23

On the convergence of the generalized asynchronous multisplitting block two-stage relaxation methods for the large sparse systems of mildly nonlinear equations

D.J. Evans[1], Zhong-Zhi Bai[2]

Introduction

Consider the mildly nonlinear system

$$Ax = G(x), \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a large sparse nonsingular matrix and $G : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear mapping of certain smooth properties.

To solve this system efficiently, Bai [1] established a class of sequential two-stage iterative methods by taking into account concrete properties of the involved matrix

1 *Department of Computing, The Nottingham Trent University Burton Street, Nottingham NG1 4BU, England Email: D.J. Evans@lboro.ac.uk*
2 Supported by the National Natural Science Foundation project 19601036.
State Key Laboratory of Scientific/Engineering Computing Institute of Computational Mathematics and Scientific/Engineering Computing The Academy of Mathematics and Systems Sciences Chinese Academy of Sciences, P.O.Box 2719, Beijing 100080, P.R.China Email: bzz@lsec.cc.ac.cn
Eleventh International Conference on Domain Decomposition Methods
Editors Choi-Hong Lai, Petter E. Bjørstad, Mark Cross and Olof B. Widlund

and mapping. Then, based on the matrix multisplitting technique, Bai [2] presented efficient parallel generalizations of the above sequential two-stage iterative methods. To exploit the parallel efficiency of the high-speed multiprocessor systems as far as possible, Bai and Huang [7] further proposed the asynchronous multisplitting two-stage iterative methods. These asynchronous methods have the potential of converging much faster than their synchronous counterparts in [2], in particular, when there is load imbalance. When the matrix $A \in \mathbb{R}^{n\times n}$ is a point H-matrix and the mapping $G : \mathbb{R}^n \to \mathbb{R}^n$ is a point P-bounded mapping, the convergence of the afore-mentioned two-stage iterative methods were discussed in detail in [1], [2] and [7], respectively, under suitable conditions imposed upon the multisplittings of the matrix $A \in \mathbb{R}^{n\times n}$. Moreover, several different models of asynchronous multisplitting block two-stage relaxation methods for solving the mildly nonlinear system (1) were proposed in [11], and their global convergence properties were studied in depth for the case when the matrix $A \in \mathbb{R}^{n\times n}$ is a block H-matrix of different types and the mapping $G : \mathbb{R}^n \to \mathbb{R}^n$ is a block P-bounded mapping. All these results favoured a variety of practical sequential and parallel methods for solving the large sparse block system of mildly nonlinear equations (1), and they also have reliable theoretical guarantees for the convergence of these methods. We remark that the parallel multisplitting two-stage iterative methods for the large sparse systems of linear equations were discussed in [8], [10] and [13].

In this paper, based on the above existing results we will discuss a class of generalized asynchronous multisplitting two-stage block relaxation methods for solving the block system of mildly nonlinear equations (1). For the convenience of our statements, we assume that the matrix $A \in \mathbb{R}^{n\times n}$ is partitioned into $N \times N$ blocks $A_{\ell j} \in \mathbb{R}^{n_\ell \times n_j}$, with $\sum_{j=1}^{N} n_j = n$, i.e.,

$$A \in \mathbb{L}_n = \{A \in \mathbb{R}^{n\times n} \mid A = (A_{\ell j}), A_{\ell j} \in \mathbb{R}^{n_\ell \times n_j}, 1 \le \ell, j \le N\}$$

and A_{jj} being nonsingular for $j = 1, 2, \ldots, N$, and the vectors x and $G(x)$ are partitioned into subvectors $x_j \in \mathbb{R}^{n_j}$ and $G_j(x) \in \mathbb{R}^{n_j}$, $j = 1, 2, \ldots, N$, in a way conformally with the partition of A, i.e.,

$$x \in V_n = \{x \in \mathbb{R}^n \mid x = (x_1^T, \ldots, x_N^T)^T, x_j \in \mathbb{R}^{n_j}, 1 \le j \le N\}.$$

We will study convergence properties of these generalized asynchronous multisplitting block two-stage relaxation methods for the nonsingular matrices including block H-matrices of different types (see e.g., [3], [4], [5], [6] and [12]), and nonlinear mappings including block P-bounded mapping (see e.g., [11]). Therefore, the two-stage relaxation methods for solving the large sparse systems of mildly nonlinear equations (1) and their corresponding global convergence are dealt with in a unified manner.

