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Domain Decomposition for Indefinite Weakly Singular Integral Equations

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INTRODUCTION

We propose and analyze a preconditioner based on domain decomposition for the p-version of the boundary element method in three dimensions. We consider indefinite weakly singular integral equations on surfaces and use quadrilateral elements for the boundary discretization. The GMRES method is used as iterative solver for the linear systems. For a non-overlapping method, we prove that the numbers of GMRES-iterations (which are required to solve the systems up to a given accuracy) grow only polylogarithmically with the polynomial degree.

Solving first-kind weakly singular integral equations by the Galerkin method does not require continuous ansatz functions. Therefore, without performing any Schur complement technique, simple non-overlapping methods are sufficient to define efficient preconditioners. In [Heua] this is shown in the symmetric, positive definite case and here we extend those results to the more general situation of indefinite operators. For this extension we follow the ideas of Cai, Widlund [CW92] (who deal with finite element systems) and Stephan, Tran [ST98] (who deal with boundary integral equations on curves). In this paper, we consider problems in the three-dimensional space. That means we have to deal with integral operators on surfaces.

As model problem we consider the Dirichlet problem for the scalar Helmholtz

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equation exterior to a screen. The problem reads as follows. *For given* $g \in H^{1/2}(\Gamma)$ *find* $u \in H^1_{\text{loc}}(\Omega)$ *with* $\Omega := I\!R^3 \setminus \bar{\Gamma}$ *satisfying*

$$(\Delta + k^2)u = 0 \quad in \quad \Omega, \qquad u = g \quad on \quad \Gamma, \tag{1}$$

$$\left.\begin{array}{rl} \frac{\partial u}{\partial r} - iku = o(r^{-1}) & for \quad k \neq 0 \quad or \\ u = O(r^{-1}) & for \quad k = 0 \end{array}\right\} \quad as \quad r := |x| \to \infty. \tag{2}$$

Here, Γ is a plane surface with polygonal boundary such that it can be decomposed into quadrilaterals. We note that our method also works for polyhedral surfaces and triangular meshes. We consider wave numbers k with $\Im(k) \geq 0$ and $|k|$ small. The latter condition is to ensure the existence of a Gårding inequality with moderate constants for the boundary integral operator V_k, i.e. the costant γ_1 in (4) is not much smaller than γ_2, cf. [Ste87]. If this were not the case our method would require a very small coarse mesh size H, cf. Lemma 3 below.

The Dirichlet screen problem (1), (2) appears in the scattering theory of acoustic fields u by obstacles. From [Ste87] we know that for $\Im(k) \geq 0$ this problem is uniquely solvable. Further, $u \in H^1_{\text{loc}}(\Omega)$ is the solution of the Dirichlet screen problem if and only if the jump $[\partial u/\partial n]$ of the normal derivative of u across Γ is the solution to the weakly singular integral equation

$$V_k \left[\frac{\partial u}{\partial n}\right](x) := \frac{1}{4\pi}\int_\Gamma \left[\frac{\partial u}{\partial n}\right](y)\frac{e^{ik|x-y|}}{|x-y|}\,dS_y = g(x) \quad (x \in \Gamma). \tag{3}$$

Moreover, by [Cos88, Ste87], V_k is continuous,

$$V_k : \tilde{H}^{-1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma) \quad (-1/2 \leq s \leq 1/2)$$

and strongly elliptic, i.e., it satisfies a Gårding inequality: *There exist constants* $\gamma_1, \gamma_2 > 0$ *such that for all* $v \in \tilde{H}^{-1/2}(\Gamma)$

$$\Re\langle V_k v, v\rangle \geq \gamma_1 \|v\|^2_{\tilde{H}^{-1/2}(\Gamma)} - \gamma_2\|v\|^2_{\tilde{H}^{-1}(\Gamma)}. \tag{4}$$

Here, $\langle\cdot,\cdot\rangle$ denotes the $L^2(\Gamma)$-inner product. The space $\tilde{H}^{-s}(\Gamma)$ for $s > 0$ is the dual space of $H^s(\Gamma)$ which is the usual Sobolev space for integer s and is obtained by interpolation for non-integer s, see, e.g., [LM72]. The operator $V := V_0$, which is the main part of V_k, is symmetric, positive definite and induces an equivalent norm in $\tilde{H}^{-1/2}(\Gamma)$. The remainder $V_k - V$ is of lower order than V_k and is, as well as its adjoint $V_k' - V$, bounded:

$$|\langle (V_k - V)v_1, v_2\rangle| \leq c\min\{\|v_1\|_{\tilde{H}^{-1}(\Gamma)}\|v_2\|_{\tilde{H}^{-1/2}(\Gamma)},\ \|v_1\|_{\tilde{H}^{-1/2}(\Gamma)}\|v_2\|_{\tilde{H}^{-1}(\Gamma)}\} \tag{5}$$

for any $v_1, v_2 \in \tilde{H}^{-1/2}(\Gamma)$, see [Ste87].

