Domain Decomposition Methods to Penalty Combinations for Singularity Problem

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Abstract

An embedding technique is presented in this paper to implement penalty combinations into parallel by the iterative substructuring method. Penalty combinations using a penalty integral are much simpler than the direct constraints to match different admissible functions. In combinations, the Ritz-Galerkin method is used in the singular subdomains Ω^+ where exist solution singularities, and the singular particular solutions are chosen to be admissible functions so that only a few of them are needed to cope well with the difference grids in the finite difference methods used in the rest of the solution domain. Consequently, while applying the iterative substructuring methods, we may attain the singular domain Ω^+ to the interface of two subregions in the domain decomposition methods, and regard Ω^+ as a "fat" interface or called an interface "zone". The matrix contribution resulting from Ω^+ can be included into the preconditioner matrix due to a few of unknown coefficients to be sought. The new embedding technique may reform penalty combination easily into parallel computing by the existing, iterative substructuring methods. Such an technique has been proven to be effective, by a brief analysis and numerical experiments of Motz's problem given in this paper.

1.1 Penalty Combinations of the Ritz-Galerkin and Finite Difference Methods

Parallel penalty combinations are presented in this paper for solving singularity problems, to join the combined methods with the domain decomposition methods (simply written DDMs). In this paper the penalty combinations to combine the finite difference method with the Ritz-Galerkin method, to regain the superconvergence rates $O(h^{2-\delta})$, where $\delta(>0)$ is an arbitrarily small number. Usually, the singular domain should be chosen as one subregion, where the mixed Neumann-Dirichlet problem in

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DDMs is solved. However, since only a few unknown coefficients are used with the total number L, having an asymptotic relation:

$$L = O(|lnh|), \tag{1.1}$$

where h is the maximal mesh spacing of difference grids, the singular subdomain may be regarded as a "fat" interface or called an interface "zone" in the renovated domain decomposition methods. Indeed, a few more unknown coefficients in the preconditioner matrices will not cause much effort in computation in the inverse preconditioner matrices.

Consider the Poisson equation with the Dirichlet boundary condition

$$-\Delta u = -(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) = f(x,y), \quad (x,y) \in \Omega, \tag{1.2}$$

$$u = 0, \quad (x,y) \in \partial\Omega, \tag{1.3}$$

where Ω is a polygon domain, and the function f is smooth enough. For simplicity, we assume that only one singular point of the solution $u(x,y)$ exists in $\bar{\Omega} = \Omega \cup \partial\Omega$. Let Ω be divided by a piecewise straight line Γ_0 into two subdomains Ω^+ and Ω^-. The Ritz-Galerkin method is used in $\Omega^+ \cup \partial\Omega$ including the singular point, and the finite difference method is used in Ω^-. The subdomain Ω^- is again split by quasiuniform difference grids into small rectangles $\square_{ij}$ and triangles $\triangle_{ij}$. Denote $u_{i,j} = u(x_i, y_i)$, as the solution on the difference nodes (i,j). The finite difference method can be regarded as a special kind of finite element method if using piecewise bilinear and linear interpolatory functions and if using special integration rules to approximate the integrals involved (also see [Li(86,95)]).

Assume that the solution u in Ω can be spanned by $u = \Psi_0 + \sum_{i=1}^{\infty} D_i \Psi_i$, where D_i are the expansion coefficients, and $\Psi_i (i = 1, 2, \cdots \infty)$ are complete and linearly independent basis functions that are known. Then the admissible functions in combinations of the RG-FDMs are written as:

$$v = \begin{cases} v^- &= v_1, \text{ in } \Omega^-, \\ v^+ &= \Psi_0 + \sum_{i=1}^{L} \tilde{D}_i \Psi_i \text{ . in } \Omega^+, \end{cases} \tag{1.4}$$

where $\tilde{D}_i$ are unknown coefficients to be sought. If the particular solutions of (1.2) and (1.3) are chosen as Ψ_i, their total number will greatly decrease for a given accuracy of solutions.

