Convergence Analysis of Parallel Domain Decomposition Algorithm for Navier-Stokes Equations

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1 Introduction

In the paper [1], we developed a Schwarz's domain decomposition algorithm and a convergence analysis for the stationary incompressible Navier-Stokes problem. But this is a serial algorithm. In this paper, we discuss a class of parallel algorithms. We consider the stationary Navier-Stokes equations with the Dirichlet boundary condition:

$$\left\{\begin{aligned} -\nu\Delta u + \sum_{j=1}^{N} u_j \frac{\partial u}{\partial x_j} + \mathrm{grad} p = f & \\ \mathrm{div} u = 0 \quad \text{in } \Omega & \\ u|_{\partial\Omega} = 0, & \end{aligned}\right. \tag{1}$$

where Ω is a bounded domain of $R^N (N = 2, 3)$ with a Lipschitz-continuous boundary $\partial\Omega$. The vector $u = \{u_i\}_{i=1}^N$ is the velocity of the fluid, $\nu > 0$ is its kinematic viscosity (assumed to be constant), p its pressure and the vector $f = \{f_i\}_{i=1}^N \in [H^{-1}(\Omega)]^N$ the density of the body forces per unit mass. We introduce the following spaces:

$$X = [H_0^1(\Omega)]^N, \quad V = \{v | v \in X; \mathrm{div} v = 0, \mathrm{in} \quad \Omega\},$$

$$M = L_0^2(\Omega) = \{q | q \in L^2(\Omega); \int_\Omega q dx = 0\}.$$

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The weak formulation of (1) is

$$(P)\qquad \begin{cases} \text{Find } u \in V \quad \text{such that} \\ a_0(u,v) + a_1(u;u,v) = < f, v >, \forall v \in V, \end{cases}$$

or

$$(Q)\qquad \begin{cases} \text{Find } \{u,p\} \in X \times M \quad \text{such that} \\ a_0(u,v) + a_1(u;u,v) - b(p,v) = < f, v >, \forall v \in X \\ b(q,u) = 0, \quad \forall q \in M, \end{cases}$$

where

$$a_0(u,v) = \nu(\text{grad}u, \text{grad}v), \quad a_1(u;v,w) = \sum_{i,j=1}^{N} \int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_i dx,$$

$$b(p,v) = (p, \text{div}v), \quad < f, v >= \int_\Omega f \cdot v dx, \quad \forall u, v, w \in X.$$

Problem (P) is equivalent to problem (Q) and there exists at least one solution[2,3].

2 The Parallel Algorithm

Assume that Ω is split into m subdomains Ω_j:

$$\Omega = \bigcup_{j=1}^{m} \Omega_j \quad (m \geq 2),$$

satisfying

$$\begin{aligned} &\exists \xi_j \in C_0^{+\infty}(\Omega_j), 0 \leq \xi_j \leq 1, \text{in} \quad \Omega_j; \xi_j = 0, \text{in} \quad \Omega \backslash \Omega_j, \\ &\text{such that} \quad \sum_{j=1}^{m} \xi_j = 1, \text{in} \quad \Omega. \end{aligned} \tag{2}$$

Later on, we denote by $r(x)$ the number of subdomains to which the point x belongs. Let

$$V_j = \{u \in [H_0^1(\Omega_j)]^N; \text{div}u = 0, \text{in} \quad \Omega_j\} \quad (j = 1, 2, \cdots, m).$$

and consider V_j as a closed subspace of V by extending its elements to $\Omega \backslash \Omega_j$ by 0. Then we have the following simple result.

Lemma 1 ([1]). *Under the condition (2), we have*

$$V = V_1 + V_2 + \cdots + V_m,$$

and, for all $v \in V$, there exist $v_j \in V_j$ such that

$$v = \sum_{j=1}^{m} v_j \quad \textit{and} \quad \max_{1 \leq j \leq m} \| v_i \|_1 \leq C_0 \| v \|_1,$$

where C_0 is some positive constant.

The algorithm is designed as follows:

Step 0. Choose an initial value $u^0 \in V$.