The rest of this paper is organized as follows: We describe the new generalized asynchronous multisplitting block two-stage relaxation methods in section 23; Some preliminary results used in this paper is presented in section 23; In section 23, we analyze the global convergence of these generalized asynchronous multisplitting block two-stage relaxation methods.

Establishments of the new methods

Assume that the multiprocessor system consists of α processors and consider the splittings $A = B_i - C_i$, $i = 1, 2, \ldots, \alpha$, and a set of block diagonal nonnegative matrices E_i, $i = 1, 2, \ldots, \alpha$, such that $\sum_{i=1}^{\alpha} E_i = I$ (the identity matrix). Let $\tau_\ell^{(i)}(p)$ be a nonnegative integer that represents the index of the ℓ-th block element of a currently available global iterate which the i-th processor uses to compute its p-th local approximation, and $J_i(p)$ be a nonempty subset of the integer set $\mathbb{N} = \{1, 2, \ldots, N\}$ that satisfies $\ell \in J_i(p)$ if and only if the i-th processor starts its computation of the ℓ-th block element of a new iterate at the p-th step. As is customary in the descriptions and analyses of asynchronous methods, we assume that the superscripts $\tau_\ell^{(i)}(p)$ and the subsets $J_i(p)$, $p \in N_0 = \{0, 1, 2, \ldots\}$, satisfy the following conditions:

(a) $\tau_\ell^{(i)}(p) \leq p$ for all $i \in \{1, 2, \ldots, \alpha\}$, $\ell \in \mathbb{N}$ and $p \in N_0$;
(b) $\lim_{p\to\infty} \tau_\ell^{(i)}(p) = \infty$ for all $i \in \{1, 2, \ldots, \alpha\}$ and $\ell \in \mathbb{N}$; and
(c) *The set* $\{p \in N_0 \mid \ell \in J_i(p)\}$ is infinite for all $i \in \{1, 2, \ldots, \alpha\}$ and $\ell \in \mathbb{N}$.

Moreover, to describe the new asynchronous multisplitting block two-stage relaxation method we let γ_ℓ and $\omega_\ell (\neq 0)$, $\ell \in \mathbb{N}$, be two groups of relaxation parameters and for $i = 1, 2, \ldots, \alpha$, $D_i = \mathrm{Diag}(B_i)$ be the block diagonal matrices of $B_i = (B_{\ell j}^{(i)}) \in \mathbb{L}_n$, $L_i = (L_{\ell j}^{(i)}) \in \mathbb{L}_n$ be strictly block lower triangular matrices and $U_i = (U_{\ell j}^{(i)}) \in \mathbb{L}_n$ block zero-diagonal matrices, satisfying $B_i = D_i - L_i - U_i$. Denote by $E_i = Diag(E_{11}^{(i)}, E_{22}^{(i)}, \ldots, E_{NN}^{(i)})$. Then the new Generalized Asynchronous Multisplitting Block Two-stage (GAMBT) Accelerated OverRelaxation (AOR) method, or in short, the GAMBT-AOR method, can be described as follows.

THE GAMBT-AOR METHOD: Given an initial vector $x^0 \in \mathbb{R}^n$. Supposing we have approximations $x^0, x^1, \ldots, x^p$ to the solution $x^* \in \mathbb{R}^n$ of the block system of mildly nonlinear system (1). Then the next approximation $x^{p+1} = ((x_1^{p+1})^T, (x_2^{p+1})^T, \ldots, (x_N^{p+1})^T)^T$ is obtained element by element from

$$x_\ell^{p+1} = \sum_{i\in\mathbb{N}_\ell(p)} E_{\ell\ell}^{(i)} x_\ell^{p+1,i} + \sum_{i\notin\mathbb{N}_\ell(p)} E_{\ell\ell}^{(i)} x_\ell^{p}, \qquad \ell = 1, 2, \ldots, N, \tag{2}$$

where for each $\ell \in J_i(p)$, $x_\ell^{p+1,i} = x_\ell^{p,i,s_i(p)}$ is computed from the recursive formula

