Schwarz Domain Decomposition Method for Multidimensional and Nonlinear Evolution Equations: Subdomains Have Overlaps

Qiming He and Lishan Kang

1 Introduction

Domain decomposition method is one of the most important approaches for solving partial differential equations numerically. This method has many merits: the size of a problem can be compressed by domain decomposition; various computing schemes can be used exploiting different geometric forms of the subdomains or the different features of the problems; parallel computing can be implemented on different subdomains, etc. Most present work focused on domain decomposition method for elliptic equations. Less effort has been devoloped to parabolic systems. In this article, we study the following initial-boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = Lu + f(t,x,u), \quad t \in (0,T),\ x \in \Omega, \\ u = 0, \quad t \in [0,T],\ x \in \partial\Omega, \\ u|_{t=0} = \varphi(x), \quad x \in \overline{\Omega}, \end{cases} \tag{1}$$

where $\Omega \subset R^m$ is a bounded convex region, with a boundary $\partial\Omega$ which is piecewise smooth, $u = (u_1, \cdots, u_J)^T$, $f = (f_1, \cdots, f_J)^T$, $\varphi = (\varphi_1, \cdots, \varphi_J)^T$, and the linear

[1] This work was supported in part by National Natural Science Foundation of China and National 863 High Technology Project of China.
[2] Institute of Software Engineering, Wuhan University, Wuhan 430072, People's Republic of China.

Domain Decomposition Methods in Sciences and Engineering, edited by R. Glowinski *et al.*

differential operator $L=(L_1,\cdots,L_J)^T$ is given by

$$L_j u=\sum_{k,l}^{m} D_k(a_{jkl}(x)D_l u_j)-c_j(x)u_j,\ D_k=\frac{\partial}{\partial x_k},\ j=1,2,\cdots,J,\ k=1,2,\cdots,m.$$

We suppose that the following conditions hold:
H_1) $a_{j,k,l}(x)$ is continuously differential on $\overline{\Omega}$, $c_j(x)\geq 0$ and is bounded on $\overline{\Omega}$, and

$$\sum_{k,l}^{m} a_{j,k,l}(x)\xi_k\xi_l\geq\gamma|\xi|^2,\ \xi\in R^m,\ \gamma>0.$$

H_2) f is continuously differentiable for $(t,x,u)\in[0,T]\times\overline{\Omega}\times R^J$, and

$$\left|\frac{\partial f}{\partial t}\right|\leq b(1+|u|),\quad \left|\frac{\partial f}{\partial u}\right|\leq b_1.$$

H_3) $\varphi(x)\in W_2^{(2)}(\Omega)\cap \dot{W}_2^{(1)}(\Omega)$.
Suppose that Ω is partitioned into $\Omega_1,\cdots,\Omega_I$, and that $\Omega_i\cap\Omega_{i_1}\neq\emptyset$ $(i\neq i_1)$, i.e. the subdomains overlap. $\partial\Omega_i$ stands for the boundary of Ω_i.

Let $t_n=n\Delta t(n=0,1,2,\cdots,N)$, i.e. $N\Delta t=T$. Let $u(t_n,x)=u^n$, and discretize (1) with respect to t. We have

$$\begin{cases}\dfrac{u^{n+1}-u^n}{\Delta t}=Lu^{n+1}+f^{n+1}, & x\in\Omega,\\ u^{n+1}|_{\partial\Omega}=0, & \\ u^0=\varphi(x), & \end{cases}\tag{2}$$

which can be written as

$$\begin{cases}(I+\Delta tL)u^{n+1}=u^n+\Delta tf^{n+1}, & x\in\Omega,\\ u^{n+1}|_{\partial\Omega}=0, & \end{cases}\tag{3}$$

where $f^{n+1}=f(t_{n+1},x,u^{n+1})$.

The paper is organized as follows: Section 2 studies the existence for generalized solution of nonlinear elliptic equations (3); Section 3 discusses the Picard-Schwarz algorithm(named P-S algorithm in brief) for problem (3); the convergence of P-S algorithm is investigated in Section 4 and a priori estimates for semidiscrete solutions are presented in Section 5. The convergence of semidiscrete solutions is discussed in Section 6.