Since there occurs discontinuity of solutions on Γ_0, i.e., $v^+ \neq v^-$ on Γ_0, we define another space

$$H = \{v \in L^2(\Omega), v \in H^1(\Omega^-), \text{ and } v \in H^1(\Omega^+)\}, \tag{1.5}$$

where $H^1(\Omega)$ is the Sobolev space. Let $V_h (\subseteq H)$ denote a finite dimensional collection of the function v in (1.4) satisfying (1.3). We will couple v^+ and v^- by the following integrals

$$\hat{D}(u,v) = \frac{P_c}{h^\sigma} \hat{\int}_{\Gamma_0} (u^+ - u^-)(v^+ - v^-) d\ell, \tag{1.6}$$

where $P_c(> 0)$ the penalty constant, and $\sigma(\geq 0)$ the penalty power. Hence, we lead to the penalty combination of RG-FDMs, to seek a solution $u_h \in V_h$ such that

$$a_h(u_h, v) = f_h(u, v), \quad \forall v \in V_h \quad where \tag{1.7}$$

$$a_h(u,v) = \widehat{\iint_{\Omega^-}} \nabla u \nabla v ds + \iint_{\Omega^+} \nabla u \nabla v ds + \frac{P_c}{h^\sigma} \hat{\int}_{\Gamma_0} (u^+ - u^-)(v^+ - v^-) d\ell \tag{1.8}$$

In virtual computation, the integrals $D(u,v)$ on Γ_0 are evaluated approximately by integration rules. When $\sigma \geq 4$, the superconvergence rates of error norms $\overline{||\epsilon||}_h = O(h^{2-\delta})$, and $\overline{||\epsilon||}_h = O(h^{3/2})$ in the solution derivatives can be achieved by penalty combinations for the quasiuniform difference grids: $S_1 = \cup_{ij} \square_{ij}$, *and* $S_1 = (\cup_{ij} \square_{ij}) \cup (\cup_{ij} \triangle_{ij})$ respectively.

1.2 Domain Decomposition Methods to Penalty Combination

We will still follow the typical approaches of Bjorstad and Widlund (86), Bramble, Pasciak and Schatz (86), Manteuffel and Parter (90) and Morz (89). Let Ω be also split into Ω_I and Ω_{II} by Γ in DDMs:

$$\bar{\Omega} = \partial\Omega \cup \Omega_I \cup \Omega_{II} \cup \Gamma, \tag{1.9}$$

where Ω_I, Ω_{II} and Γ are two subregions and their interface. To distinguish partitions between DDMs and combinations, we call *subregions* Ω_I, Ω_{II} and *interface* Γ in DDMs, but *subdomains* Ω^+, Ω^- and *common boundary* Γ_0 in combinations respectively, where

$$\bar{\Omega} = \partial\Omega \cup \Omega^+ \cup \Omega^- \cup \Gamma_0. \tag{1.10}$$

The key treatments of the DDMs applied to the combinations are how to embed Ω^+, Ω^- and Γ_0 into Ω_I, Ω_{II} and Γ.

It should be noted from Eq. (1.1) that the number of the coefficients D_l is much less than that of the variables $v_{ij} \in \{\Gamma \cap \partial\Omega^+\}$. Hence we may simply attain thė singular domain Ω^+ to interface Γ, and call the interface "zone" Γ^* instead,

$$\Gamma^* = \Gamma \cup \Gamma_0 \cup \Omega^+. \tag{1.11}$$

So we have

$$\Omega^+ \subset \Gamma^* \ and\, \Gamma \subset \Gamma^*. \tag{1.12}$$

The subdomain Ω^- is also divided into Ω_I and Ω_{II}.

Below let us describe more in details the renovated DDMs of combinations. Denote the variables by x_1, x_2 and x_3 in Ω_I, Ω_{II} and their interface Γ^*, respectively. Then the unknown expansion coefficients $\{D_i\}$ are included in x_3. Denote $\tilde{x} = (x_1, x_2, x_3)^T$. The equations (1.7) are then reduced to a linear algebraic equation system

$$A\tilde{x} = \tilde{b}, \tag{1.13}$$

where $\tilde{b} = (b_1, b_2, b_3)^T$, and

$$A = \begin{bmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{bmatrix}. \tag{1.14}$$

In (1.14), A_{11}, A_{22} and A_{33} are the coefficient matrices from Ω_I, Ω_{II} and Γ^*. Also A_{13} denotes the relation matrices between x_1 and x_3, and A_{23} denotes those between x_2 and x_3.

