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Numerical Simulation of Wave Propagation Phenomena in Vocal Tract and Domain Decomposition Method

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INTRODUCTION

We develop a finite element approximation method for the Helmholtz equation in some unbounded region Ω_0:

$$-\Delta u - \omega^2 u = 0 \text{ in } \Omega_0 \tag{1}$$

and apply the method to the wave propagation phenomena in a vocal tract. For various time frequencies ω, we solve the Helmholtz equation in an unbounded acoustic region Ω_0 a part of which forms a vocal tract. Then we investigate the resonance phenomena of the sound wave propagation which is important to characterize vowels and consonants. We use a two dimensional model as well as one dimensional model called Webster's horn equation.

In this research, we confine our study to the case in which the outer region consists of a semi-infinite cylinder, and we assume that the original three dimensional acoustic region is planar which enables us to reduce the problem into a two dimensinal one after the Fourier mode decomposition with respect to the perpendicular direction to the corresponding plane of the planar region.

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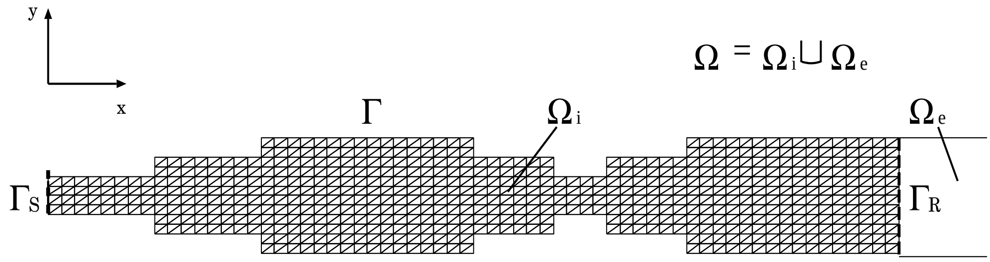

Figure 1 Acoustic region in 2D case

Assuming the radiation condition at infinity, we derive a radiation boundary condition on the artificial boundary which forms a part of the boundary of the bounded sub-region of the original unbounded acoustic region. In the radiation condition, we use the Dirichlet to Neumann map for the Helmholtz problem in the unbounded outer cylindrical region.

We introduce a one dimesional Webster's model which corresponds to the 2-D plane region using the width of the vocal tract. There is a good coincidence between the results of 1-D and 2-D models as far as the magnitude of the modulus of a wave number is not large, i.e. less than some constant. The results justify the use of the 1-D Webster's model as the approximation of the 2-D model in case that the incident wave from the vocal cord contains only low time frequency modes.

MATHEMATICAL MODELS

We consider the case where the outer unbounded region consists of a semi-infinite cylinder $\Omega_{0,e}$ with a bounded 2-D cross section $S_{0,e}$:

$$\Omega_{0,e} = \{\mathbf{x} = (x, y, z) \mid x_0 < x < +\infty,\ (y, z) \in S_{0,e}\} \tag{2}$$

and we assume that the original three dimensional acoustic region is planer:

$$\Omega_0 = (\Omega_i \cup \Omega_e) \times (0, z_0) \text{ with } \Omega_{0,e} = \Omega_e \times (0, z_0) \text{ and } S_{0,e} = (0, y_0) \times (0, z_0), \tag{3}$$

which enables us to reduce the problem into a two dimensinal one after the Fourier mode decomposition with respect to the perpendicular z-direction to the corresponding xy plane of the planar region.

Now, consider the following wave equation:

$$(\frac{\partial^2}{\partial t^2} - \Delta)u(t, \mathbf{x}) = f(t, \mathbf{x}) \text{ in } R_t \times \Omega_{\mathbf{x}},\ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \tag{4}$$

$$(\alpha \frac{\partial}{\partial n} + \beta)u(t, \mathbf{x}) = g(t, \mathbf{x}) \text{ on } R_t \times \partial\Omega_{\mathbf{x}}, \tag{5}$$

with some initial condition for u and $\partial u/\partial t$, and consider a stationary time harmonic solution: $u(t, \mathbf{x}) = e^{i\omega t}u(\mathbf{x})$ for inhomogeneous data: $f(t, \mathbf{x}) = e^{i\omega t}f(\mathbf{x})$ and $g(t, \mathbf{x}) =$

$e^{i\omega t}g(\mathbf{x})$. Then u satisfies the Helmholtz equation:

$$(-\Delta - \omega^2)u(\mathbf{x}) = f(\mathbf{x}) \ \text{ in } \Omega, \tag{6}$$

$$(\alpha\frac{\partial}{\partial n} + \beta)u(\mathbf{x}) = g(\mathbf{x}) \ \text{ on } \partial\Omega, \tag{7}$$

with some radiation condition at infinity ($|\mathbf{x}| \to +\infty$).