$$\begin{aligned} x_\ell^{p,i,k+1} \;=\; & B_{\ell\ell}^{(i)^{-1}} \Bigg\{ \gamma_\ell \sum_{j<\ell;\, j\in J_i(p)} L_{\ell j}^{(i)} x_j^{p,i,k} + (\omega_\ell - \gamma_\ell) \sum_{j<\ell;\, j\in\mathbb{N}} L_{\ell j}^{(i)} x_j^{\tau_j^{(i)}(p)} \\ & + \gamma_\ell \sum_{j<\ell;\, j\in\mathbb{N}\setminus J_i(p)} L_{\ell j}^{(i)} x_j^{\tau_j^{(i)}(p)} + \omega_\ell \sum_{j\neq\ell} U_{\ell j}^{(i)} x_j^{\tau_j^{(i)}(p)} \\ & + \omega_\ell \left(\sum_{j=1}^{N} C_{\ell j}^{(i)} x_j^{\tau_j^{(i)}(p)} + G_\ell(\ldots, x_j^{\tau_j^{(i)}(p)}, \ldots) \right) \Bigg\} \\ & + (1 - \omega_\ell) x_\ell^{\tau_\ell^{(i)}(p)}, \quad k = 0, 1, \ldots, s_i(p) - 1, \end{aligned} \tag{3}$$

with the starting point $x_\ell^{p,i,0} = x_\ell^{\tau_\ell^{(i)}(p)}$; and $\mathbb{N}_\ell(p) = \{i \mid \ell \in J_i(p), i = 1,2,\dots,\alpha\}$ for all $\ell \in \mathbb{N}$ and $p \in N_0$. Here, γ_ℓ and ω_ℓ, $\ell \in \mathbb{N}$, are relaxation and acceleration parameters, respectively.

Some important special cases of the GAMBT-AOR method are:

(a) the asynchronous multisplitting block two-stage Gauss-Seidel method, which corresponds to the case of taking the relaxation parameter pair $(\gamma_\ell, \omega_\ell)$ to be $(1,1)$;
(b) the asynchronous multisplitting block two-stage SOR method, which corresponds to the case of taking the relaxation parameter pair $(\gamma_\ell, \omega_\ell)$ to be (ω, ω);
(c) the asynchronous multisplitting block two-stage AOR method, which corresponds to the case of taking the relaxation parameter pair $(\gamma_\ell, \omega_\ell)$ to be (γ, ω); and
(d) the generalized asynchronous multisplitting block two-stage SOR method, which corresponds to the case of taking the relaxation parameter pair $(\gamma_\ell, \omega_\ell)$ to be $(\omega_\ell, \omega_\ell)$.

Define $x_\ell^{p,i,k} = x_\ell^{\tau_\ell^{(i)}(p)}$ for all $\ell \in \mathbb{N}\backslash J_i(p)$, $i \in \{1,2,\dots,\alpha\}$ and $k \in \{0,1,\dots,s_i(p)-1\}$. If we introduce the projection operators $\mathcal{P}_\ell : V_n \to \mathbb{R}^{n_\ell}(\ell = 1,2,\dots,N)$ by $\mathcal{P}_\ell(x) = x_\ell$ for any $x \in V_n$, and the matrices

$$\left\{\begin{array}{lcl} M_i(\gamma_\ell,\omega_\ell) &=& \frac{1}{\omega_\ell}(D_i - \gamma_\ell L_i), \\ N_i(\gamma_\ell,\omega_\ell) &=& \frac{1}{\omega_\ell}\left((1-\omega_\ell)D_i + (\omega_\ell - \gamma_\ell)L_i + \omega_\ell U_i\right), \end{array}\right. \qquad i = 1,2,\dots,\alpha, \quad \ell \in \mathbb{N},$$

then after direct calculations, the GAMBT-AOR method can be briefly expressed in the matrix-vector form:

$$x_\ell^{p+1} = \sum_{i\in\mathbb{N}_\ell(p)} E_{\ell\ell}^{(i)} \mathcal{P}_\ell(y^{p+1,i,\ell}) + \sum_{i\notin\mathbb{N}_\ell(p)} E_{\ell\ell}^{(i)} x_\ell^p, \quad \ell = 1,2,\dots,N, \tag{4}$$

where

$$\begin{array}{rcl} y^{p+1,i,\ell} &=& \left(M_i(\gamma_\ell,\omega_\ell)^{-1}N_i(\gamma_\ell,\omega_\ell)\right)^{s_i(p)} x^{\tau^{(i)}(p)} \\ && + \sum_{k=0}^{s_i(p)-1} \left(M_i(\gamma_\ell,\omega_\ell)^{-1}N_i(\gamma_\ell,\omega_\ell)\right)^k M_i(\gamma_\ell,\omega_\ell)^{-1}(C_i x^{\tau^{(i)}(p)} + G(x^{\tau^{(i)}(p)})) \end{array} \tag{5}$$

and

$$x^{\tau^{(i)}(p)} = (x_1^{\tau_1^{(i)}(p)}, x_2^{\tau_2^{(i)}(p)}, \dots, x_N^{\tau_N^{(i)}(p)})^T. \tag{6}$$

