega_i\}_{i=1}^m$ are quasi-uniform quadrilaterals with size H.

For any i, trace average operator α_i is defined by

$$\begin{cases}\forall v_h\in V_{0,h},\alpha_i(v_h)\in\Phi_h \text{ and}\\ \alpha_i(v_h)(x)=\dfrac{1}{2}v_h(x),\ \forall \text{ nodes } x \text{ on } \partial\Omega_i\backslash\Omega,\ x \text{ which are not a vertex of } \Omega_i\\ \alpha_i(v_h)(x)=\dfrac{1}{N}v_h(x),\ \forall \text{ common vertex } x\ of\ N \text{ subdomains},\ x\notin\partial\Omega\\ \alpha_i(v_h)(x)=0\ \forall \text{ node } x \text{ on } \Gamma\backslash\partial\Omega_i\end{cases}$$

obviously

$$\sum_{i=1}^m\alpha_i(v_h)=Tr(v_h),\quad \forall v_h\in V_{0,h} \tag{3.3}$$

where $Tr(v_h)$ is the trace of v_h on Γ.

Now, we describe our parallel algorithm.

Algorithm 2.

Given $\lambda_h^0\in\Phi_h$

Step 1. Compute $(u_{i,h}^n,\sigma_{i,h}^n)\in V_{i,h}\times\Sigma_{i,h}$ by solving the equations

$$\begin{cases}a_i[(u_{i,h}^n,\sigma_{i,h}^n),(v_h,\varphi_h)]=(f,v_h)_{\Omega_i}+\beta(g,\varphi_h)_{\Omega_i}\quad \forall(v_h,\varphi_h)\in V_{i,h}^0\times\Sigma_{i,h}\\ u_{i,h}^n=\lambda_h^n \text{ on } \Gamma\cap\partial\Omega_i\end{cases}$$

Step 2. Compute $(\omega_{i,h}^n,\psi_{i,h}^n)\in V_{i,h}\times\Sigma_{i,h}$ by solving the equations

$$a_i[(\omega_{i,h}^n,\psi_{i,h}^n),(v_h,\varphi_h)]=\sum_{j=1}^m\{a_j[(u_{j,h}^n,\sigma_{j,h}^n),(v_{j,h},\varphi_{j,h})]-(f,v_{j,h})_{\Omega_j}-\beta(g,\varphi_{j,h})_{\Omega_j}\},$$

$\forall (v_h, \varphi_h) \in V_{i,h} \times \Sigma_{i,h}, (v_{j,h}, \varphi_{j,h}) \in V_{j,h} \times \Sigma_{j,h}, v_{j,h}|_\Gamma = \alpha_i(v_h), j = 1, \ldots, m.$

Step 3. $\lambda_h^{n+1} = \lambda_h^n - \rho \sum_{j=1}^{m} \alpha_j(\omega_{j,h}^n)$ on Γ, where ρ is a positive constant.

Remark 3.1. $(\omega_{i,h}^n, \psi_{i,h}^n)$ in Step 2 has a unique solution, and Algorithm 3 can be implemented in parallel.

Theorem 3.1. *For Algorithm 2, there exists a constant* τ

$$1 \le \tau \le O((1 + H^{-2})(1 + \ln^2(H/h))$$

such that if $\rho < 2/\tau$, *then*

$$\sum_{i=1}^{m} \|(\varepsilon_i^{n+1}, \delta_i^{n+1})\|_{a_i}^2 \le (1 - 2\rho + \tau\rho^2) \sum_{i=1}^{m} \|(\varepsilon_i^n, \delta_i^n)\|_{a_i}^2.$$

4 The analysis of the preconditioner corresponding to Algorithm 2

In Section 3, we have analyzed Algorithm 2, which is a preconditioned Richardson iterative method. Our purpose in this section is to analyze the preconditioner related to Algorithm 2, and to estimate the condition number of the preconditioned system.

It is known that $Q = S(I - P)^{-1}$ is the preconditioner in the iterative scheme $x^{n+1} = Px^n + q$ for solving system $Sx = b$. From the point of view of parallel computation, the preconditioner Q should satisfy

1) Q^{-1} should be easy to obtain in parallel;

2) The condition number of $Q^{-1}S$ should not be large.