We define boundary element spaces $S^0_p(\Gamma_h)$ of piecewise polynomials of degree p on quasi-uniform quadrilateral meshes $\bar{\Gamma}_h = \sum_{j=1}^J \bar{\Gamma}_j$ of size h:

$$S^0_p(\Gamma_h) := \{f \in L^2(\Gamma);\ f|_{\Gamma_j} \in P_p(\Gamma_j), j = 1, \ldots, J\} \subset \tilde{H}^{-1/2}(\Gamma)$$

Here, $P_p(\Gamma_j)$ denotes the set of polynomials on Γ_j which are at most of degree p in each of the variables. The Galerkin scheme for the numerical solution of the integral equation (3) then is: *Find $u_N \in S_p^0(\Gamma_h)$ such that*

$$\langle V_k u_N, w\rangle = \langle g, w\rangle \quad \textit{for any} \quad w \in S_p^0(\Gamma_h). \tag{6}$$

Here, the subscript N is used to denote the dimension of $S_p^0(\Gamma_h)$. By the strong ellipticity of V_k this method converges quasi-optimally.

Since V_k is a first kind integral operator of order minus one the spectral condition number of the stiffness matrix A in (6) behaves (when using standard basis functions) like $O(h^{-1}p^2)$, cf., e.g., [HW77, Heu96]. Especially for the p-version we are considering, a preconditioner is necessary for the efficient solution of (6). The stiffness matrix A is in general non-Hermitian and indefinite and we use the GMRES method [SS86] as iterative solver. For real linear systems the rate of convergence of the GMRES method depends on the two numbers

$$\Lambda_0 = \inf_{v\in\mathbb{R}^N} \frac{a(v,Av)}{a(v,v)}, \quad \text{and} \quad \Lambda_1 = \sup_{v\in\mathbb{R}^N} \frac{||Av||_a}{||v||_a}.$$

The quantity Λ_0 is the minimum eigenvalue of the symmetric part of A and Λ_1 is the matrix norm of A which is induced by the norm $||\cdot||_a$. In the complex case we have to consider, instead of Λ_0 and Λ_1 as defined above, the expressions

$$\Lambda_0 = \inf_{v\in\mathbb{C}^N} \Re\frac{a(v,Av)}{a(v,v)}, \quad \text{and} \quad \Lambda_1 = \sup_{v\in\mathbb{C}^N} \frac{||Av||_a}{||v||_a}.$$

For $a(\cdot,\cdot)$ we use the inner product which is given by V, i.e., $a(\cdot,\cdot) = \langle V\cdot,\cdot\rangle \simeq ||\cdot||^2_{\tilde{H}^{-1/2}(\Gamma)}$. From [EES83] we cite the following result: *If $\Lambda_0 > 0$ then the GMRES method converges and for the residual corresponding to the jth iterate $r_j = g - Au_j$ there holds*

$$||r_j||_a \le \Big(1 - \frac{\Lambda_0^2}{\Lambda_1^2}\Big)^{j/2} ||r_0||_a.$$

Here, g is the right hand side vector of the linear system (6). In order to reduce the number of iterations of the GMRES method we need to find a preconditioner for the matrix A which bounds the ratio Λ_0/Λ_1 from below.

Throughout the paper c denotes a positive generic constant.

