32

A non-overlapping DDM of Robin-Robin type for parabolic problems

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Introduction

We consider a time-dependent advection-reaction-diffusion model. An A-stable implicit semidiscretization in time leads to a sequence of elliptic problems. Here we use the discontinuous Galerkin method DG(r). It allows time-step control based on a-posteriori estimates.
Domain decomposition methods (later referred to as DDM) provide a promising tool for solving the arising elliptic problems. Overlapping methods require a minimal overlap which depends on the recent time step [Kus90], [Ran94]. This results in a cumbersome implementation if the time step varies. Therefore we prefer non-overlapping DDM. There is some recent progress in such methods for elliptic problems using interface conditions of Robin type [Gas96], [Nat95], [Aug98], [Alo98].
In Sec. 2 we introduce the semidiscrete problem. In Sec. 3 we apply, for DG(0), the DDM to the fully discretized, and SUPG-stabilized problem. The main result is an a-posteriori error estimate of the discrete DDM using the interface error, which can be used as a stopping criterion. Additionally we get a bound for a parameter in the interface condition which depends on the (variable) time steps. Then we discuss briefly the extension to DG(1). Numerical results are given in Sec. 4.
For a domain G we denote by $W^{k,p}(G)$ the Sobolev space with norm $\|\cdot\|_{k,p,G}$ and seminorm $|\cdot|_{k,p,G}$. $(\cdot,\cdot)_G$ and $\|\cdot\|_G$ are the inner product and the norm in $L^2(G)$. In

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case of $G = \Omega$ we usually omit the index. For a sufficiently smooth curve $S \subset \overline{\Omega}$, we denote by $\langle\cdot,\cdot\rangle_S$ the inner product in $L^2(S)$ or, whenever needed, the duality product between $H^{-1/2}(S)$ and $H^{1/2}(S)$. C is a generic constant not depending on relevant parameters.

Stable semidiscretization in time

In a bounded domain $\Omega \subset \mathbf{R}^d, d \leq 3$, we consider

$$\frac{\partial u}{\partial t} + L_\epsilon u := \frac{\partial u}{\partial t} - \epsilon\Delta u + \vec{b}\cdot\nabla u + cu = f \quad \text{in } (0,T]\times\Omega, \tag{1}$$

$$u = 0 \quad \text{on } (0,T]\times\partial\Omega; \qquad u = u_0(x) \quad \text{in } \{0\}\times\Omega. \tag{2}$$

Assume that $\nabla\cdot\vec{b} \equiv 0$ and $c \geq 0$.

The discontinuous Galerkin method allows a systematic construction of A-stable, implicit, and high-order schemes for problem (1)-(2). Galerkin orthogonality is a basic ingredient of a-posteriori estimates used for time-step control. We set $\partial_t^q v := (\frac{\partial}{\partial t})^q v$, $\mathbf{V} := W_0^{1,2}(\Omega)$ with norm $||\cdot||_{1,\Omega}$. $\mathbf{V}^*$ is the dual space.

The weak formulation of (1)-(2) reads :

Find $u \in W(0,T) \equiv \{z \in L^2(0,T;\mathbf{V}) |\ \partial_t z \in L^2(0,T;\mathbf{V}^*)\}$ *s.t.* $\forall v \in L^2(0,T;V)$

$$\int_0^T (\partial_t u, v)\ dt + \int_0^T B_G(u,v)\ dt = \int_0^T (f,v)\ dt, \quad u(0) = u_0, \tag{3}$$

$$B_G(u,v) \equiv (\epsilon\nabla u, \nabla v) + \frac{1}{2}\Big((\vec{b}\cdot\nabla u, v) - (\vec{b}\cdot\nabla v, u)\Big) + (cu,v). \tag{4}$$

Let $0 = t_0 < t_1 < \ldots < t_{M+1} = T$ be a sequence of discrete time levels with time step $\tau(t) \equiv \tau_m := t_{m+1} - t_m, t \in I_m \equiv [t_m, t_{m+1})$. $\Pi_r(I_m)$ denotes the space of polynomials of degree $r \in \mathbf{N_0}$ with coefficients in $\mathbf{V}$. Furthermore we set $v_\pm^m := \lim_{s\to+0} v(t_m \pm s)$, $[v^m] := v_+^m - v_-^m$.