DOMAIN DECOMPOSITION INTO INNER AND OUTER REGIONS

We consider the case of outgoing wave and derive a radiation condition on the artificial boundary $S_e = \{x_0\} \times (0, y_0)$ which forms a part of the boundary of the bounded subregion Ω_i of the original unbounded acoustic region $\Omega = \Omega_i \cup \Omega_e$ (Figure 1).

The radiation condition is then given by the Dirichlet to Neumann map Λ_y for the Helmholtz problem in the unbounded outer cylindrical region Ω_e (see [Mas87] in the case of obstacle scattering):

$$\frac{\partial}{\partial x}u(x_0, y) = \Lambda_y u(x_0, y), \tag{8}$$

where

$$\Lambda_y u = \Lambda u \equiv \sum_{n=0}^{\infty} \gamma_n C_n(u)\cos(\frac{n\pi}{y_0}y)$$

with

$$C_n(u) = \begin{cases} \frac{1}{y_0}\int_0^{y_0} u(x_0, y)dy & (n = 0), \\ \frac{2}{y_0}\int_0^{y_0} u(x_0, y)\cos(\frac{n\pi}{y_0}y)dy & (n \geq 1), \end{cases}$$

$$\gamma_n = \begin{cases} i\zeta_n, & \zeta_n = \{\omega^2 - (\frac{n\pi}{y_0})^2\}^{1/2}, \quad 0 < \frac{n\pi}{y_0} < \omega, \\ -\eta_n, & \eta_n = \{(\frac{n\pi}{y_0})^2 - \omega^2\}^{1/2}, \quad \omega \leq \frac{n\pi}{y_0}. \end{cases}$$

Then, we have the following Helmholtz equation in the inner region Ω_i:

$$(-\omega^2 - \Delta)u = 0 \ \text{ in } \Omega_i, \tag{9}$$

$$\frac{\partial u}{\partial n} = 0 \ \text{ on } \Gamma, \ \frac{\partial u}{\partial n} = g_S \ \text{ on } \Gamma_S, \ \frac{\partial u}{\partial n} = \Lambda u \ \text{ on } \Gamma_R \equiv S_e.$$

WEBSTER'S HORN EQUATION

Now, we introduce a one dimensional approximation of the original problem which is called Webster's horn equation:

$$-\frac{\partial v}{\partial t} = \frac{A(x)}{\rho}\frac{\partial u}{\partial x}, \tag{10}$$

$$-\frac{\partial u}{\partial t} = \frac{\rho c^2}{A(x)}\frac{\partial v}{\partial x}, \tag{11}$$

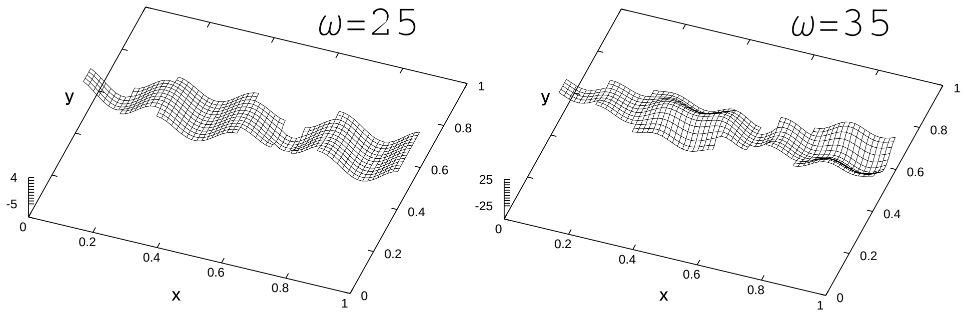

Figure 2 Acoustic field for some wave numbers

which can be reduced to the form of the second order equation:

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{A(x)} c^2 \frac{\partial}{\partial x}(A(x)\frac{\partial u}{\partial x}) = 0, \tag{12}$$

where $A(x)$ corresponds to the width of the original 2-D or 3-D region of the vocal tract. The corresponding time harmonic equation becomes to be

$$-A(x)\frac{d}{dx}\{\frac{1}{A(x)}\frac{d}{dx}u(x)\} - \omega^2 u(x) = 0. \tag{13}$$

NUMERICAL RESULTS AND ERROR ANALYSIS

As for the discretization, we adopt the finite element method for respective 1-D and 2-D models using a finite element subspace of piecewise linear continuous functions, and compare the numerical results for various wave numbers ω. There is a good coincidence between the results of 1-D and 2-D cases for the frequency responce as far as the magnitude of the modulus of a wave number (frequency) is not large (see Figure 2,3). The results justify the use of the 1-D Webster's equation model as the approximation of the 2-D model in case that the incident wave from the vocal cord contains only low frequency modes