NON-OVERLAPPING ADDITIVE SCHWARZ METHOD

In order to define our preconditioner we need to decompose the boundary element space. This is done by introducing a coarse mesh $\bar{\Gamma}_H = \cup_{j=1}^n \bar{G}_j$ $(H \ge h)$ which must be compatible with Γ_h. We note that the case $\Gamma_H = \Gamma_h$ is included. Now we decompose

$$S_p^0(\Gamma_h) = X_0 + X_1 + \cdots + X_n \tag{7}$$

where X_0 is the space of piecewise constant functions on the coarse mesh Γ_H and

$$X_j := \{v|_{G_j};\ v \in S_p^0(\Gamma_h),\ \langle v, 1\rangle_{L^2(G_j)} = 0\}, \quad j = 1,\dots,n.$$

We define the projection operators $\mathcal{P}_j : S_p^0(\Gamma_h) \to X_j$, $j = 0, \ldots, n$, and $P_j : S_p^0(\Gamma_h) \to X_j$, $j = 1, \ldots, n$, such that, for given $v \in S_p^0(\Gamma_h)$,

$$\langle V_k \mathcal{P}_j v, \phi \rangle = \langle V_k v, \phi \rangle \quad \forall \phi \in X_j \quad \text{and} \quad \langle V P_j v, \phi \rangle = \langle V_k v, \phi \rangle \quad \forall \phi \in X_j.$$

Two types of the additive Schwarz operator are now defined by $\mathcal{P} = \mathcal{P}_0 + \mathcal{P}_1 + \cdots + \mathcal{P}_n$ and $P = \mathcal{P}_0 + P_1 + \cdots + P_n$. The additive Schwarz method consists in solving, instead of the system (6), one of the equations

$$\mathcal{P} u_N = f_1 \quad \text{or} \quad P u_N = f_2.$$

The right hand sides $f_1 = (\mathcal{P}_0 + \mathcal{P}_1 + \cdots + \mathcal{P}_n) u_N$ and $f_2 = (\mathcal{P}_0 + P_1 + \cdots + P_n) u_N$ can be computed without knowing the solution u_N of the original Galerkin system (6) since only $\langle V_k u_N, \phi \rangle$ for $\phi \in S_p^0(\Gamma_h)$ needs to be known. But $V_k u_N = g$ when testing against functions of the ansatz space $S_p^0(\Gamma_h)$. The operators $\mathcal{P}_j$ $(j > 0)$ are computed by locally inverting the indefinite operator V_k whereas for P_j we only need to solve positive definite problems for V.

The efficiency of the additive Schwarz operators $\mathcal{P}$ and P is proved by the following theorem.

Theorem 1 *There exist positive constants c_1, c_2, c_3, h_0 such that, if $0 < H \leq h_0$, there holds for $p > 0$ and for any $u \in S_p^0(\Gamma_h)$*

$$\Re \langle V \mathcal{P} u, u \rangle \geq c_1 \Big(c_2 \big(1 + \log(\frac{H}{h}(p+1))\big)^{-2} - (1 + \log 1/H)^{1/2} H^{1/2} \Big) \langle V u, u \rangle$$

and

$$\|\mathcal{P} u\|_{\tilde{H}^{-1/2}(\Gamma)} \leq c_3 \|u\|_{\tilde{H}^{-1/2}(\Gamma)}.$$

Analogous estimates hold for the operator P.

Technical details for the proof of this theorem are given in the next section.

TECHNICAL DETAILS

We collect all the technical details which are needed to investigate the decomposition (7) within $\tilde{H}^{-1/2}(\Gamma)$ and we show how these results apply to indefinite operators. For the generalization of efficient decompositions within $\tilde{H}^{-1/2}(\Gamma)$ (which is the symmetric, positive definite case) to efficient preconditioners for indefinite operators on surfaces we extend the theory for integral operators on curves of Stephan and Tran [ST98], see also [CW92]. The sketch of a proof of Theorem 1 is given at the end of this section.

The first lemma, which we cite from [Heua], proves that the non-overlapping decomposition (7) is almost optimal within the space $\tilde{H}^{-1/2}(\Gamma)$. This is equivalent to the boundedness of the spectral condition number of the additive Schwarz operator (defined by this decomposition) when considering symmetric, positive definite weakly singular integral operators.

Lemma 1 [Heua] *Let $X_0, \dots, X_n$ be the subspaces of the non-overlapping decomposition (7). There exist constants $c_1, c_2 > 0$ such that for any $u \in S_p^0(\Gamma_h)$ with arbitrary representation $u = \sum_{j=0}^n v_j$ (with $v_j \in X_j$) there exist $u_j \in X_j$ with $u = \sum_{j=0}^n u_j$ and*

$$c_1\big(1+\log(\frac{H}{h}(p+1))\big)^{-2}\sum_{j=0}^n \|u_j\|^2_{\tilde{H}^{-1/2}(\Gamma)} \le \|u\|^2_{\tilde{H}^{-1/2}(\Gamma)} \le c_2 \sum_{j=0}^n \|v_j\|^2_{\tilde{H}^{-1/2}(\Gamma)}.$$

The constants $c_1, c_2 > 0$ do not depend on the mesh size h or the polynomial degree p.