The DG(r)-method requires the solution of elliptic problems resp. systems on each time slab $I_m\times\Omega,\ m = 0,\ldots,M$

$$\begin{aligned} \textit{Find } u_\tau^m \in \Pi_r(I_m) \quad \textit{s.t.} \quad & A_G^m(u_\tau^m, v) = F_G^m(v) \quad \forall v \in \Pi_r(I_m), \\ A_G^m(v,w) \quad \equiv \quad & \frac{1}{\tau_m}\{(\partial_t v, w)_{I_m} + B_G^m(v,w) + (v_+^m, w_+^m)\}, \\ F_G^m(w) \quad \equiv \quad & \frac{1}{\tau_m}\{(f,w)_{I_m} + (v_-^m, w_+^m)\}, \quad u_-^0 = u_0 \end{aligned} \tag{5}$$

where $(v,w)_{I_m} \equiv \int_{I_m}(v,w)\ dt$ and $B_G^m(v,w) \equiv \int_{I_m} B_G(v,w)\ dt$.

We apply the DG(r)-method with $r = 0$ resp. $r = 1$ for long time resp. time accurate calculations. Error estimates can be derived for problem (5), say for time-independent coefficients $\vec{b}$ and c, which extend the results in [Tho97], Ch.12.

Proposition 1 *Assume that* $\partial_t^{r+1}u$ *is smooth. Then we obtain for the semidiscrete* $DG(r)-$*method the a-priori error estimate*

$$||u_\tau^m(t_m) - u(t_m)||^2 \leq C\sum_{i=0}^{m-1}\tau_i^{2(r+1)}\int_{I_i}||\partial_t^{r+1}u||_1^2\ dt$$

and for $r = 0$ *an a-posteriori estimate with* $C \sim \left(1 + \|b\|_\infty \sqrt{t_m}\right) \sqrt{\log \frac{t_m}{\tau_m}} + 1$

$$\|u_\tau^m(t_m) - u(t_m)\| \leq C \max_{i \leq m-1} \tau_i \left(\sup_{t \in I_i} \|f(t)\| + \left\| \frac{u^i - u^{i-1}}{\tau_i} \right\| \right).$$

The a-posteriori estimate is useful for an adaptive time-step control. As a first step for $r = 0$, one can take the jumps $u^i - u^{i-1}$ as an error indicator.

Robin-Robin DDM for the elliptic problems

DD iteration for the semidiscrete problem

Now we apply a non-overlapping DDM to the semidiscrete problems (5) arising from the DG(0)-method, i.e. with constant (in time) u^m on I_m. Setting $\vec{b}_m = \frac{1}{\tau_m} \int_{I_m} \vec{b}\, dt$, $c_m = \frac{1}{\tau_m} \int_{I_m} c\, dt$, $\tilde{c}_m = c_m + \frac{1}{\tau_m}$, $f_\tau^m = \frac{1}{\tau_m} \left(\int_{I_m} f\, dt + u^{m-1} \right)$, the variational problem (5) is related to the elliptic problem

$$L_{\epsilon,\tau}^m u^m \equiv -\epsilon \Delta u^m + \vec{b}_m \cdot \nabla u^m + \tilde{c}_m u^m = f_\tau^m \quad \text{in } \Omega; \qquad u^m = 0 \quad \text{on } \partial\Omega. \qquad (6)$$

Let $\overline{\Omega} = \cup_k \overline{\Omega}_k$ be a non-overlapping partition with $\Gamma_{kj} \equiv \partial\Omega_k \cap \partial\Omega_j, j \neq k$, $\Gamma_k \equiv \partial\Omega_k \setminus \partial\Omega$. Now we seek for $n \in \mathbf{N}$ a sequence of approximations $u_{n;k}^m$ to $u^m|_{\Omega_k}$ on Ω_k using the additive iteration-by-subdomain method of Robin-Robin type

$$\begin{aligned} L_{\epsilon,\tau}^m u_{n;k}^m &= f_\tau^m \quad \text{in } \Omega_k; \qquad u_{n;k}^m = 0 \quad \text{on } \partial\Omega_k \cap \partial\Omega, & (7) \\ \epsilon \frac{\partial u_{n;k}^m}{\partial n_k} + \rho_k^m u_{n;k}^m &= \epsilon \frac{\partial u_{n-1;j}^m}{\partial n_k} + \rho_k^m u_{n-1;j}^m \quad \text{on } \Gamma_{kj},\ j \neq k. & (8) \end{aligned}$$