2 Existence of a generalized solution of the boundary value problem

We will now demonstrate that problem (3) has a generalized solution. The following lemma will be used in the proofs. Consider the transform:

$$z=T(y,\lambda),$$

where $y, z \in X$, X is a Banach space, $\lambda \in [a, b]$ is a parameter, and $[a, b]$ is a bounded interval. We assume that

(i) $T(y, \lambda)$ is defined for $y \in X, \lambda \in [a, b]$. For fixed $\lambda \in [a, b]$, $T(y, \lambda)$ is continuous in X. For y in a bounded set of X, $T(y, \lambda)$ is uniformly continuous with respect to λ.

(ii) For any fixed λ, $T(y, \lambda)$ is a compact transform.

(iii) There exists a constant $M > 0$ such that all possible solutions of $z - T(z, \lambda) = 0$, $\lambda \in [a, b]$, satisfy $\|z\| \leq M$.

(iv) Equation $z - T(z, a) = 0$ has a unique solution.

Then we have

Lemma 1(Leary-Schauder fixed point theorem). *Under the conditions* (i)$-$(iv), *transform* $z = T(y, \lambda)$ *has a fixed point for* $\lambda \in [a, b]$, *i.e. equation* $z - T(z, \lambda) = 0$ *has solution for* $\lambda \in [a, b]$. *In particular, the solution of* $z - T(z, b) = 0$ *exists.*

Detailed proofs can be found in [2].

Now we start to prove that problem (3) has a generalized solution. For this purpose, we first consider the following boundary value problem containing the parameter $0 \leq \lambda \leq 1$:

$$\begin{cases} (I + \Delta t L)u^{n+1} = u^n + \lambda \Delta t f^{n+1}, & x \in \Omega, \\ u^{n+1}|_{\partial\Omega} = 0. \end{cases} \tag{4}$$

Taking $v \in \dot{W}_2^{(0)}(\Omega)$, and denoting $f(t_{n+1}, x, v) = h(x, v)$, we find using (Theorem 8. 3) that it is easy to see that the linear problem

$$\begin{cases} (I + \Delta t L)u^{n+1} = u^n + h(x, v), & x \in \Omega, \\ u^{n+1}|_{\partial\Omega} = 0 \end{cases} \tag{5}$$

has a unique generalized solution $u^{n+1} \in \dot{W}_2^{(1)}(\Omega)$. Thus we can define a transform from $v \in \dot{W}_2^{(0)}(\Omega)$ to $u^{n+1} \in \dot{W}_2^{(1)}(\Omega)$: $u^{n+1} = T(v, \lambda)$. What remains is to test this transform satisfies all conditions of Lemma 1; We these arguments. Hence a fixed point exists for $\lambda \in [0, 1]$, especially for $\lambda = 1$, problem (3) has a generalized solution $u^{n+1} \in \dot{W}_2^{(1)}(\Omega)$. Therefore we have

Theorem 1. *If conditions* $H_1) - H_3)$ *are satisfied, then the nonlinear boundary value problem (3) has a generalized solution* $u^{n+1} \in \dot{W}_2^{(1)}(\Omega)$.

3 P-S algorithm for the boundary value problem

The algorithm proposed in this section is a combination of a Picard iteration and Schwarz iteration, which is named the P-S algorithm. The procedure can be described as follows:

I. *Constructing the Picard Iteration.* Taking $u_0^{n+1} \in \dot{W}_2^{(1)}(\Omega)$, and making successive approximation for fixed n:

$$\begin{cases} (I + \Delta t L)u_{q+1}^{(n+1)} = u^n + \Delta t f(t_{n+1}, x, u_q^{n+1}), & x \in \Omega, \\ u_{q+1}^{(n+1)}|_{\partial\Omega} = 0, \quad q = 0, 1, 2, \cdots. \end{cases} \tag{6}$$

II. *Constructing Schwarz Iteration.* For every fixed q, (6) is a linear boundary value problem. Denoting by $u_{q+1}^{n+1} = v$, its Schwarz iteration can be specified as follows:

Step 1. Taking an initial guess $v^{(0)} \in \dot{W}_2^{(1)}(\Omega)$, $p := 0$.