Preliminaries

The orderings and the point absolute values in $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ are defined according to elements, respectively. A nonsingular matrix $A \in \mathbb{R}^{n\times n}$ is called an M-matrix if it has non-positive off-diagonal entries and it is monotone (i.e., $A^{-1} \geq 0$). Let $D_A = \mathrm{diag}(A)$ be the point diagonal matrix of A and $B_A = D_A - A$. Then A is an M-matrix if and only if D_A is positive diagonal, B_A is nonnegative and $\rho(D_A^{-1}B_A) < 1$.

Define

$$\mathbb{L}_{n,I} = \{A = (A_{\ell j}) \in \mathbb{L}_n \mid A_{\ell\ell} \in \mathbb{R}^{n_\ell \times n_\ell} \text{ nonsingular}, \ell = 1, \ldots, N\},$$

which is a subset of $\mathbb{L}_n$. For a matrix $A \in \mathbb{L}_n$, let $D(A) = \text{Diag}(A_{11}, \ldots, A_{NN})$, i.e., its block-diagonal part. Thus, $A \in \mathbb{L}_{n,I}$ if and only if $A \in \mathbb{L}_n$ and $D(A)$ is nonsingular.

For any matrix $A \in \mathbb{L}_{n,I}$ we define its type-I and type-II comparison matrices $\langle A \rangle = (\langle A \rangle_{\ell j}) \in \mathbb{R}^{N\times N}$ and $\langle\langle A \rangle\rangle = (\langle\langle A \rangle\rangle_{\ell j}) \in \mathbb{R}^{N\times N}$ as $\langle A \rangle_{\ell\ell} = \|A_{\ell\ell}^{-1}\|^{-1}$, $\langle A \rangle_{\ell j} = -\|A_{\ell j}\|$, $\ell \neq j$, and $\langle\langle A \rangle\rangle_{\ell\ell} = 1$, $\langle\langle A \rangle\rangle_{\ell j} = -\|A_{\ell\ell}^{-1} A_{\ell j}\|$, $\ell \neq j$, $\ell, j = 1, 2, \ldots, N$, respectively. For $A \in \mathbb{L}_n$, we define its block absolute value by $[A] = (\|A_{\ell j}\|) \in \mathbb{R}^{N\times N}$. The definition for a vector $v \in V_n$ is analogous. Here $\|\cdot\|$ is any consistent matrix norm satisfying $\|I\| = 1$. Note that these concepts directly reduce to the corresponding point ones in the literature when they are understood in the pointwise sense. We refer to [3] for the detailed properties about these block absolute values.

$A \in \mathbb{L}_{n,I}$ is said to be a Type-I (Type-II) block H-matrix if $\langle A \rangle$ ($\langle\langle A \rangle\rangle$) is an M-matrix in $\mathbb{R}^{N\times N}$ ($A \in H_B^I$ ($A \in H_B^{II}$)). It follows that $H_B^I \subset H_B^{II}$ with the inclusion being strict. For $A \in \mathbb{L}_{n,I}$, a splitting $A = M - N$ is called an H_B^I-compatible (H_B^{II}-compatible) splitting if

$$\langle A \rangle = \langle M \rangle - [N] \quad (\langle\langle A \rangle\rangle = \langle\langle M \rangle\rangle - [D(M)^{-1}N]).$$

A mapping $G : \mathbb{R}^n \to \mathbb{R}^n$ is called block P-bounded if there exists a nonnegative matrix $P \in \mathbb{R}^{N\times N}$ such that

$$[G(x) - G(y)] \leq P[x - y]$$

holds for all $x, y \in \mathbb{R}^n$. We refer to [3] for the concrete properties about the H_B^I and H_B^{II} matrix classes.

The following two lemmas are very useful for proving the global convergence of the GAMBT-AOR method.

Lemma 1 [11] *Let $A \in \mathbb{R}^{n\times n}$ be nonsingular. Then the block system of weakly nonlinear equations (1) has a unique solution provided either of the following two conditions holds:*
(a) $A \in H_B^I$, G is block P-bounded, and $\rho(\langle A \rangle^{-1} P) < 1$.
(b) $A \in H_B^{II}$, G is block P-bounded, and $\rho(\langle\langle A \rangle\rangle^{-1}[D(A)^{-1}]P) < 1$.