By using the block Gaussian elimination, we can again reduce (1.13) to a system of algebraic equations of x_3 only:

$$S\,x_3 = \hat{b}_3\,, \tag{1.15}$$

where the Schur matrix

$$S = A_{33} - A_{13}^T A_{11}^{-1} A_{13} - A_{23}^T A_{22}^{-1} A_{23}, \tag{1.16}$$

and the vector $\hat{b}_3 = b_3 - A_{13}^T A_{11}^{-1} b_1 - A_{23}^T A_{22}^{-1} b_2$. We may solve x_3 from (1.15) by the preconditioner conjugate gradient method (PCGM) (see [Golub and Loan (89)]). Then x_1 in Ω_I and x_2 in Ω_{II} can be solved in parallel. For the penalty combinations, we choose the preconditioning matrix

$$S_1 = A_{33}^{(1)} - A_{13}^T A_{11}^{-1} A_{13}, \tag{1.17}$$

where $A_{33}^{(1)}$ excludes the matrix components of the contribution of Ω_{II} to A_{33}, but includes that of Ω^+ when using (1.12) in the embedding techniques.

The penalty integral

$$\hat{D}(v,v) = \frac{P_c}{h^\sigma} \int_{\Gamma_0} (v^+ - v^-)^2 \, d\ell\,, \tag{1.18}$$

plays a role to match the admissible functions v^+ and v^-. The variables v_{ij} on Γ_0 and the coefficients $\{D_\ell\}$ are also included in x_3. We then have [Bjorstad and Widlund (86)]

$$\begin{bmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{bmatrix} \begin{bmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{13}^T & 0 & A_{33}^{(1)} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ SS_1^{-1} y \end{bmatrix}, \tag{1.19}$$

where y are the sequential solutions of x_3 on the interface boundary (or "zone"). The operation involving the inverse matrix can be carried out by two steps [Bramble, Pasciak and Schatz (86)].

Step I. Solve the Dirichlet problem on $\Omega_{II} \cup \Gamma$:

$$A_{22}x_2 + A_{23}y = 0. \tag{1.20}$$

Step II. Solve the Neumann-Dirichlet problem on $\Omega_I \cup \Gamma$

$$\begin{bmatrix} A_{11} & A_{13} \\ A_{13}^T & A_{33}^{(1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}. \tag{1.21}$$

In fact, the preconditioner matrix S_1 in (1.17) is the Schur matrix of the above equation.

1.3 Analysis of Preconditioning Condition Number

Let the matrices A and B be positive definitive and symmetric, and B be the preconditioner matrix. Then the preconditioning number is defined by the following ratios of the maximal and minimal eigenvalues of $B^{-1}A$.

$$Con.\ (B^{-1}A) = Con.\ (A\,B^{-1}) = \lambda_{\max}(B^{-1}A)/\lambda_{\min}(B^{-1}A)\,, \tag{1.22}$$

$$\lambda_{\max}(B^{-1}A) = \max_{\|x\|\neq 0} \frac{A(x,x)}{B(x,x)}\,, \quad \lambda_{\min}(B^{-1}A) = \min_{\|x\|\neq 0} \frac{A(x,x)}{B(x,x)}\,, \tag{1.23}$$

where $A(x,x) = x^T Ax$, and the Euclidean norm $\|x\| = (\sum_{i=1}^n x_i^2)^{1/2}$.

Consider the Poisson equation on Ω^- in Fig. 1.1b with the mixed Neumann-Dirichlet condition:

$$-\Delta u = f \text{ in } \Omega^-, u = 0 \text{ on } \partial\Omega \setminus \Gamma_0, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_0, \tag{1.24}$$

where Γ_0 is the common boundary of Ω^+ and Ω^-. Suppose that the finite difference method (or the finite element method) is used in Ω^-. Let A_a and B_a be the associated matrix and the preconditioner matrix in the traditional DDMs. Many reports on DDMs such as [Bjorstad and Widlund (86), Bramble et al. (86), Manteuffel and Parter (90) and Mroz (89)] have obtained the $B-D$ bounds

$$\alpha_0 B_a(x_a, x_a) \le A_a(x_a, x_a) \le \alpha_1 B_a(x_a, x_a), \tag{1.25}$$

where $\alpha_0 = 1$, $\alpha_1 = C(1 + \ell n^2 h)$, and C is a bounded constant independent of h. We can prove the following theorem.

