Domain Decomposition and Multilevel Techniques for Preconditioning Operators

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1 INTRODUCTION

In recent years, domain decomposition methods have been used extensively to efficiently solve boundary value problems for partial differential equations in complex shape domains [4, 13, 16]. On the other hand, multilevel techniques on hierarchical data structures also have developed into an effective tool for the construction and analysis of fast solvers [2, 5, 15, 17]. But the direct realization of multilevel techniques on a parallel computer system for the global problem in the original domain involves difficult communication problems. In this paper, we present and analyze a combination of these two approaches: domain decomposition and multilevel decomposition on hierarchical structures to design optimal preconditioning operators.

Let $\Omega \subset R^2$ be a polygon. In the domain Ω we consider the boundary value problem

$$\begin{cases} -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a_0(x)u = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \Gamma_0, \\ \frac{\partial u}{\partial n_a} + \sigma(x)u = 0, & x \in \Gamma_1. \end{cases} \tag{1.1}$$

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where

$$\frac{\partial u}{\partial n_a} = \sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial u}{\partial x_j} \cos(n, x_i)$$

is the conormal derivative, n denotes the outward normal to Γ, and Γ_0 is a union of a finite number of curvilinear segments, $\Gamma = \Gamma_0 \cup \Gamma_1, \Gamma_0 = \bar{\Gamma}_0$. Here $\bar{\Gamma}_0$ denotes the closure of Γ_0.

By $H^1(\Omega, \Gamma_0)$ we denote the subspace of the Sobolev space $H^1(\Omega)$

$$H^1(\Omega, \Gamma_0) = \left\{ v \in H^1(\Omega) \mid v(x) = 0, \;\; x \in \Gamma_0 \right\}.$$

We introduce the bilinear form $a(u, v)$ and the linear functional $l(v)$:

$$a(u, v) = \int_{\Omega} \left(\sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + a_0(x) uv \right) dx + \int_{\Gamma_1} \sigma(x) uv dx,$$

$$l(v) = \int_{\Omega} f(x) v dx.$$

Let us suppose that the operator coefficients and the right-hand side of the problem (1.1) are such that the bilinear form $a(u, v)$ is symmetric, elliptic, and continuous on $H^1(\Omega, \Gamma_0) \times H^1(\Omega, \Gamma_0)$, i.e.

$$a(u, v) = a(v, u) \quad \forall u, v \in H^1(\Omega, \Gamma_0),$$

$$\alpha_0 ||u||^2_{H^1(\Omega)} \leq a(u, u) \leq \alpha_1 ||u||^2_{H^1(\Omega)}, \quad \forall u \in H^1(\Omega, \Gamma_0)$$

and the linear functional $l(v)$ is continuous on $H^1(\Omega, \Gamma_0)$:

$$|l(u)| \leq \alpha ||u||_{H^1(\Omega)}, \quad \forall u \in H^1(\Omega, \Gamma_0).$$

The generalized solution $u \in H^1(\Omega, \Gamma_0)$ of (1.1) is, by definition, a solution to the projection problem [1]

$$u \in H^1(\Omega, \Gamma_0) \; : \; a(u, v) = l(v), \quad \forall v \in H^1(\Omega, \Gamma_0). \tag{1.2}$$

We know that under these assumptions for $a(u, v)$ and $l(v)$ there exists a unique solution of (1.2).

Let Ω be a union of n nonoverlapping subdomains Ω_i,

$$\bar{\Omega} = \bigcup_{i=1}^{n} \bar{\Omega}_i, \quad \Omega_i \cap \Omega_j = \emptyset, \;\; i \neq j,$$

where Ω_i are polygons with diameters on the order of H. Let us consider a coarse grid triangulation of Ω

$$\Omega_0^h = \bigcup_{i=1}^{n} \Omega_{0,i}^h, \quad \Omega_{0,i}^h = \bigcup_{l=1}^{M_i^{(0)}} \bar{\tau}_{i,l}^{(0)},$$

$$\text{diam}\ (\tau_i^{(0)}) = 0(H)$$

and we refine $\Omega_{0,i}^h$ several times. This results in a sequence of nested triangulations

$$\Omega_{i,0}^h, \Omega_{i,1}^h, \cdots, \Omega_{i,J}^h$$

such that

$$\bar{\Omega}_{i,k}^h = \bigcup_{l=1}^{M_i^{(k)}} \bar{\tau}_{i,l}^{(k)}, k = 0, 1, \cdots, J,$$

where the triangles $\tau_{i,l}^{(k+1)}$ are generated by subdividing triangles $\tau_{i,l}^{(k)}$ into four congruent subtriangles by connecting the midpoints of the edges.