Step 1. For $j = 1, 2, \cdots, m$ and $n \geq 0$, solve in parallel the following subproblems:

$$\begin{cases} \text{Find } u_j^{n+1} \in u^n + V_j \text{ and } p_j^{n+1} \in L_0^2(\Omega_j) \text{ such that} \\ a_0(u_j^{n+1}, v)_{\Omega j} + a_1(u_j^{n+1}; u_j^{n+1}, v)_{\Omega_j} - b(p_j^{n+1}, v)_{\Omega_j} \\ \quad =< f, v >_{\Omega_j}, \quad \forall v \in [H_0^1(\Omega_j)]^N, \\ b(q, u_j^{n+1})_{\Omega_j} = 0, \qquad \forall q \in L_0^2(\Omega_j), \\ p_j^{n+1} = 0, \qquad \text{in } \Omega \backslash \Omega_j. \end{cases} \tag{3}$$

Step 2. Choose $\theta_j \in (0,1)$ such that $\sum\limits_{j=1}^{m} \theta_j = 1$ and for $n \geq 0$ set

$$u^{n+1} = \sum_{j=1}^{m} \theta_j u_j^{n+1}, \quad p^{n+1}(x) = \frac{1}{r(x)} \sum_{j=1}^{m} p_j^{n+1}. \tag{4}$$

Set $n = n + 1$, go to Step 1.

Here the notations are given by:

$$a_0(u,v)_{\Omega_j} = \nu \int\limits_{\Omega_j} \text{grad}u \cdot \text{grad}v dx, \quad a_1(u; v, w)_{\Omega_j} = \sum_{i,j=1}^{N} \int\limits_{\Omega_j} u_j \frac{\partial v_i}{\partial x_j} w_i dx,$$

$$b(p,v)_{\Omega_j} = \int\limits_{\Omega_j} p\text{div}v dx, \quad < f, v >_{\Omega_j} = \int\limits_{\Omega_j} f \cdot v dx.$$

3 The Convergence

Let H be a Hilbert space, F a differentiable mapping from H into H' (the dual space of H), $DF(\cdot)$ its derivative, and let $u \in H$ be a solution of the equation $F(u) = 0$. We say that u is a nonsingular solution if there exists a constant $\gamma_0 > 0$ such that

$$\| DF(u) \cdot v \|_* \geq \gamma_0 \| v \|_H \quad \forall v \in H.$$

For the Navier-Stokes problem, we define a C^2-mapping $F(\cdot) : V \to V'$ as follows:

$$< F(u), v >= a_0(u,v) + a_1(u; u, v) - < f, v > \quad \forall u, v \in V.$$

Clearly,$F(\cdot)$ is infinitely differentiable in V and its derivative $DF(u) \in \mathcal{L}(V; V')$ is given by:

$$< DF(u)v, w >= a_0(v,w) + a_1(u; v, w) + a_1(v; u, w).$$

As a consequence, problem (P) can be rewritten as:

$$\begin{cases} \text{Find} \quad u \in V \quad \text{such that} \\ < F(u), v >= 0, \quad \forall v \in V. \end{cases} \tag{5}$$

In addition, we introduce the abstract Stokes operator: $A \in \mathcal{L}(V; V')$ defined by

$$< Au, v >= a_0(u, v), \quad \forall u, v \in V.$$

Obviously[3] A is symmetric, V-elliptic, and there exists $A^{-1} \in \mathcal{L}(V'; V)$. Furthermore, the mapping $f \in V' \to \| f \|_{V'} = < A^{-1}f, f >^{\frac{1}{2}}$ is a norm on V' that is equivalent to the dual norm[3].