Next, we give the convergence results for the finite element calculation. First, we write the two domensional problem in the following weak form:

Find $u \in \mathcal{V} \equiv H^1(\Omega)$:

$$a(u,v) + b(u,v) = (f,v)(= a_0(g,v)) \qquad \forall v \in \mathcal{V} \tag{14}$$

where

$$a(u,v) = a_0(u,v) + b_2^{\infty}(u,v), \quad b(u,v) = b_1(u,v) + b_2^0(u,v)$$

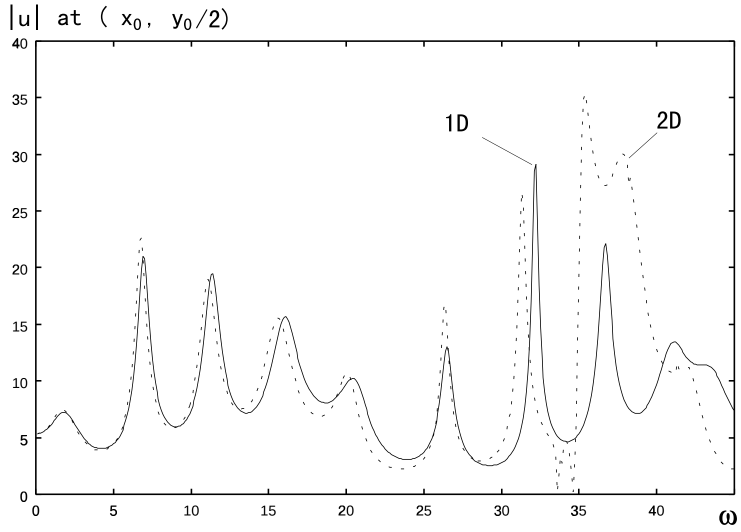

Figure 3 Comparison between the results of 1-D and 2-D cases for the frequency responce

with

$$
\begin{aligned}
a_0(u,v) &= \int_\Omega \nabla u \cdot \overline{\nabla v} + u\overline{v} dxdy, \\
b_1(u,v) &= -(\omega^2+1)\int_\Omega u\overline{v} dxdy, \\
b_2(u,v) &= (\Lambda u(x_R,\cdot), v(x_R,\cdot)) = b_2^0(u,v) + b_2^\infty(u,v), \\
b_2^0(u,v) &= \sum_{0\le \frac{n\pi}{L} < \omega} \gamma_n C_n(u) C_n(v), \\
b_2^\infty(u,v) &= \sum_{\omega \le \frac{n\pi}{L}} \gamma_n C_n(u) C_n(v).
\end{aligned}
$$

Based on this formulation we introduce a finite element method:
Find $u_h \in \mathcal{V}_h \subset \mathcal{V}$:

$$a(u_h, v_h) + b(u_h, v_h) = (f, v_h)(= a_0(g, v_h)) \qquad \forall v_h \in \mathcal{V}_h. \tag{15}$$

Concerning the above finite element methods, we have the followings:

1. $b_2^\infty(u,v)$ is a nonnegative sesquilinear form, and hence $a(u,v)$ is an inner product in $\mathcal{V}$;
2. $b_2^0(u,v)$ and hence b is a compact form with respect to $a(u,v)$ in $\mathcal{V}$;
3. Applying the results of Mikhlin [Mik64], we get the convergence of the approximation (see [Kak81],[LK98]);

4. After the finite element approximation, we can approximate $b_2^\infty(u, v)$ by the one with a finite summation up to the N-th Fourier mode, and the approximation converges to the original finite element solution in the finite dimensional space $\mathcal{V}_h$ as N tends to infinity.

CONCLUDING REMARKS

We summerise our results as follows:

1. Mathematical models for the speech production problem are formulated. In the outer infinite region, the exact solution is given and the radiation boundary condition is proposed by using the Dirichlet to Neumann mapping on the artificial boundary between inner and outer regions;
2. A comparison between 1-D and 2-D calculations is given. When the time frequency of a sound source is low enough, the coincidence of the two calculations is good and 1-D Webster's horn equation model can be used to simulate the resonance phenomena of the vocal tract. A discrepancy becomes large when the frequency is larger than the value above which the single mode approximation is violated due to transversal higher modes (see Figure 2);
3. Numerical analysis of the 2-D problem is given. Using the property of the Dirichlet to Neumann mapping, we can prove the convergence of the finite element approximation, and then the approximation of the Dirichlet to Neumann mapping by its finite Fourier mode summation can be justified in the finite dimensional approximation subspace.

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