Introduce the spaces

$$\begin{aligned} W_{i,0} \subset W_{i,1} \subset \cdots \subset W_{i,J} &= H_h(\Omega_i), \\ V_{i,0} \subset V_{i,1} \subset \cdots \subset V_{i,J} &= H_h(\Gamma_i), \\ \Gamma_i = \partial\Omega_i, \quad i = 1, 2, \cdots, n. \end{aligned} \tag{1.3}$$

Here the space $W_{i,k}$ consists of real-valued functions which are continuous on Ω and linear on the triangles in $\Omega_{i,k}^h$. The space $V_{i,k}$ is the space of traces on Γ_i of functions from $W_{i,k}$:

$$V_{i,k} = \left\{ \varphi^h \mid \varphi^h = u^h|_{\Gamma_i}, \text{ with } u^h \in W_{i,k} \right\}.$$

We define the space $H_h(\Omega)$ of real continuous functions which are linear on each triangle of Ω^h and vanish at Γ_0.

Let us consider the projection problem

$$u^h \in H_h(\Omega) : a(u^h, v^h) = l(v^h), \quad \forall v^h \in H_h(\Omega) \tag{1.4}$$

which is an approximation of the problem (1.2).

Each function $u^h \in H_h(\Omega)$ is put in correspondence with a real column vector $u \in R^N$ whose components are values of the function u^h at the corresponding nodes of the triangulation Ω^h. Then (1.4) is equivalent to the system of mesh equations

$$\begin{aligned} Au &= f, \\ (Au, v) &= a(u^h, v^h), \quad \forall u^h, v^h \in H_h(\Omega), \\ (f, v) &= l(v^h), \quad \forall v^h \in H_h(\Omega), \end{aligned} \tag{1.5}$$

where u^h and v^h are the respective interpolations of vectors u and v; (f, v) is the Euclidean scalar product inR^N.

The goal of this work is to construct a symmetric positive definite preconditioning operator B for (1.5) so as to satisfy the inequalities

$$c_1(Bu, u) \leq (Au, u) \leq c_2(bu, u) \tag{1.6}$$

where the positive constants c_1 and c_2 are independent of h and H; the multiplication of a vector by B^{-1} should be easy to implement.

Using a combination of Additive Schwarz and Fictitious Space Methods, optimal preconditioning operators have been constructed in [11, 12, 13] for the case of arbitrary (unstructured) grids. However, that construction involves explicit extension operators whose implementation for three dimensional problems is optimal from the arithmetic cost and the condition number points of view but difficult for practical realization. The main goal of this work is to construct, using the hierarchical structure (1.3), a robust optimal preconditioning operator. One of the crucial points in [11, 12, 13] and this paper is the use of non–exact solvers in subdomains and explicit extension operators.

It means, to construct optimal preconditioning operators, we can design norm preserving operators of functions given at Γ_i into Ω_i with the optimal arithmetic cost (a number of arithmetic operations should be proportional to a number degrees of freedom) and then, instead of exact solvers in subdomains, we can use any spectrally equivalent preconditioning operators. Optimal extension operators have been presented in [8, 9, 11] for unstructured grids and robust explicit extension operators on hierarchical data structures in [5, 14].

The paper is organized as follows. In Section 2, using Additive Schwarz Method, we describe general construction of a preconditioning operator with local multilevel preconditioning operators. In Section 3, we present an optimal multilevel extension of grid functions from boundaries subdomains into inside subdomains. In Section 4, we propose an optimal interface preconditioning operator at the boundaries of the subdomains which involves a multilevel decomposition and corresponding explicit extension operators at interfaces.