We define the functional

$$J(v) = \frac{1}{2} \| F(v) \|_{V'}^2$$

Then problem (5) is equivalent to:

$$\begin{cases} \text{Find} \quad u \in V \quad \text{such that} \\ J(u) = \inf_{v \in V} J(v). \end{cases} \tag{6}$$

Lemma 2. *Let u^* be a nonsingular solution of Navier-Stokes problem (1). Then the functional*

$$J(v) = \frac{1}{2} < A^{-1}F(v), F(v) >$$

is strictly convex in a neighborhood of u^* and weakly lower semicontinuous. This means that there exist two constants, $\rho > 0$ and $\alpha > 0$, such that

$$D^2 J(v) \cdot (w, w) \geq \alpha \| w \|_1^2, \quad \forall v \in S(u^*; \rho), \forall w \in V. \tag{7}$$

Here

$$S(u^*; \rho) = \{v \in V; \| v - u^* \|_1 \leq \rho\}.$$

Proof. The strict convexity of $J(v)$ in a neighborhood of a nonsingular solution u^* can be found in [3].

Now, we will prove the weak lower semicontinuity. To do this, let $\{v_i\}$ be a weakly convergent sequence in V. Assume that $v_i \to v_*$ weakly in V. We then have

$$\lim_{i \to \infty} < F(v_i), v >=< F(v_*), v > \quad \forall v \in V.$$

(see chapter 9 in [4]. Let $A^{-1}F(v_*) = g_*, A^{-1}F(v_i) = g_i$, i.e.

$$\begin{aligned} & Ag_* = F(v_*), Ag_i = F(v_i), \\ & < A(g_* - g_i), v >=< F(v_*) - F(v_i), v >= a_0(g_* - g_i, v). \end{aligned}$$

Hence

$$\lim_{i \to \infty} a_0(g_* - g_i, v) = \lim_{i \to \infty} < F(v_*) - F(v_*), v > - < F(v_*) - F(v_*), v >= 0 \quad \forall v \in V.$$

This implies $g_i \to g_*$ weakly in V.

Also, with $Ag = F(v)$

$$J(v) =< A^{-1}F(v), F(v) >=< g, Ag >= a_0(g, g),$$

since

$$a_0(g_i - g_*, g_i - g_*) \geq 0, \quad a_0(g_i, g_i) \geq 2a_0(g_i, g_*) - a_0(g_*, g_*),$$
$$\lim_{i\to\infty} \inf a_0(g_i, g_i) \geq a_0(g_*, g_*).$$

It follows that

$$\lim_{i\to\infty} \inf J(v_i) \geq J(v).$$

$J(v_i)$ is weakly lower semicontinuous. □

Let u^0 be an initial value in $S(u^*, \rho)$, and let $R = J(u^0)$. It is clear that $D = \{v \in V; J(v) \leq R\} \cap S(u^*; \rho)$ is not empty because of $u^0 \in S(u^*, \rho)$, and functional $J(v)$ is strictly convex in D. Its derivative $DJ(v)$ is uniformly continuous in D[3]. Therefore there exist constants $\alpha, C > 0$ such that (see (7))

$$J(v) - J(u) - < DJ(u), v - u > \geq \alpha \parallel v - u \parallel_1^2, \quad \forall u, v \in D \tag{8),}$$

and

$$\parallel DJ(v) - DJ(u) \parallel_{V'} \leq C \parallel v - u \parallel_1 \quad \forall u, v \in D. \tag{9}$$

Now, we consider a local minimization problem:

$$\begin{cases} \text{Find} \quad u \in D \qquad \text{such that} \\ J(u) = \inf\limits_{v\in D} J(v) \end{cases} \tag{10}$$

Theorem 1. *There exists a unique solution u^* of (10) which satisfies (6).*

Proof By using lemma 2, $J(v)$ is weakly lower semicontinuous in V and D is a closed subset in $V, J(v) \geq 0$. Then the generalized Weierstrass theorem shows that (10) has a unique solution u^*. □

Clearly, the domain decomposition algorithm for (10) consists of

$$(P_n) \begin{cases} \text{Find} \quad u_j^{n+1} \in \{u^n + V_j\} \cap D \quad \text{such that} \\ a_0(u_j^{n+1}, v)_{\Omega_j} + a_1(u_j^{n+1}; u_j^{n+1}, v)_{\Omega_j} = < f, v >_{\Omega_j} \quad \forall v \in V_j. \end{cases}$$