Theorem 1. *Let the $B-D$ condition hold. Then for the DDMs of penalty combination (1.7) by the embedding technique (1.12), there exist the bounds*

$$Con.(S_1^{-1}S) \le \alpha_1/\alpha_0\,. \tag{1.26}$$

This theorem displays that the renovated DDMs in this paper will not cause deterioration of preconditioning condition numbers of the combinations, compared with the existing DDMs.

1.4 Numerical Experiments for Motz's Problem

Consider Motz's problem which solves the Laplace equation on a rectangle Ω $(-1 < x < 1, 0 < y < 1)$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ in } \Omega, \tag{1.27}$$

with the mixed Neumann-Dirichlet boundary conditions

$$u|_{x<0\wedge y=0} = 0, \qquad u|_{x=1} = 500, \quad \frac{\partial u}{\partial y}|_{y=0} = \frac{\partial u}{\partial y}|_{y=0\wedge x>0} = \frac{\partial u}{\partial x}|_{x=-1} = 0. \tag{1.28}$$

Note that there exists an angular singularity at the origin (0,0) due to the intersection point of the Neumann-Dirichlet boundary conditions.

We split Ω by Γ_0 into two subdomains Ω^+ and Ω^-, where the singular subdomain Ω^+ is a small rectangle $(-\frac{1}{2} < x < \frac{1}{2}, 0 < y < \frac{1}{2})$, and Ω^- is the rest of Ω (see Fig. 1.2a). The Ritz-Galerkin method and the finite difference method are used in Ω^+ and Ω^- respectively, with the admissible functions [Li(89,95)]

$$v = \begin{cases} v^- & \text{in } \Omega^- \\ v^+ = \sum_{\ell=0}^{L} D_\ell r^{\ell+\frac{1}{2}} \cos(\ell+\frac{1}{2})\theta\,, \end{cases} \tag{1.29}$$

where (r, θ) are the polar coordinates, and D_ℓ are the coefficients to be sought. The subdomain Ω^- is again divided into small squares as shown in Fig. 1.2b.

For DDMs of penalty combinations, we choose Fig. 1.3 as the computational model, where Ω^- is divided into three subregions

$$\Omega^- = \Omega_I^- \cup \Omega_{II,1}^- \cup \Omega_{II,2}^-, \quad \Omega_I = \Omega_I^-, \quad \Omega_{II} = \Omega_{II,1}^- \cup \Omega_{II,2}^- \tag{1.30}$$

and the interface "zone" Γ^* in (1.12), where Γ is the common boundary Ω_I and Ω_{II}. Choose the zero initial values: $D_\ell^{(0)} = 0$ and $v_{ij}^{(0)} = 0$ on $\Gamma \cup \Gamma_0$. Then the sequences $D_\ell^{(k)}$ and $v_{ij}^{(k)}$ on $\Gamma \cup \Gamma_0$ $D_\ell^{(k)}$ can be obtained from the DDMs. To measure the iterative errors we may compute the following sequential errors,

$$\Delta_{\max}^{(k)} = \max\{\max_{(i,j)\in\Gamma\cup\Gamma_0} |u_{ij}^{(k)} - u_{i,j}^{(k-1)}|, \max_\ell |D_\ell^{(k)} - D_\ell^{(k-1)}|\} \tag{1.31}$$

Table 1 presents the sequential errors (1.31) until the reduced ratios $\Delta_{maix}^{(k)}/\Delta_{max}^{(1)} \leq 10^{-6}/2$. It can be seen that only 7–8 iterations are needed in the renovated DDMs. Table 2 provides the approximate coefficients in the iterations. The above computational results show an effectiveness of parallel penalty combinations for singularity problems.