2 DOMAIN DECOMPOSITION – ADDITIVE SCHWARZ-METHOD

To design the preconditioning operator for system (1.5), we use the additive Schwarz–Method [7] and employ the main idea of the construction of preconditioners from [13] for the hierarchical grids. Denote by $\overset{\circ}{H}_h(\Omega_i)$ the subspace of $H_h(\Omega_i)$

$$\overset{\circ}{H}_h(\Omega_i) = \{u^h \in H_h(\Omega_i) \mid u^h(x) = 0, \quad x \in \Gamma_i\}$$

and define the local preconditioning operators B_i such that

$$B_i : \overset{\circ}{H}_h(\Omega_i) \rightarrow \overset{\circ}{H}_h(\Omega_i),$$

$$c_3\|u^h\|^2_{H^1(\Omega_i)} \leq (B_i u, u) \leq c_4\|u^h\|^2_{H^1(\Omega_i)} \quad \forall u^h \in \overset{\circ}{H}_h(\Omega_i),$$

where c_3, c_4 are independent of h and H. We hereafter use the same notation for an operator and its matrix representation. For instance, to define B_i, we can use the so-called BPX–preconditioners [3]. To do it, denote by $\{f_l^{(k)}\}$ nodal basis functions from the k–th level and define

$$B_i^{-1}u^h = \sum_{k=0}^{J} \sum_{f_l^{(k)} \in \overset{\circ}{H}_h(\Omega_i)} (u^h, f_l^{(k)})_{L_2(\Omega_i)} f_l^{(k)}. \tag{2.1}$$

Let us assume that we can define the extension operators t_i

$$t_i \,:\, V_{i,J} \longrightarrow W_{i,J}$$

such that

$$\begin{aligned} t\varphi^h &= u^h, \\ u^h(x) &= \varphi^h(x), \;\; x \in \Gamma_i, \\ \|t_i\varphi^h\|_{H^1(\Omega_i)} \leq c_5 \|\varphi^h\|_{H^{1/2}(\Gamma_i)} & \quad \forall \varphi^h \in V_{i,J}, \end{aligned} \tag{2.2}$$

with c_5 independent of h and H. Here $\|\varphi^h\|_{H^{1/2}(\Gamma_i)}$ is the norm [10] in the Sobolev space $H^{1/2}(\Gamma_i)$

$$\|\varphi^h\|^2_{H^{1/2}(\Gamma_i)} = H\int\limits_{\Gamma_i}(\varphi^h(x))^2 dx + \int\limits_{\Gamma_i}\int\limits_{\Gamma_i}\frac{(\varphi^h(x)-\varphi^h(y))^2}{|x-y|^2}dxdy.$$

Then, we can define the extension operator t

$$t:\; H_h(S) \to H_h(\Omega),$$

where $H_h(S)$ is the space of traces of functions from $H_h(\Omega)$ at S

$$S = \bigcup_{i=1}^{n} \Gamma_i$$

and for any $\varphi^h \in H_h(S)$

$$\begin{aligned} t\varphi^h &= u^h, \\ u^h(x) &= \varphi^h(x), \quad x \in S, \\ \|t\varphi^h\|_{H^1(\Omega)} &\leq c_5\|\varphi^h\|_{H^{1/2}(S)}. \end{aligned}$$

Here

$$\|\varphi^h\|^2_{H^{1/2}(S)} = \sum_{i=1}^{n}\|\varphi^h\|^2_{H^{1/2}(\Gamma_i)}.$$

The operator t_i from (2.2) is constructed in Section 3.

Let $\sum$ satisfies the following inequalities

$$c_6\|\varphi^h\|^2_{H^{1/2}(S)} \leq (\sum \varphi, \varphi) \leq c_7\|\varphi^h\|^2_{H^{1/2}(S)} \quad \forall\varphi^h \in H_h(S), \tag{2.3}$$

where c_6, c_7 are independent of h and H. Then, according to [11], we can define the preconditioning operator B as follows

$$B^{-1} = \begin{bmatrix} 0 & & & \\ & B_1^{-1} & & \\ & & \ddots & \\ & & & B_n^{-1} \end{bmatrix} + t{\sum}^{-1}t^*. \tag{2.4}$$

Here 0 is the null-matrix which corresponds to nodes of the triangulation Ω^h at S and B_i is from (2.1).