Lemma 2 [9] *Let $\{H_p^{(i)}\}_{p\in N_0} (i = 1, 2, \ldots, \alpha)$ be sequences of nonnegative matrices in $\mathbb{R}^{N\times N}$, $\widetilde{E}_i = diag(\widetilde{e}_1^{(i)}, \widetilde{e}_2^{(i)}, \ldots, \widetilde{e}_N^{(i)}) (i = 1, 2, \ldots, \alpha)$ be nonnegative diagonal matrices in $\mathbb{R}^{N\times N}$ satisfying $\sum_{i=1}^{\alpha} \widetilde{E}_i \leq I$, and $\{\epsilon^p\}_{p\in N_0}$ be sequence in $\mathbb{R}^N$ defined by*

$$\epsilon_\ell^{p+1} = \sum_{i\in\mathbb{N}_\ell(p)} \widetilde{e}_\ell^{(i)} \epsilon_\ell^{\tau_\ell^{(i)}(p)} + \sum_{i\notin\mathbb{N}_\ell(p)} \widetilde{e}_\ell^{(i)} \epsilon_\ell^p, \qquad \ell = 1, 2, \ldots, N, \quad p = 0, 1, 2, \ldots$$

with $\{\mathbb{N}_\ell(p)\}_{p\in N_0} (\ell = 1, 2, \ldots, N)$ and $\{\tau_\ell^{(i)}(p)\}_{p\in N_0} (\ell = 1, 2, \ldots, N, i = 1, 2, \ldots, \alpha)$ being described as in section 23. Then $\lim_{p\to\infty} \epsilon^p = 0$ holds for any $\epsilon^0 \in \mathbb{R}^N$, provided there exist a constant $\theta \in [0, 1)$ and a positive vector $v \in \mathbb{R}^N$ such that $H_p^{(i)} v \leq \theta v$ $(i = 1, 2, \ldots, \alpha, p \in N_0)$.

The global convergence theory

In this section, we will prove the global convergence of the GAMBT-AOR method when the matrix $A \in \mathbb{L}_n$ is a block H-matrix of different types and when the mapping $G : \mathbb{R}^n \to \mathbb{R}^n$ is a block P-bounded mapping.

Theorem 1 *Let $A \in H_B^I$ $(H_B^{II}) \subset \mathbb{L}_{n,I}$. Let the splittings $A = B_i - C_i$, $i = 1, 2, \dots, \alpha$, be H_B^I-compatible (H_B^{II}-compatible) such that $D(B_i) = D(A)$, $i = 1, 2, \dots, \alpha$, the splittings $B_i = D_i - L_i - U_i$, $i = 1, 2, \dots, \alpha$, satisfy $\langle B_i \rangle = \langle D_i \rangle - [L_i] - [U_i] (i = 1, 2, \dots, \alpha)$ for Type-I case (and $\langle\langle B_i \rangle\rangle = I - [D_i^{-1} L_i] - [D_i^{-1} U_i] (i = 1, 2, \dots, \alpha)$ for Type-II case), and the weighting matrices E_i, $i = 1, 2, \dots, \alpha$, satisfy $\sum_{i=1}^{\alpha} [E_i] \leq I$. Assume further that $G : \mathbb{R}^n \to \mathbb{R}^n$ is a block P-bounded mapping such that*

$$\rho(\langle A \rangle^{-1} P) < 1 \quad (\rho(\langle\langle A \rangle\rangle^{-1} [D(A)^{-1}] P) < 1).$$

Then, the GAMBT-AOR method converges to the unique solution of the system of mildly nonlinear equations (1), for any initial vector $x^0 \in \mathbb{R}^n$ and any sequence of numbers of inner iterations $s_i(p) \geq 1$, $i = 1, 2, \dots, \alpha$, $p \in N_0$, provided the relaxation parameters γ_ℓ and ω_ℓ, $\ell \in \mathbb{N}$, satisfy $0 \leq \gamma_\ell \leq \omega_\ell$ $(\ell \in \mathbb{N})$ and

$$0 < \omega_\ell < \frac{2}{1 + \rho(D_{\langle A \rangle}^{-1} (B_{\langle A \rangle} + P))} \quad \left(0 < \omega_\ell < \frac{2}{1 + \rho(B_{\langle\langle A \rangle\rangle} + [D(A)^{-1}] P)} \right), \quad \ell \in \mathbb{N}.$$