Step 2. Solving the boundary value problems on subdomains Ω_i in parallel:

$$\begin{cases} (I + \Delta t L)v_i^{(p+1)} = u^n + \Delta t f(t_{n+1}, x, u_q^{n+1}), & x \in \Omega_i, \\ v_i^{(p+1)} = v^{(p)}, \quad x \in \partial\Omega_i, \ p = 0, 1, 2, \cdots, \ i = 1, 2, \cdots, I. \end{cases} \tag{7}$$

Step 3. Continuing $v_i^{(p+1)}$ to Ω, i.e. setting

$$\hat{v}_i^{(p+1)} = \begin{cases} v_i^{(p+1)}, & x \in \Omega_i, \\ v_p, & x \in \Omega \setminus \Omega_i, \end{cases}$$

and then taking the mean value:

$$v^{(p+1)} = \frac{1}{I} \sum_{i=1}^{I} \hat{v}_i^{(p+1)},$$

and then setting $p := p + 1$ and going to Step 2. From I the Picard sequence (named the P-sequence) has been formed, and the Schwarz sequence (named the S-sequence) is given by II. In the next section we will prove that both sequences converge.

4 Convergence of the P-S iteration

The convergence of S-sequence can be obtained by using a method similar to that of [4]. Let $V_i = \dot{W}_2^{(1)}(\Omega_i)$, $V = \dot{W}_2^{(1)}(\Omega)$. Then, we have

Theorem 2. *Suppose conditions $H_1) - H_3)$ are satisfied. If $V = \overline{\sum_{i=1}^{I} V_i}$, then the S-sequence $\{v^{(p)}\}$ converges to u_{q+1}^{n+1} in the $\dot{W}_2^{(1)}(\Omega)$ norm; if $V = \sum_{i=1}^{I} V_i$, then $\{v^{(p)}\}$ converges geometrically to u_{q+1}^{n+1} in the $\dot{W}_2^{(1)}(\Omega)$ norm.*

We now prove that P-sequence $\{u_{q+1}^{n+1}\}$ converges to u^{n+1}. As $q \to \infty$, we denote the error $e_q = u^{n+1} - u_q^{n+1}$. From (3) and (6), we have

$$\begin{cases} (I + \Delta t L)e_{q+1} = \Delta t(f(t_{n+1}, x, u^{n+1}) - f(t_{n+1}, x, u_q^{n+1})), \\ e_{q+1}|_{\partial\Omega} = 0. \end{cases} \tag{8}$$

It follows that

$$(e_{q+1}, v) + \Delta t a(e_{q+1}, v) = \Delta t(f(t_{n+1}, x, u^{n+1}) - f(t_{n+1}, x, u_q^{n+1}), v), \ \forall v \in \dot{W}_2^{(1)}(\Omega),$$

where

$$a(u, v) = \sum_{j=1}^{J} \int_\Omega \{\sum_{k,l}^{m} a_{jkl} D_k u_j D_l v_j + c_j u_j v_j\} dx.$$

Taking $v = e_{q+1}$ and making use of $H_2)$, we can get

$$\|e_{q+1}\|_2^2 + \Delta t\|e_{q+1}\|_a^2 \le b_1\Delta t\|e_q\|_2\|e_{q+1}\|_2, \tag{9}$$

where $\|\cdot\|_a^2 = a(\cdot,\cdot)$. It follows that

$$\|e_{q+1}\|_2 \le b_1\Delta t\|e_q\|_2.$$

By induction, we obtain

$$\|e_q\|_2 \le (b_1\Delta t)^q\|e_0\|_2. \tag{10}$$

Then from (9) it follows that

$$\|e_{q+1}\|_2^2 + \Delta t\|e_{q+1}\|_a^2 \le \left(\frac{b_1\Delta t}{2}\right)^2 \|e_q\|_2^2 + \|e_{q+1}\|_2^2).$$

Hence,

$$\begin{aligned}\|e_{q+1}\|_a^2 &\le \frac{b_1^2}{4}\Delta t\|e_q\|_2^2\\ &\le \frac{b_1}{2}\|e_q\|_2^2\end{aligned}$$

for Δt small enough.

Notice that $\|\cdot\|_a^2 \ge \gamma\|\cdot\|^2_{\dot{W}_2^{(1)}(\Omega)}$. It follows that

$$\|e_{q+1}\|_{\dot{W}_2^{(1)}(\Omega)} \le \sqrt{\frac{b_1}{2\gamma}}\|e_q\|_2 \le \sqrt{\frac{b_1}{2\gamma}}(b_1\Delta t)^q\|e_0\|_2. \tag{11}$$

(10) and (11) are the error estimates for the Picard iteration. It shows that the P-sequence is geometrically convergent in $\dot{W}_2^{(1)}(\Omega)$ norm.