The following theorem holds

Theorem 2.1 *If operator B is obtained from (2.4), then the constants c_1, c_2 in (1.6) are independent of h and H.*

3 MULTILEVEL EXPLICIT EXTENSION OPERATORS

The main goal of this section is to construct a robust operator t_i from (2.2). In this section, we omit the subscript i.

To design the extension operator

$$t\,:\,V_J \to W_J,$$

we follow to [5, 14]. Denote by $\varphi_i^{(k)}, i = 1, 2, \cdots, N_k$, the nodal basis of V_k and denote by $\Phi_i^{(k)}$ the one-dimensional subspace spanned by this function $\varphi_i^{(k)}$. Define

$$Q_i^{(k)}\,:\,L_2(\Gamma) \to \Phi_i^{(k)}$$

the L_2 orthogonal projection from $L_2(\Gamma)$ onto $\Phi_i^{(k)}$ and denote

$$\tilde{Q}_k = \sum_{i=1}^{N_k} Q_i^{(k)}, \quad k = 0, 1, \cdots, J-1.$$

For $k = J$ we define $\tilde{Q}_J$ as the L_2 orthogonal projection from $L_2(\Gamma)$ onto V_j.
The following lemmas hold [14].

Lemma 3.1 *There exists a positive constant c_8, independent of h and H, such that for any $\varphi^h \in V_J$ we have*

$$||\varphi_0^h||^2_{H^{1/2}(\Gamma)} + \frac{1}{H}||\varphi_1^h||^2_{L_2(\Gamma)} + |\varphi_1^h|^2_{H^{1/2}(\Gamma)} \le c_8||\varphi^h||^2_{H^{1/2}(\Gamma)},$$

where

$$\varphi_0^h = \tilde{Q}_0\varphi^h, \quad \varphi_1^h = \varphi^h - \varphi^h. \tag{3.1}$$

Here

$$|\varphi^h|^2_{H^{1/2}(\Gamma)} = \int_\Gamma\int_\Gamma \frac{(\varphi^h(x) - \varphi^h(y))^2}{|x-y|^2}dxdy.$$

Lemma 3.2 *There exists a positive constant c_9, independent of h and H, such that*

$$||\varphi_0^h||^2 + \frac{1}{H}\left(||\tilde{Q}_0\varphi_1^h||^2_{L_2(\Gamma)} + \sum_{k=1}^{J} 2^k||(\tilde{Q}_k - \tilde{Q}_{k-1})\varphi_1^h||^2_{L_2(\Gamma)}\right) \le c_9||\varphi^h||^2_{H^{1/2}(\Gamma)},$$

where φ_0^h, φ_1^h are defined by (3.1).

The construction of operator t is based on the decomposition from Lemma 3.2. Denote by $x_i^{(k)}, i = 1, 2, \cdots, L_k$, the nodes of the triangulation Ω_k^h (we assume that nodes $x_i^{(k)}$ are enumerated first on Γ and then inside Ω) and define the extension operator t in the following way. For any $\varphi^h \in V_J$ set

$$\begin{aligned} \psi_0^h &= \tilde{Q}_0 \varphi^h, \\ \psi_k^h &= (\tilde{Q}_k - \tilde{Q}_{k-1})\varphi^h, \ k = 1, 2, \cdots, J. \end{aligned} \tag{3.2}$$