Proof: By Lemma 1 we know that there exists a unique vector $x^* \in \mathbb{R}^n$ such that $Ax^* = G(x^*)$. Thus, if we let $\varepsilon^p = x^p - x^*$ be the error at the p-th iteration of the GAMBT-AOR method, then according to (4)-(6), $\{\varepsilon^p\}_{p \in N_0}$ satisfies

$$\varepsilon_\ell^{p+1} = \sum_{i \in \mathbb{N}_\ell(p)} E_{\ell\ell}^{(i)} \mathcal{P}_\ell (y^{p+1,i,\ell} - x^*) + \sum_{i \notin \mathbb{N}_\ell(p)} E_{\ell\ell}^{(i)} \varepsilon_\ell^p \tag{7}$$

for all $\ell \in \mathbb{N}$, where

$$\begin{aligned} y^{p+1,i,\ell} - x^* &= \left(M_i(\gamma_\ell, \omega_\ell)^{-1} N_i(\gamma_\ell, \omega_\ell) \right)^{s_i(p)} \varepsilon^{\tau^{(i)}(p)} \\ &+ \sum_{k=0}^{s_i(p)-1} \left(M_i(\gamma_\ell, \omega_\ell)^{-1} N_i(\gamma_\ell, \omega_\ell) \right)^k M_i(\gamma_\ell, \omega_\ell)^{-1} \\ &\times \left(C_i \varepsilon^{\tau^{(i)}(p)} + G(x^{\tau^{(i)}(p)}) - G(x^*) \right). \end{aligned} \tag{8}$$

Let us denote by $\widetilde{P} = [D(A)^{-1}] P$, and for $i = 1, 2, \dots, \alpha$,

$$\widetilde{B}_i = D(A)^{-1} B_i, \quad \widetilde{C}_i = D(A)^{-1} C_i, \quad \widetilde{L}_i = D(A)^{-1} L_i, \quad \widetilde{U}_i = D(A)^{-1} U_i$$

and

$$\begin{cases} \widetilde{M}_i(\gamma_\ell, \omega_\ell) &= \frac{1}{\omega_\ell} (I - \gamma_\ell \widetilde{L}_i), \\ \widetilde{N}_i(\gamma_\ell, \omega_\ell) &= \frac{1}{\omega_\ell} ((1 - \omega_\ell) I + (\omega_\ell - \gamma_\ell) \widetilde{L}_i + \omega_\ell \widetilde{U}_i), \end{cases} \quad \ell \in \mathbb{N}.$$

Then we have

$$\begin{cases} \langle \widetilde{M}_i(\gamma_\ell, \omega_\ell)\rangle & = & \langle\langle \widetilde{M}_i(\gamma_\ell, \omega_\ell)\rangle\rangle = \frac{1}{\omega_\ell}(I - \gamma_\ell[\widetilde{L}_i]) \equiv \mathcal{M}_i(\gamma_\ell, \omega_\ell), \\ [\widetilde{N}_i(\gamma_\ell, \omega_\ell)] & \leq & \frac{1}{\omega_\ell}(|1-\omega_\ell|I + (\omega_\ell - \gamma_\ell)[\widetilde{L}_i] + \omega_\ell[\widetilde{U}_i]) \equiv \mathcal{N}_i(\gamma_\ell, \omega_\ell) \end{cases} \tag{9}$$

for $i = 1, 2, \ldots, \alpha$ and $\ell \in \mathbb{N}$. By using these last inequalities, taking block absolute values on both sides of (7), applying the block P-bounded property of G, writing $\mathcal{C}_i = [\widetilde{C}_i]$ and $\mathcal{P} = \widetilde{P}$, and noticing that $\mathcal{M}_i(\gamma_\ell, \omega_\ell)$ $(i = 1, 2, \ldots, \alpha, \ell \in \mathbb{N})$ are M-matrices in $\mathbb{R}^{N\times N}$, we know from (8)-(9) that the error $\{\varepsilon^p\}_{p\in N_0}$ of GAMBT-AOR method satisfies

$$[\varepsilon_\ell^{p+1}] \leq \sum_{i\in\mathbb{N}_\ell(p)} [E_{\ell\ell}^{(i)}] e_\ell^T T_p^{(i,\ell)} [\varepsilon^{\tau^{(i)}(p)}] + \sum_{i\notin\mathbb{N}_\ell(p)} [E_{\ell\ell}^{(i)}][\varepsilon_\ell^p], \quad \ell \in \mathbb{N}, \quad p \in N_0 \tag{10}$$