Theorem 3. *If the conditions of Theorem 2 are satisfied, then the error estimates (10) and (11) hold, i.e. the P-sequence converges geometrically to the generalized solution of problem (3) in $\dot{W}_2^{(1)}(\Omega)$ norm.*

5 A priori estimates for the semidiscrete solution

In this section we will prove a priori estimates for the generalized solution of problem (3) in preparation for the convergence proofs in the next section.

First, from (3), we have

$$(u^{n+1}, v) + \Delta t a(u^{n+1}, v) = (u^n, v) + \Delta t(f^{n+1}, v),\ \forall v \in \dot{W}_2^{(1)}(\Omega). \tag{12}$$

Taking $v = u^{n+1}$ and applying conditions $H_1) - H_3)$, and a computation, we can obtain

$$\|u^n\|_2 \le K_0. \tag{13}$$

Second, from (3), we have

$$(u^{n+1},v)+\Delta t a(u^{n+1},v)=(u^n,v)+\Delta t(f(t_{n+1},x,u^{n+1}),v),$$
$$(u^n,v)+\Delta t a(u^n,v)=(u^{n-1},v)+\Delta t(f(t_n,x,u^n),v).$$

Let $w^{n+1}=\frac{u^{n+1}-u^n}{\Delta t}$. It follows that

$$(w^{n+1}-w^n,v)+\Delta t a(w^{n+1},v)=(f(t_{n+1},x,u^{n+1})-f(t_n,x,u^n),v), \quad (14)$$

$n=1,2,\cdots,N-1$. We define

$$w^0=\frac{\varphi-u^{-1}}{\Delta t}=L\varphi+f(0,x,\varphi).$$

Then (14) is also definited for $n=0$. Taking $v=w^{n+1}$ in (14), and by a computation, we can get

$$\|w^n\|_2=\left\|\frac{u^n-u^{n-1}}{\Delta t}\right\|_2\le K_1, \quad n=1,2,\cdots,N. \quad (15)$$

Finally, we estimate $\|D_k u^n\|_2$. From (3), we have

$$(w^{n+1},v)+a(u^{n+1},v)=(f^{n+1},v).$$

Taking $v=u^{n+1}$ and noting estimates (13) and (15), it can be established that

$$\|u^n\|_{\dot{W}_2^{(1)}(\Omega)}\le K_2, \quad (16)$$

where the constants K_0-K_2 are independent of Δt.

Theorem 4. *If the conditions of Theorem 3 are satisfied, then estimates (13), (15) and (16) hold.*

6 Convergence of semidiscrete solution

The following lemma will be used in our proofs of the rest of our theorems.

Lemma 2. *Under the conditions of Theorem 3, let*

$$W_{\Delta t}(t,x)=\frac{t-t_n}{\Delta t}u^{n+1}+\frac{t_{n+1}-t}{\Delta t}u^n,\ t_n\le t\le t_{n+1},\ x\in\overline{\Omega}, \quad (17)$$

and let $Q_T=\{0\le t\le T, x\in\overline{\Omega}\}$. Then $\{W_{\Delta t}\}$ is strongly compact in $L_2(Q_T)$, $\{D_k W_{\Delta t}\}$ and and $\{D_t W_{\Delta t}\}$ are weakly compact in $L_2(Q_T)$.

The proof can be found in [5-6].

Now we prove that the semidiscrete solution $\{u^n\}$ is convergent (as $\Delta t\to 0$). Its limit is the generalized solution of problem (1). Let $Q_T^n=\{t_n<t\le t_{n+1}, x\in\overline{\Omega}\}$. Then, for $(t,x)\in Q_T^n$, we define

$$u_{\Delta t}(t,x)=u^{n+1},\ \overline{u}_{k\Delta t}(t,x)=D_k u^{n+1}\ (k=1,2,\cdots,m),\ \tilde{u}_{\Delta t}(t,x)=\frac{u^{n+1}-u^n}{\Delta t}.$$