The following lemma is used to estimate the coarse grid component of the additive Schwarz operator.

Lemma 2 *There exists $h_0 > 0$ such that, for all $H \in (0, h_0)$, the projection operator $\mathcal{P}_0 : S_p^0(\Gamma_h) \to X_0$ is well defined. Moreover, there exists a constant $c > 0$ such that, for $H \in (0, h_0)$, there holds for any $u \in S_p^0(\Gamma_h)$*

$$\|\mathcal{P}_0 u\|_{\tilde{H}^{-1/2}(\Gamma)} \le c\|u\|_{\tilde{H}^{-1/2}(\Gamma)}$$

and

$$\|u - \mathcal{P}_0 u\|_{\tilde{H}^{-1}(\Gamma)} \le c(1+\log 1/H)^{1/2} H^{1/2} \|u\|_{\tilde{H}^{-1/2}(\Gamma)}.$$

Proof. The projection operator $\mathcal{P}_0$ is the standard h-version Galerkin projection operator (restricted onto $S_p^0(\Gamma_h)$) with respect to V_k. Since V_k is strongly elliptic the boundedness of $\mathcal{P}_0$ and the convergence

$$\|u - \mathcal{P}_0 u\|_{\tilde{H}^{-1}(\Gamma)} \le c(\epsilon) H^{1/2-\epsilon} \|u\|_{\tilde{H}^{-1/2}(\Gamma)} \quad \text{for any} \quad u \in S_p^0(\Gamma_h)$$

($\epsilon > 0$) follows by standard arguments, see, e.g., [Wen90, Ste87]. Here, $c(\epsilon)$ is a constant which is not bounded when $\epsilon \to 0$, in general. This is due to the reduced regularity of the solution to the equation $V_k u = f$ near the boundary of a screen, even for smooth right hand side f. This regularity is used in the Aubin-Nitsche trick which give error estimates in lower norms than the energy norm $\tilde{H}^{-1/2}(\Gamma)$, cf. [CS88, Wen90]. When one uses precise norm estimates of the singularities appearing in the images of V_k^{-1}, then the Aubin-Nitsche trick yields $c(\epsilon) = O(\epsilon^{-1/2})$ (see [Heub]) and taking $\epsilon := -1/\log H$ this proves the second assertion of the lemma.

The next lemma will be used to bound the lower order norm in the Gårding inequality (4) for functions with small support. This gives the positive definiteness of the real part of V_k on local subspaces of $S_p^0(\Gamma_h)$.

Lemma 3 *Let u be a function from any one of the local subspaces X_1, X_2, …, X_n in (7), that means* $\mathrm{diam}(\mathrm{supp}(u)) = O(H)$. *Then there exist $c, C, h_0 > 0$ such that, for any $H \in (0, h_0)$, there holds*

$$\|u\|_{\tilde{H}^{-1}(\Gamma)} \le C H^{1/2} \|u\|_{\tilde{H}^{-1/2}(\Gamma)} \tag{8}$$

and

$$\Re\langle V_k u, u\rangle \ge c\|u\|^2_{\tilde{H}^{-1/2}(\Gamma)}.$$

Proof. The subspaces X_j are constructed such that $\int_\Gamma u\,dS = 0$ for $u \in X_j$ $(j > 0)$. For functions with integral mean zero the $\tilde{H}^{-s}$-norms $(0 \le s \le 1)$ are scalable with respect to the diameter of the supports, see Lemma 3 in [Heua], and this proves assertion (8). The second assertion follows from (8) by Gårding's inequality (4) choosing H small enough.

The following result is used to prove Theorem 1. It is based on the previous lemmas and extends the almost-optimality result of Lemma 1 to the indefinite situation of the operators $\mathcal{P}$ and P.

Lemma 4 *There exist $c_1, c_2, h_0 > 0$ such that, for $H \in (0, h_0)$ and $p > 0$, there holds for any $u \in S_p^0(\Gamma_h)$*

$$c_1\bigl(1 + \log(\frac{H}{h}(p+1))\bigr)^{-2}\|u\|^2_{\tilde{H}^{-1/2}(\Gamma)} \le \sum_{j=0}^{N}\|\mathcal{P}_j u\|^2_{\tilde{H}^{-1/2}(\Gamma)} \le c_2\|u\|^2_{\tilde{H}^{-1/2}(\Gamma)}.$$

The same estimates hold if one replaces $\mathcal{P}_j$ by P_j for $j > 0$.