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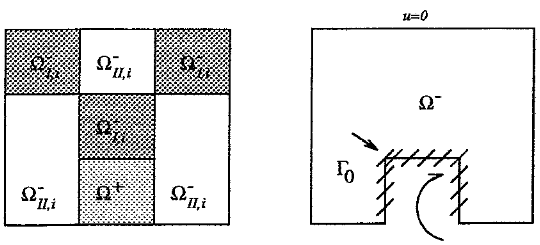
Table 1.1 Error norms in iterations the by Penalty Combination using preconditioner S_1 with $P_c = 1$ and $\sigma = 4$ by $\Delta^{(k)}_{max}/\Delta^{(k)}_{max} \le 10^{-6}/2$

Iteration number	Δ_{Max}				
	$MS = 2$	4	6	8	10
1	401.1	403.3	403.8	403.9	403.9
2	80.17	78.29	77.88	79.73	79.65
3	5.802	7.444	7.300	7.467	7.436
4	0.3281	0.2061	0.1640	0.1760	0.1917
5	$0.5317 * 10^{-1}$	0.1460	0.1282	$0.2953 * 10^{-1}$	$0.2902 * 10^{-1}$
6	$0.1186 * 10^{-1}$	$0.1309 * 10^{-2}$	$0.3653 * 10^{-2}$	$0.1517 * 10^{-2}$	$0.1433 * 10^{-2}$
7	$0.4412 * 10^{-4}$	$0.1914 * 10^{-3}$	$0.2300 * 10^{-3}$	$0.6330 * 10^{-3}$	$0.6161 * 10^{-3}$
8	/	$0.1162 * 10^{-4}$	$0.3964 * 10^{-4}$	$0.3516 * 10^{-5}$	$0.3794 * 10^{-5}$

Table 1.2 Leading coefficients by the DDMs of Penalty Combinations as $Ms = 8$,$L + 1 = 6$,$Pc = 1$ and $\sigma = 4$.

Iteration number	$\tilde{D_0}$	$\tilde{D_1}$	$\tilde{D_2}$	$\tilde{D_3}$	$\tilde{D_4}$	$\tilde{D_5}$
1	403.8726	9.8300	34.6974	−45.6345	0.5574	−16.3848
2	404.9640	89.5558	24.3091	−3.1285	5.9365	3.7177
3	401.0731	87.7007	16.8425	−8.6168	1.6305	0.7451
4	401.0927	87.6446	16.9661	−8.7928	1.7377	0.6990
5	401.0925	87.6467	16.9644	−8.7924	1.7295	0.7285
6	401.0926	87.6469	16.9636	−8.7939	1.7306	0.7295
7	401.0926	87.6470	16.9636	−8.7940	1.7312	0.7299
8	401.0926	87.6470	16.9636	−8.7940	1.7312	0.7299
Com. Coeffs.	401.0926	87.6470	16.9636	−8.7939	1.7312	0.7299
True Coeffs.	401.1625	87.6559	17.2379	−8.0712	1.4403	0.3311



(a) (b)

Figure 1.1 The partition of the DDMs when $\Omega^+ \subset \Gamma^*$

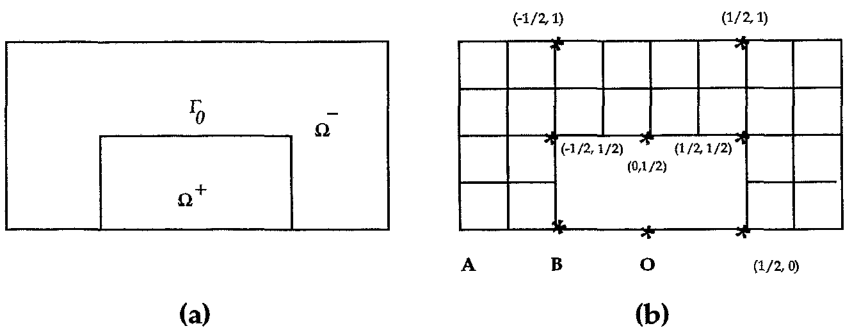


Figure 1.2 Partitions of Motz's problem in the combinations and partition in DDMs

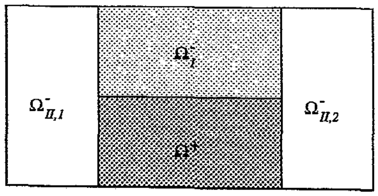


Figure 1.3 Partitions of Motz's problem in the DDMs while $\Omega^+ \subset \Omega_I$,

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