Then

$$\varphi^h = \psi_0^h + \psi_1^h + \cdots + \psi_J^h.$$

Define the extension $u_k^h \in W_k$ as follows

$$\begin{aligned} u_0^h(x_i^{(0)}) &= \begin{cases} \psi_0^h(x_i^{(0)}), & x_i^{(0)} \in \Gamma, \\ \bar{\psi}, & x_i^{(0)} \in \Gamma, \end{cases} \\ u_k^h(x_i^{(k)}) &= \begin{cases} \psi_k^h(x_i^{(k)}), & x_i^{(k)} \in \Gamma, \\ 0, & x_i^{(k)} \notin \Gamma, \end{cases} \end{aligned} \tag{3.3}$$

$$k = 1, 2, \cdots, J.$$

Here $\bar{\psi}$ is, for instance, the meanvalue of the function ψ_0^h on Γ, namely

$$\bar{\psi} = \frac{1}{N_0} \sum_{i=1}^{N_0} \psi_0^h(x_i^{(0)}).$$

Define

$$t\varphi^h = u^h \equiv u_0^h + u_1^h + \cdots + u_J^h. \tag{3.4}$$

Remark 3.1 *We can use the L_2 orthogonal projections from $L_2(\Gamma)$ onto V_k instead of $\tilde{Q}_k, k = 0, 1, \cdots, J-1$. But in this case the cost of the decomposition (3.2) is expensive (especially for three dimensional problems).*

Theorem 3.1 *There exists a positive constant c_{10}, independent of h and H, such that*

$$\|t\varphi^h\|_{H^1(\Omega)} \leq c_{10} \|\varphi^h\|_{H^{1/2}(\Gamma)} \quad \forall \varphi^h \in V_J.$$

Here the operator t is from (3.2)-(3.4).

Remark 3.2 *It is obvious that*

$$Q^{(k)}\varphi^h = \frac{(\varphi^h, \varphi_i^{(k)})_{L_2(\Gamma)}}{(\varphi_i^k, \varphi_i^k)_{L_2(\Gamma)}} \varphi_i^{(k)}$$

and the cost of the action of t and t^* is proportional to the number of nodes of the grid domain.

4 INTERFACE PRECONDITIONING OPERATORS

In this section, we construct an optimal interface preconditioner in the space $H_h(S)$ which satisfies (2.3). To do it, we use the idea of Additive Schwarz Method at interface S from [13]. Let S be a union of K nonoverlapping edges E_i of the triangulation Ω_0^h

$$S = \bigcup_{j=1}^{K} \bar{E}_j, \quad E_j \cap E_i = \emptyset, \quad i \neq j.$$

Split $H_h(S)$ into a vector sum of subspaces

$$H_h(S) = U_0 + U_1 + \cdots + U_k, \tag{4.1}$$

where U_0 is the coarse space which consists of continuous functions linear on the edges E_j, $j = 1, 2, \ldots, K$, and U_j, $j = 1, 2, \ldots, K$, correspond to E_j and are defined below.

Denote by

$$\mathring{U}_j = \{\varphi^h \in H_h(S) \mid \varphi^h(x) = 0,\ x \notin E_j\},$$
$$\tilde{U}_j^{(k)} = V_k\big|_{E_j}, \quad k = 0, 1, \ldots, J.$$

For any edge E_j we define the explicit extension operator τ_j

$$\tau_j \,:\, \tilde{U}_j^{(J)} \to H_h(S)$$

as follows. Denote by $\varphi_{j,i}^{(k)}$, $i = 1, 2, \ldots, I_j^{(k)}$, the nodal basis of $\tilde{U}_j^{(k)}$ (the functions $\varphi_{j,i}^{(k)}$ differ from the functions $\varphi_i^{(k)}$ from Section 3 only at the end points of E_j) and denote by $\Phi_{j,i}^{(k)}$ the one-dimensional subspace spanned by this function $\varphi_{j,i}^{(k)}$. Denote by

$$Q_{j,i}^{(k)} \,:\, L_2(E_j) \to \Phi_{j,i}^{(k)}$$

corresponding L_2 orthogonal projection. Set

$$\tilde{Q}_j^{(k)} = \sum_{i=1}^{I_j^{(k)}} Q_{j,i}^{(k)}, \quad k = 0, 1, \cdots, J-1\,,$$

and define $\tilde{Q}_J^{(k)}$ as the L_2 orthgonal oprojection from $L_2(E_j)$ onto $\tilde{U}_j^{(J)}$. Now we can define the extension operator τ_j according to (3.2)–(3.4). For any $\varphi^h \in \tilde{U}_j^{(J)}$ set