The main problem is the design of the ρ_k^m. In [Aug98] we obtained convergence to the solution of (6) for $\rho_k^m \equiv -\frac{1}{2} \vec{b}_m \cdot \vec{n}_k + z_k^m,\ z_k^m > 0$.

Denoting by $A_G^{m;k}(\cdot,\cdot)$ and $F_G^{m;k}(\cdot)$ the restrictions of $A_G^m(\cdot,\cdot)$ resp. $F_G^m(\cdot)$ to Ω_k, the weak formulation of (7), (8) can be written as :

$$\begin{aligned} \textit{Find } u_{n;k}^m &\in \mathbf{V}(\Omega_k) := \mathbf{V}|_{\Omega_k} \textit{ s.t. } \forall v \in \mathbf{V}(\Omega_k),\ \forall \phi \in \mathbf{V}|_{\Gamma_{kj}} \\ A_G^{m;k}(u_{n,k}^m, v) &+ \left\langle z_k^m u_{n,k}^m, v \right\rangle_{\Gamma_k} = F_G^{m;k}(v) + \sum_{j(\neq k)} \left\langle \lambda_{n-1;jk}^m, v \right\rangle_{\Gamma_{kj}}, & (9) \\ \left\langle \lambda_{n;kj}^m, \phi \right\rangle_{\Gamma_{kj}} &= \left\langle \left(z_k^m + z_j^m\right) u_{n;k}^m - \lambda_{n-1;jk}^m, \phi \right\rangle_{\Gamma_{kj}} & (10) \end{aligned}$$

Stabilized Galerkin method and domain decomposition

Now we consider a fully discrete version of the DDM using a SUPG-stabilized FEM. Let $\mathcal{T}_h$ be an admissible triangulation of Ω with simplicial elements K. Furthermore assume that the macroelement partition $\{\Omega_k\}_k$ is aligned with $\mathcal{T}_h$. Let $\mathbf{V}_h \subset \mathbf{V}$ be the subspace of piecewise polynomials of degree $l \in \mathbf{N}$. The discrete space $\Pi_{\tau,h}(I_m)$ in each slab $I_m \times \Omega$ is the set of polynomials of degree r w.r.t. t with coefficients in $\mathbf{V}_h$. In the singularly perturbed case we consider the stabilized Galerkin FEM:

$$\textit{Find } U_{\tau,h}^m \in \Pi_{\tau,h}(I_m) \quad \textit{s.t.} \quad A_{SG}^m(U_{\tau,h}^m, v) = F_{SG}^m(v) \quad \forall v \in \Pi_{\tau,h}(I_m), \qquad (11)$$

$$A^m_{SG}(v,w) \equiv A^m_G(v,w) + \sum_K \delta^m_K \left(L^m_{\epsilon,\tau} v, \vec{b}_m \cdot \nabla w\right)_K ,$$
$$F^m_{SG}(w) \equiv F^m_G(w) + \sum_K \delta^m_K \left(f^m_\tau, \vec{b}_m \cdot \nabla w\right)_K .$$

The usual Galerkin FEM corresponds to $\delta^m_K = 0$. (11) is approximately consistent to the continuous problem (6). The parameter set $\{\delta^m_K\}$ is determined in such a way that (11) yields a stable and accurate method. If only steady state calculations (with time-independent coefficients b and c) are considered, one can replace the semidiscrete residual $L^m_{\epsilon,\tau} v - f^m_\tau$ by $L_\epsilon v - f$, i.e. the approximation $\frac{1}{\tau_m}(u^m - u^{m-1})$ of $\partial_t u$ is not used in the stabilizing terms.
We denote by $A^{m;k}_{SG}(\cdot,\cdot)$ and $F^{m;k}_{SG}(\cdot)$ the restrictions of $A^m_{SG}(\cdot,\cdot)$ resp. $F^m_{SG}(\cdot)$ to Ω_k. Furthermore, $\Pi^k_{\tau,h}(I_m)$ is the restriction of $\Pi_{\tau,h}(I_m)$ to $I_m \times \Omega_k$. The discrete DDM (for $n \in \mathbf{N}$) consists in the parallel solution of problems on $I_m \times \Omega_k$:

$$\textit{Find } U^m_{n;k} \in \Pi^k_{\tau,h}(I_m) \quad \textit{s.t.} \quad \forall v \in \Pi^k_{\tau,h}(I_m), \, \forall \phi \in \Pi_{\tau,h}(I_m)|_{\Gamma_{kj}}$$
$$A^{m;k}_{SG}(U^m_{n;k}, v) + \langle z^m_k U^m_{n;k}, v\rangle_{\Gamma_k} = F^{m;k}_{SG}(v) + \sum_{j(\neq k)} \langle \Lambda^m_{n-1;jk}, v\rangle_{\Gamma_{kj}}, \qquad (12)$$
$$\langle \Lambda^m_{n;kj}, \phi\rangle_{\Gamma_{kj}} = \langle (z^m_k + z^m_j) U^m_{n;k} - \Lambda^m_{n-1;jk}, \phi\rangle_{\Gamma_{kj}}. \qquad (13)$$

The subproblems are well-posed: Standard arguments for stabilized FEM for elliptic problems yield for all $v \in \Pi^k_{\tau,h}(I_m)$ the coercivity estimate

$$2A^{m;k}_{SG}(v,v) \geq |||v|||^2_{SG;\Omega_k} \equiv \epsilon |v|^2_{1,2,\Omega_k} + ||\sqrt{\tilde{c}_m} v||^2_{0,2,\Omega_k} + \sum_{K \in \Omega_k} \delta^m_K ||\vec{b}_m \cdot \nabla v||^2_{0,2,K}.$$

Existence and uniqueness of $U^m_{n;k}$ follow from Lax-Milgram Lemma.

A-priori and a-posteriori estimates for the Robin-Robin method

Here we consider the simplified case $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$ with $\Omega_1 \cap \Omega_2 = \emptyset$, $\partial\Omega_i \cap \partial\Omega \neq \emptyset$ and $H_i \equiv \operatorname{diam}(\Omega_i)$. The interface is $\Gamma \equiv \partial\Omega_1 \cap \partial\Omega_2$. Furthermore, set $\mathbf{V}(\Omega_i) \equiv \mathbf{V}|_{\Omega_i}$, $\mathbf{V}_{i,h} \equiv \mathbf{V}_h|_{\Omega_i}$, $W \equiv H^{1/2}_{00}(\Gamma)$, $W_h \equiv \mathbf{V}_h|_\Gamma$. We can prove [Lub98]

Theorem 1 *Let $0 < Z \leq z^m_1 = z^m_2 \in L^\infty(\Gamma)$, $\Lambda^m_0 \in L^2(\Gamma)$. Then the solutions $(U^m_{n;k})$ of (12)-(13) converge to the solution of (11) as*

$$|||U^m_{n;k} - U^m|_{\Omega_k}|||_{SG,\Omega_k} \to 0, \quad n \to \infty, \quad k = 1, 2.$$

In order to have a stopping criterion for the DD iteration in each time-step, we derive an a-posteriori error estimate which controls the subdomain error $U^m_{n;k} - U^m_{\tau,h}|_{\Omega_k}$ w.r.t. $|||\cdot|||_{SG;\Omega_k}$ in terms of the interface error. Later on we apply

Lemma 1 *[Lub98] The trace mapping $Tr_i : V(\Omega_i) \to L^2(\Gamma)$, the extension operator $Tr_i^{-1} : W \to \mathbf{V}(\Omega_i)$ and the injection of $L^2(\Gamma)$ into W are continuous:*

$$||Tr_i v||_{0,2,\Gamma} \leq C_{Tr_i} |v|_{1,2,\Omega_i} \quad \forall v \in \mathbf{V}(\Omega_i), \quad C_{Tr_i} \approx \sqrt{H_i}, \qquad (14)$$
$$|Tr_i^{-1}\phi|_{1,2,\Omega_i} \leq C_{Tr_i^{-1}} ||\phi||_W \quad \forall \phi \in W, \quad C_{Tr_i^{-1}} \approx 1, \qquad (15)$$
$$||\phi||_{0,2,\Gamma} \leq C_I ||\phi||_W \quad \forall \phi \in W, \quad C_I \approx \sqrt{H_i}. \qquad (16)$$