(P_n) is equivalent to

$$\begin{cases} v_j^{n+1} = \operatorname*{arginf}\limits_{\substack{v \in V_j \\ u^n + v \in D}} \; J(u^n + v) \quad \text{in } \Omega_j \\ v_j^{n+1} = 0, \qquad\qquad \text{in} \quad \Omega\backslash\Omega_j \\ u_j^{n+1} = u^n + v_j^{n+1} \qquad \text{in} \quad \Omega. \end{cases} \tag{11}$$

Similarly, (11) has a unique solution sequence $\{u_j^n\}$.

Theorem 2. *Assume that $\{u^*, p^*\}$ is an isolated solution of the N-S Eqs. Then the sequence $\{u^n, p^n\}$, defined by (11) and (4), converges strongly in $X \times M$ to $\{u^*, p^*\}$.*

Proof It follows from (11) that

$$J(u_j^{n+1}) \leq J(u^n), \quad u_j^{n+1} \in D \quad (n \geq 0, j = 1, 2, \cdots, m).$$

Since $J(v)$ is convex in D, we have

$$J(u^{n+1}) = J(\sum_{j=1}^{m} \theta_j u_j^{n+1}) \leq \sum_{j=1}^{m} \theta_j J(u_j^{n+1}) \leq J(u^n), \quad (n \geq 0). \tag{12}$$

Therefore there exists a $q \in R^1$ such that $J(u^n) \to q$, as $n \to +\infty$. Thus, $\{u^n\} \subset D$. Furthermore, (11) yields

$$\begin{cases} < DJ(u_j^{n+1}), v_j >= 0 \quad \forall v_j \in V_j \\ u_j^{n+1} - u^n \in V_j \quad (j = 1, 2, \cdots, m; n \geq 0). \end{cases} \tag{13}$$

Combining (11) with (7), we deduce

$$J(u^n) - J(u_j^{n+1}) \geq \alpha \parallel u^n - u_j^{n+1} \parallel_1^2 .$$

For $\theta_j \in (0, 1)$ satisfying $\sum_{j=1}^{m} \theta_j = 1$, we have

$$\alpha \sum_{j=1}^{m} \theta_j \parallel u^n - u_j^{n+1} \parallel_1^2 \leq J(u^n) - \sum_{j=1}^{m} \theta_j J(u_j^{n+1})$$
$$\leq J(u^n) - J(u^{n+1}) \to 0, \text{as} \quad n \to +\infty.$$

Therefore,

$$\parallel u^n - u_j^{n+1} \parallel_1 \to 0 \quad (j = 1, 2, \cdots, m), \quad \text{as} \quad n \to +\infty.$$

By using Lemma 1, for each $v \in V$, there exist $v_j \in V_j$ such that

$$v = \sum_{j=1}^{m} v_j \quad \text{and} \quad \max_{1 \leq j \leq m} \parallel v_j \parallel_1 \leq C_0 \parallel v \parallel_1 .$$

Combining this with (7) and (8), we deduce

$$| < DJ(u^n), v > | = | \sum_{j=1}^{m} < DJ(u^n), v_j > | = | \sum_{j=1}^{m} < DJ(u^n) - DJ(u_j^{n+1}), v_j > |$$

$$\leq C_0 \sum_{j=1}^{m} \parallel DJ(u^n) - DJ(u_j^{n+1}) \parallel_{V'} \parallel v \parallel_1 \leq C_0 C \sum_{j=1}^{m} \parallel u^n - u_j^{n+1} \parallel_1 \parallel v \parallel_1 .$$

Hence

$$\parallel DJ(u^n) \parallel_{V'} \leq C_0 C \sum_{j=1}^{m} \parallel u^n - u_j^{n+1} \parallel_1 \to 0, \quad \text{as} \quad n \to +\infty.$$