$$\begin{aligned} \psi_0^h &= \tilde{Q}_j^{(0)}\varphi^h, \\ \psi_k^h &= (\tilde{Q}_j^{(k)} - \tilde{Q}_j^{(k-1)})\varphi^h, \quad k = 1, 2, \cdots, J, \end{aligned} \tag{4.2}$$

and

$$u_k^h = \begin{cases} \psi_k^h(x_i^{(k)}), & x_i^{(k)} \in E_j \\ 0, & x_i^{(k)} \notin E_j, \quad k = 0, 1, \cdots, J, \end{cases} \tag{4.3}$$

$$\tau_j \varphi^h = u_0^h + u_1^h + \cdots + u_J^h.$$

Define

$$U_j = \mathring{U}_j + \tau_j \tilde{U}_j.$$

Then from Theorem 3.1 and [13] we have the following theorem concerning decomposition (4.1).

Theorem 4.1 *There exists a positive constant c_{11}, dependent of h and H, such that for any function $\varphi^h \in H_h(S)$ there exist $\varphi_j^h \in U_j$, $j = 0, 1, \ldots, K$, such that*

$$\varphi_0^h + \varphi_1^h + \cdots + \varphi_k^h = \varphi^h,$$

$$\|\varphi_0^h\|^2_{H^{1/2}(S)} + \|\varphi_1^h\|^2_{H^{1/2}(S)} + \cdots + \|\varphi_K^h\|^2_{H^{1/2}(S)} \leq c_{11}\|\varphi^h\|^2_{H^{1/2}(S)}$$

Let the operator $\sum_0$ generates an equivalent norm in U_0, namely

$$c_{12}\|\varphi^h\|^2_{H^{1/2}(S)} \leq (\textstyle\sum_0 \varphi, \varphi) \leq c_{13}\|\varphi^h\|^2_{H^{1/2}(S)} \quad \forall \varphi^h \in U_0, \tag{4.4}$$

where c_{12}, c_{13} independent of h and H. Define local preconditioners for U_j, $j = 1, 2, \ldots, K$. Denote by $\mathring{\sum}_j$ and $\widetilde{\sum}_j$ the BPX-like preconditioners in the spaces $\mathring{U}_j$ and $\tilde{U}_j$, respectivly

$$\mathring{\sum}_i^{-1} \varphi^h = \sum_{k=0}^{J} \sum_{\sup \varphi_{j,i}^{(k)} \subset E_j} (\varphi^h, \varphi_{j,i}^{(k)})_{L_2(E_j)} \varphi_{j,i}^{(k)} \quad \forall \varphi^h \in \mathring{U}_j,$$

$$\widetilde{\sum}_i^{-1} \varphi^h = \sum_{k=0}^{J} \sum_{\sup \varphi_{j,i}^{(k)} \cap E_j \neq \emptyset} (\varphi^h, \varphi_{j,i}^{(k)})_{L_2(E_j)} \varphi_{j,i}^{(k)} \quad \forall \varphi^h \in \tilde{U}_j.$$

Then, define the interface preconditioning operator $\sum$ in the following way

$$\sum^{-1} = \sum_0^{+} + \sum_{j=0}^{K} (\mathring{\sum}_j^{-1} + \tau_j \widetilde{\sum}_j^{-1} \tau_j^*). \tag{4.5}$$

Here $\sum_0^+$ is a pseudo-inverse of $\sum_0$ from (4.4), τ_j is obtained from (4.2), (4.3), and we extend operator $\mathring{\sum}_j^{-1}$ by zero outside E_j. The following theorem holds.

Theorem 4.2 *If operator $\sum$ is defined from (4.5) then the constants c_6, c_7 from (2.3) are independent of h and H.*

Remark 4.1 *The method suggested in this paper can be generalized evidently for three dimensional problems.*

Remark 4.2 *Using combinations of the presented techniques and techniques from [10], effective preconditoning operators for elliptic problems with discontinuous coefficients can be constructed.*

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