Proof. This lemma is the counter part to Lemma 2.5 in [ST98] and for a proof we refer to that reference. The main ingredients are provided by the previous Lemmas 1, 2, 3 and by Gårding's inequality (4).

Proof of Theorem 1. Using the technical details provided in this section Theorem 1 follows analogously as in [Heub] where hypersingular operators are considered. In fact, one only needs to substitute $\tilde{H}^{1/2}(\Gamma)$ by $\tilde{H}^{-1/2}(\Gamma)$ and $L^2(\Gamma)$ by $\tilde{H}^{-1}(\Gamma)$ in the proof of the main theorems in [Heub]. All the needed details are then given by Lemmas 1, 2, 4 and by (5) and (8) above. See also the proof of Theorem 2.1 in [ST98] where the details for integral operators on curves are given.

NUMERICAL RESULTS

We consider the Dirichlet screen problem (1), (2) for the homogeneous Helmholtz equation with $\Gamma = (0,1)^2 \times \{0\} \subset \mathbb{R}^3$. As right hand side g in (1) we simply take the function 1. This choice does not influence the behavior of the resulting stiffness matrix. For the definition of the ansatz space $S_p^0(\Gamma_h)$ we use uniform meshes Γ_h consisting of squares and take piecewise polynomials of degree p that need not be continuous across element interfaces. As basis functions we use tensor products of Legendre polynomials which are scaled to have unit L^2-norm.

For the non-overlapping additive Schwarz method we simply take the coarse mesh $\Gamma_H = \Gamma_h$, i.e., $H/h = 1$. Figure 1 shows the parameters Λ_0 and Λ_1 for different wave numbers and full local solvers ("I", additive Schwarz operator $\mathcal{P}$) and positive definite local solvers ("P", additive Schwarz operator P). In these cases Theorem 1 proves an asymptotic bound like $(1 + \log p)^{-2}$ for the minimum eigenvalue Λ_0 of the real Hermitian part of the preconditioned stiffness matrix. The norm Λ_1 of the preconditioned stiffness matrix is proven to be bounded. Both theoretical estimates seem to be confirmed by the numerical results. The requirement of Theorem 1 that the mesh size h must be small enough does depend on the actual polynomial degree. In particular, to ensure convergence of the GMRES method, we must have $\Lambda_0 > 0$, i.e., h must be small enough such that

$$c_2(1+\log p)^{-2} - (1+\log 1/h)^{1/2}h^{1/2} > 0.$$

Here, c_2 is an unknown constant and, nevertheless, there is no problem to satisfy this condition since the dependence on the polynomial degree is only logarithmic. Our numerical experiments do not reflect the theoretically needed dependence of h on p in this situation. The condition on the mesh size is of asymptotic nature and may become a restriction for higher polynomial degrees and for larger wave numbers.

Table 1 presents the numbers of iterations of the GMRES method for different values of k which are needed to reduce the initial residual by a factor of 10^{-6}. The results demonstrate that the dependence of the iteration numbers of the GMRES method on the wave number is rather weak. Indeed, the numbers increase only slightly with k (and p) and are almost constant in h.

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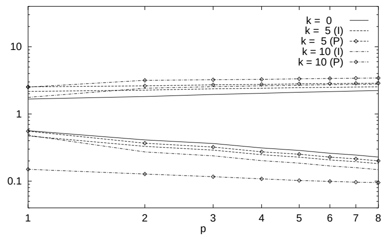

Figure 1 Values of Λ_0 and Λ_1 (p-version, $h^{-1} = 2$) with full local solvers (I) and positive definite local solvers (P).

Table 1 Numbers of GMRES iterations (different wave numbers, full local solvers and positive definite local solvers (I, P)).

		k:	0	5		10	
N	h^{-1}	p	(P)	(I)	(P)	(I)	(P)
4	2	0	2	2	2	2	2
36	2	2	7	7	7	7	7
100	2	4	9	9	9	9	11
196	2	6	9	10	9	10	11
324	2	8	10	10	10	10	11
36	2	2	7	7	7	7	7
144	4	2	13	14	14	15	15
324	6	2	13	15	15	17	17
576	8	2	13	14	14	17	17
900	10	2	13	14	14	16	16