for both Type-I and Type-II cases, where $e_\ell \in \mathbb{R}^N$ is the ℓ-th unit basis vector, and

$$\begin{aligned} T_p^{(i,\ell)} &= \left(\mathcal{M}_i(\gamma_\ell, \omega_\ell)^{-1}\mathcal{N}_i(\gamma_\ell, \omega_\ell)\right)^{s_i(p)} \\ &\quad + \sum_{k=0}^{s_i(p)-1} \left(\mathcal{M}_i(\gamma_\ell, \omega_\ell)^{-1}\mathcal{N}_i(\gamma_\ell, \omega_\ell)\right)^k \mathcal{M}_i(\gamma_\ell, \omega_\ell)^{-1}(\mathcal{C}_i + \mathcal{P}). \end{aligned} \tag{11}$$

For Type-I case, similarly to [11] we can demonstrate that there exist a constant $\theta \in [0, 1)$ and a positive vector $v \in \mathbb{R}^N$ such that

$$T_p^{(i,\ell)} v \leq \theta v, \qquad i = 1, 2, \ldots, \alpha, \quad \ell \in \mathbb{N}, \quad p \in N_0.$$

Now, defining the sequence $\{\epsilon^p\}_{p\in N_0}$ according to $\epsilon^0 = [\varepsilon^0]$ and

$$\epsilon_\ell^{p+1} = \sum_{i\in\mathbb{N}_\ell(p)} \widetilde{e}_\ell^{(i)} e_\ell^T T_p^{(i,\ell)} \epsilon^{\tau^{(i)}(p)} + \sum_{i\notin\mathbb{N}_\ell(p)} \widetilde{e}_\ell^{(i)} \epsilon_\ell^p, \qquad \ell = 1, 2, \ldots, N, \quad p \in N_0,$$

where

$$\widetilde{E}_i = diag(\widetilde{e}_1^{(i)}, \widetilde{e}_2^{(i)}, \ldots, \widetilde{e}_N^{(i)}) = [E_i], \quad i = 1, 2, \ldots, \alpha,$$

we can immediately deduce that $\{\epsilon^p\}_{p\in N_0}$ is a majorizing sequence of $\{[\varepsilon^p]\}_{p\in N_0}$. That is to say, $[\varepsilon^p] \leq \epsilon^p$ holds for all $p \in N_0$. By making use of Lemma 2 we know that $\lim_{p\to\infty} \epsilon^p = 0$. Therefore, $\lim_{p\to\infty} [\varepsilon^p] = 0$ and then, $\lim_{p\to\infty} \varepsilon^p = 0$.

Quite analogous to the proof of Type-I case we can also get $\lim_{p\to\infty} \varepsilon^p = 0$ for Type-II case.

Theorem 1 immediately leads to the following convergence theories for the asynchronous multisplitting block two-stage Gauss-Seidel, SOR and AOR methods, as well as the generalized asynchronous multisplitting block two-stage SOR method.

Theorem 2 *Let $A \in H_B^I$ $(H_B^{II}) \subset \mathbb{L}_{n,I}$. Let the splittings $A = B_i - C_i$, $i = 1, 2, \ldots, \alpha$, be H_B^I-compatible (H_B^{II}-compatible) such that $D(B_i) = D(A)$, $i = 1, 2, \ldots, \alpha$, the splittings $B_i = D_i - L_i - U_i$, $i = 1, 2, \ldots, \alpha$, satisfy $\langle B_i\rangle = \langle D_i\rangle - [L_i] - [U_i] (i = 1, 2, \ldots, \alpha)$ for Type-I case (and $\langle\langle B_i\rangle\rangle = I - [D_i^{-1}L_i] - [D_i^{-1}U_i] (i = 1, 2, \ldots, \alpha)$ for Type-II case), and the weighting matrices E_i, $i = 1, 2, \ldots, \alpha$, satisfy $\sum_{i=1}^{\alpha}[E_i] \leq$*

I. Assume further that $G : \mathbb{R}^n \to \mathbb{R}^n$ is a block P-bounded mapping such that $\rho(\langle A\rangle^{-1}P) < 1$ ($\rho(\langle\langle A\rangle\rangle^{-1}[D(A)^{-1}]P) < 1$). Then for any initial vector $x^0 \in \mathbb{R}^n$ and any sequence of numbers of inner iterations $s_i(p) \geq 1$, $i = 1, 2, \ldots, \alpha$, $p \in N_0$:

(a) the asynchronous multisplitting block two-stage Gauss-Seidel method converges to the unique solution of the system of mildly nonlinear equations (1);

(b) the asynchronous multisplitting block two-stage SOR method converges to the unique solution of the system of mildly nonlinear equations (1), provided the relaxation parameter ω satisfies

$$0 < \omega < \frac{2}{1+\rho(D_{\langle A\rangle}^{-1}(B_{\langle A\rangle}+P))} \quad \left(0 < \omega < \frac{2}{1+\rho(B_{\langle\langle A\rangle\rangle}+[D(A)^{-1}]P)}\right);$$

(c) the asynchronous multisplitting block two-stage AOR method converges to the unique solution of the system of mildly nonlinear equations (1), provided the relaxation parameters γ and ω satisfy $0 \leq \gamma \leq \omega$, and

$$0 < \omega < \frac{2}{1+\rho(D_{\langle A\rangle}^{-1}(B_{\langle A\rangle}+P))} \quad \left(0 < \omega < \frac{2}{1+\rho(B_{\langle\langle A\rangle\rangle}+[D(A)^{-1}]P)}\right);$$

(d) the generalized asynchronous multisplitting block two-stage SOR method converges to the unique solution of the system of mildly nonlinear equations (1), provided the relaxation parameters ω_ℓ ($\ell \in \mathbb{N}$) satisfy

$$0 < \omega_\ell < \frac{2}{1+\rho(D_{\langle A\rangle}^{-1}(B_{\langle A\rangle}+P))} \quad \left(0 < \omega_\ell < \frac{2}{1+\rho(B_{\langle\langle A\rangle\rangle}+[D(A)^{-1}]P)}\right), \quad \ell \in \mathbb{N}.$$

REFERENCES

Z.-Z. Bai. A class of two-stage iterative methods for systems of weakly nonlinear equations. *Numerical Algorithms*, 14:295–319, 1997.

Z.-Z. Bai. Parallel multisplitting two-stage iterative methods for large sparse systems of weakly nonlinear equations. *Numerical Algorithms*, 15:347–372, 1997.

Z.-Z. Bai. Parallel matrix multisplitting block relaxation methods. *Mathematica Numerica Sinica*, 17:238–252, 1995. In Chinese.

Z.-Z. Bai. A class of blockwise matrix multi-splitting multi-parameter block relaxation methods. *Proceeding of the Third China-Japan Joint Seminar on Numerical Mathematics*, Z. C. Shi and M. Mori eds., Science Press of China, Beijing/New York, 1998.

Z.-Z. Bai. Asynchronous parallel matrix multisplitting block relaxation iteration methods. *Numerical Mathematics, A Journal of Chinese Universities (Chinese Series)*, 19:28–39, 1997. In Chinese.

Z.-Z. Bai. A class of asynchronous parallel multisplitting blockwise relaxation methods. *Parallel Computing*, accepted for publication, 1999.

Z.-Z. Bai and Y.-G. Huang. Asynchronous multisplitting two-stage iterations for systems of weakly nonlinear equations. *Journal of Computational and Applied Mathematics* , 93:13-33, 1998.

Z.-Z. Bai, J.-C. Sun and D.-R. Wang. A unified framework for the construction of various matrix multisplitting iterative methods for large sparse system of linear equations. *Computers and Mathematics with Applications*, 32(12):51-76, 1996.

Z.-Z. Bai, D.-R. Wang and D. J. Evans. A class of asynchronous matrix multi-splitting multi-parameter relaxation iterations. *Journal of Computational Mathematics*, 16(3):221-238, 1998.

Z.-Z. Bai, D.-R. Wang and D. J. Evans. On the convergence of asynchronous nested matrix multisplitting methods for linear systems. *Journal of Computational Mathematics*, accepted for publication, 1997.

Z.-Z. Bai, V. Migallón, J. Penadés and D. B. Szyld. Block and asynchronous two-stage methods for mildly nonlinear systems. *Numerische Mathematik*, accepted for publication, 1998.

D. J. Evans and Z.-Z. Bai. Blockwise matrix multi-splitting multi-parameter block relaxation methods. *International Journal of Computer Mathematics*, 64:103-118, 1997.

D. J. Evans and Z.-Z. Bai. On the convergence of two-stage multisplitting methods for linear systems. *Neural, Parallel and Scientific Computations*, 6:279-295, 1998.