Since u^* is a solution of problem (6), we find

$$J(u^*) \leq J(u^n) \quad (n \geq 0).$$

Using again (7), we get

$$\alpha \parallel u^n - u^* \parallel_1^2 \leq J(u^*) - J(u^n) + < DJ(u^n), u^n - u^* > .$$

$$\leq < DJ(u^n), u^n - u^* > \leq \| DJ(u^n) \|_{V'} \| u^n - u^* \|_1 .$$

Therefore

$$\| u^n - u^* \|_1 \leq \frac{1}{\alpha} \| DJ(u^n) \|_{V'} \to 0, \quad \text{as} \quad n \to +\infty.$$

Thus, the sequence $\{u^n\}$ converges strongly in V to the isolated solution u^* of problem (6).

Next, we consider the convergence of the pressure sequence $\{p^n\}$.

Let the pair $\{u^*, p^*\}$ be a solution of Problem (Q). We know that

$$\| u^n - u^* \|_1 \to 0, \| u_j^{n+1} - u^* \|_1 \to 0, \quad \text{as} \quad n \to +\infty (j = 1, 2, \cdots, m).$$

Moreover, using (Q), we have for any v_j in $[H_0^1(\Omega_j)]^N$ with $v_j = 0$ in $\Omega \backslash \Omega_j$:

$$b(p_j^{n+1} - p, v_j) = a_0(u_j^{n+1} - u^*, v_j) + a_1(u_j^{n+1}; u_j^{n+1}, v_j)$$

and

$$-a_1(u^*; u^*, v_j) \to 0^{[3]}, \quad \text{as} \quad n \to +\infty.$$

Similarly, if $\Omega_i \cap \Omega_j \neq \Phi$, then we have for any v_{ij} in $[C_0^\infty(\Omega_i \cap \Omega_j)]^N$ with $v_{ij} = 0$ in $\Omega \backslash (\Omega_i \cap \Omega_j)$

$$b(p_i^{n+1} - p_j^{n+1}, v_{ij}) \to 0, \quad \text{as} \quad n \to +\infty.$$

Since $C_0^{+\infty}(\Omega_i \cap \Omega_j)$ is dense in $L^2(\Omega_i \cap \Omega_j)$, we get

$$b(p_i^{n+1} - p_j^{n+1}, v)_{\Omega_j \cap \Omega_j} \to 0, \forall v \in [H^1(\Omega_i \cap \Omega_j)]^N, \text{as} \quad n \to +\infty.$$

Hence, we get for all $v_j \in [H_0^1(\Omega_j)]^N$, with $v_j = 0$ in $\Omega \backslash \Omega_j$:

$$b(p^{n+1} - p^*, v_j)_{\Omega_j} = b(p_j^{n+1} - p^*; v_j)_{\Omega_j} + b(\frac{1}{r(x)} \sum_{k=1}^{M} p_k^{n+1} - p_j^{n+1}, v_j)_{\Omega_j}$$

$$= b(p_j^{n+1} - p^*, v_j)_{\Omega_j} + \frac{1}{r(x)} \sum_{k=1}^{m} b(p_k^{n+1} - p_j^{n+1}, v_j)_{\Omega_j \cap \Omega_k} \to 0, \quad \text{as} \quad n \to +\infty.$$

Set $v_j = \xi_j v$, for all $v \in [H_0^1(\Omega)]^N$. We assert that

$$b(p^{n+1} - p^*, v) = \sum_{j=1}^{m} b(p^{n+1} - p^*, v_j)_{\Omega_j} \to 0, \quad \text{as} \quad n \to +\infty,$$

i.e. $\forall v \in [H_0^1(\Omega)]^N$, $(p^{n+1} - p^*, divv) \to 0$ as $n \to +\infty$. Since div maps $[H_0^1(\Omega)]^N$ onto $L_0^2(\Omega)$([3]), we obtain

$$p^n \to p^* \quad \text{weakly in } M \quad \text{as } n \to +\infty.$$

The inf-sup condition [3] yields

$$p^n \to p^* \quad \text{strongly in} \quad M \quad \text{as } n \to +\infty.$$

Hence, Theorem 2 is valid. □

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