Thus these functions are all definited on Q_T. From Lemma 2, the following estimate holds

$$\sup_{0\le t\le T}\|u_{\Delta t}\|_2+\sup_{0\le t\le T}\|\overline{u}_{k\Delta t}\|_2+\sup_{0\le t\le T}\|\tilde{u}_{\Delta t}\|_2\le K_3,\quad k=1,2,\cdots,m, \tag{18}$$

where the constant K_3 is independent of Δt. Thus we can choose a subsequence $\{\Delta t_i\}$ such that $\{u_{\Delta t_i}\}$, $\{\overline{u}_{k\Delta t_i}\}$, $\{\tilde{u}_{\Delta t_i}\}$ converge weakly to $u(t,x)$, $\overline{D}_k(t,x)$, $\tilde{u}(t,x)$, respectively. Similar to [1], we can prove

$$\sup_{0\le t\le T}\|u\|_2+\sup_{0\le t\le T}\|\overline{u}_k\|_2+\sup_{0\le t\le T}\|\tilde{u}\|_2\le K_3,\quad k=1,2,\cdots,m,$$

and

$$\overline{u}_k(t,x)=D_ku,\quad \tilde{u}(t,x)=u_t.$$

Then using the definition of $u_{\Delta t}(t,x)$ and expression (17) of $W_{\Delta t}(t,x)$, it is not difficult to prove that $\{u_{\Delta t}\}$ is also strongly compact in $L_2(Q_T)$. Moreover, it and $\{u_{\Delta t_i}\}$ and $\{W_{\Delta t_i}\}$ converge strongly to the limit $u(t,x)$.

Now we prove that limit $u(t,x)$ is just the generalized solution of problem (1). Let $v(t,x)$ be a smooth function with compact support in Q_T. Then from (2) it is not difficult to show that

$$\int_0^T\{(\tilde{u}_{\Delta t},v_{\Delta t})+a(u_{\Delta t},v_{\Delta t})-(f_{\Delta t},v_{\Delta t})\}dt=0, \tag{19}$$

where $f_{\Delta t}=f(t_{n+1},x,u_{\Delta t})$. We choose a subsequence $\{\Delta t_i\}$ ($i\to\infty$ as $\Delta t_i\to 0$) and take the limit for

$$\int_0^T\{(\tilde{u}_{\Delta t_i},v_{\Delta t_i})+a(u_{\Delta t_i},v_{\Delta t_i})-(f_{\Delta t_i},v_{\Delta t_i})\}dt=0.$$

we obtain

$$\int_0^T\{(u_t,v)+a(u,v)-(f,v)\}dt=0. \tag{20}$$

This shows that $u(t,x)$ is the generalized solution of problem (1) and it is not difficult to prove that it is unique. Hence $\{u^n\}$ converges strongly to the unique generalized solution $u(t,x)\in L_\infty((0,T),\dot{W}_2^{(1)}(\Omega))\cap W_\infty^{(1)}((0,T),L_2(\Omega))$ of (1).

Theorem 5. *If the conditions of Theorem 4 are satisfied, then $\{u^n\}$ converges strongly to the generalized solution $u\in L_\infty((0,T),\dot{W}_2^{(1)}(\Omega))\cap W_\infty^{(1)}((0,T),L_2(\Omega))$ of problem (1).*

We have carried our some numerical experiments using our proposed algorithm and our numerical results demonstrate that our method is effective for solving the nonlinear evolution equations. Because space limitations, we do not discuss these computations in detail in this paper.

REFERENCES

[1] Zhou Yulin(1991) *Applications of discrete function analysis to the finite deference method,* Inter. Acad. Publishers.

[2] Friedman A. (1964) *Partial differential equations of parabolic type,* Prentice-Hall.

[3] Gilbarg D. and Trudinger, N. (1977) *Elliptic partial differential equations of second order,* Springer-Verlag.

[4] Lu Tao, Shih T. M. and Liem C. B. (1992) *Domain decomposition method —new numerical techniques for solving PDE,* Science Press, Beijing (in Chinese).

[5] He Qiming and Kang Lishan(1994) The domain decomposition method for multidimensional and large scale nonlinear systems—case of subdomains overlapping, *Parallel Algorithms and Applications,* Vol. 4, No. 1-2, pp. 77-90.

[6] He Qiming and Evans D. J. (1995) Symmetric domain decomposition method for large scale and nonlinear evolution systems, *Parallel Algorithms and Applications,* Vol. 8, No. 1-2.