Define discrete Steklov-Poincaré operator S_h as

$$(S_h\lambda_h, v_h)_\Omega \equiv \sum_{i=1}^{m} a_i[E_{i,h}\lambda_h, E_{i,h}v_h], \quad \forall (\lambda_h, v_h) \in \Phi_h$$

and let $(u_{i,h}^*, \sigma_{i,h}^*) \in V_{i,h}^0 \times \Sigma_{i,h}$, be the solution of

$$a_i[(u_{i,h}^*, \sigma_{i,h}^*), (v_h, \varphi_h)] \equiv (f, v_h)_{\Omega_i} + \beta(g, \varphi_h)_{\Omega_i}, \quad \forall (v_h, \varphi_h) \in V_{i,h}^0 \times \Sigma_{i,h}$$

$$(b_h, v_h)_\Omega \equiv \int_\Omega ((f, E_h v_h)_\Omega - a[(u_h^*, \sigma_h^*), E_h v_h]) dxdy, \quad \forall v_h \in \Phi_h$$

where

$$(E_h v_h)|_{\Omega_i} \equiv E_{i,h} v_h, \ (u_h^*, \sigma_h^*)|_{\Omega_i} = (u_{i,h}^*, \sigma_{i,h}^*)$$

and

$$S_h \lambda_{h,\Gamma} = b_h. \tag{4.1}$$

Here $b_h \in (\Phi_h)'$ is an element in the space dual to Φ_h. For S_h, we have

Lemma 4.1. *The discrete Steklov-Poincaré operator* S_h *is symmetric and positive definite, and the solution to (3.2) is given by* $(u_h, \sigma_h) = E_h\lambda_{h,\Gamma} + (u_h^*, \sigma_h^*)$.

According to Lemma 4.1 and considering $(u^*_{i,h}, \sigma^*_{i,h})$, b_h can be obtained in parallel. So, if $\lambda_{h,\Gamma}$ can be solved, then we can easily obtain (u_h, σ_h), which is the solution of equation (3.2).

Algorithm 2 is an iterative algorithm to solve (4.1). Define the discrete Steklov-Poincaré operator $S_{i,h}$ by

$$(S_{i,h}\lambda_h, v_h)_\Omega \equiv a_i[E_{i,h}\lambda_h, E_{i,h}v_h], \quad \forall \lambda_h, v_h \in \Phi_h.$$

As S_h, $S_{i,h}$ is symmetric and positive definite. Assume that R_i is the matrix form of the trace average operator α_i(see Section 3). P is the iterative matrix of the iterative scheme from λ^n to λ^{n+1} in Algorithm 2. It can be verified that

$$P = I - \rho(\sum_{i=1}^{m} R_i S_{i,h}^{-1} R_i^T) S_h.$$

It can be shown that Q has the form

$$Q^{-1} = \rho \sum_{i=1}^{m} R_i S_{i,h}^{-1} R_i^T \tag{4.2}$$

which is the preconditioner contained in Algorithm 3.

(4.2) shows that Q is symmetric and positive definite, and that Q satisfies property 1).

In order to show that Q satisfies property 2), we prove the following theorem.

Theorem 4.1. *There exists a positive constant c, which is independent of H and h, such that*

$$\rho \le \frac{(S_h Q^{-1} S_h \lambda_h, \lambda_h)_\Omega}{(S_h \lambda_h, \lambda_h)_\Omega} \le c\rho(1 + H^{-2})(1 + \ln^2(H/h)), \quad \forall \lambda_h \in \Phi_h.$$

Equivalently, the condition number of matrix $Q^{-1}S_h$ satisfies

$$\kappa(Q^{-1}S_h) \le O((1 + H^{-2})(1 + \ln^2(H/h))).$$

It is well-known that the conjugate gradient method(CG) is an efficient technique for symmetric positive definite system, but its convergence rate depends on the condition number of the coefficient matrix. So, Theorem 4.2 shows that the matrix Q satisfies property 2). The above analysis shows that Q is an efficient preconditioner of S_h. It will converge very fast when system (4.1) is solved by preconditioned CG.

5 Conclusions

In the above sections, we have studied the generalized Stokes problem using multi-subdomain decomposition methods, and proposed and analyzed Algorithm 1, Algorithm 2, and Algorithm 3, and estimated the condition number of the preconditioned system. The derived result shows that these algorithms are nearly optimal and are efficient for solving the generalized Stokes problem.

Our algorithms are also suitable for domains with $\Gamma^+_\infty \neq \emptyset$ and the domain studied in [2] and [3], with boundary condition (2.2).

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