Furthermore we need Friedrichs inequality

$$||v||_{0,2,\Omega_i} \leq C_{F,i} |v|_{1,2,\Omega_i} \qquad \forall v \in \mathbf{V}(\Omega_i) \qquad C_{F,i} \approx H_i. \qquad (17)$$

For convenience, we skip index m of the time slab, e.g. $A^k_{SG}(\cdot,\cdot) := A^{m;k}_{SG}(\cdot,\cdot)$ resp. $F^k_{SG}(\cdot) := F^{m;k}_{SG}(\cdot)$. Let U_i be the restriction of the global discrete solution of (11) to Ω_i. Then the *a-posteriori estimate* reads

Theorem 2 *Let $U \in \mathbf{V}_h$ with $U|_{\Omega_i} =: U_i \in \mathbf{V}_{i,h}$ resp. (U_1^n, U_2^n) be the solutions of the global discrete problem (11) resp. of the two domain variant of the discrete DDM (12)-(13). Then there exists a constant $C > 0$ such that for all $n \in \mathbf{N}$ holds*

$$|||U_1^{n+1} - U_1|||_{SG,\Omega_1} + |||U_2^n - U_2|||_{SG,\Omega_2} \leq C \max_{i=1,2} K_{i,i} \; \|U_1^{n+1} - U_2^n\|_W, \tag{18}$$

$$K_{i,1} := \epsilon^{-1/2}\|z_i\|_{0,\infty,\Gamma} \; C_I \; C_{Tr_i}, \tag{19}$$

$$K_{i,2} := \frac{1}{2\sqrt{\epsilon}} \sup_{t\in I_m} \|\vec{b}\cdot\vec{n}_i\|_{0,\infty,\Gamma} C_I \; C_{Tr_i} \tag{20}$$

$$+ C_{Tr_i^{-1}} \left(\sqrt{\epsilon} + \left(\frac{1}{\sqrt{\tau_m}} + \sup_{t\in I_m} \sqrt{\|c\|_{0,\infty,\Omega_i}} \right) C_{F,i} + \sqrt{\tau_m} \sup_{t\in I_m} \|\vec{b}\|_{0,\infty,\Omega_i} \right).$$

The estimate remains valid if Ω_1 and Ω_2 are interchanged.

Sketch of the proof: (i) Problem (11) is equivalent to find $(U_1, U_2) \in \mathbf{V}_{1,h} \times \mathbf{V}_{2,h}$ and $\Lambda_1, \Lambda_2 \in W_h^*$ s.t. $U_1 = U_2$ on Γ and

$$A^i_{SG}(U_i, V_i) + \langle z_i U_i, V_i\rangle_\Gamma = F^i_{SG}(V_i) + \langle \Lambda_j \; , V_i\rangle_\Gamma \qquad \forall V_i \in \mathbf{V}_{i,h}, \tag{21}$$

$$\langle \Lambda_i, \phi\rangle_\Gamma = \langle (z_i + z_j)\, U_i - \Lambda_j, \phi\rangle_\Gamma \qquad \forall \phi \in W_h. \tag{22}$$

(ii) Set $E_i^n := U_i^n - U_i$ and $\eta_i^n := \Lambda_i^n - \Lambda_i$. Taking the difference of (21),(22) and (12),(13), we obtain for $i \neq j$ the error problem

$$A^i_{SG}(E_i^{n+1}, V_i) + \left\langle z_i E_i^{n+1}, V_i\right\rangle_\Gamma = \left\langle \eta_j^n, V_i\right\rangle_\Gamma \quad \forall V_i \in \mathbf{V}_{i,h}, \tag{23}$$

$$\left\langle \eta_i^{n+1}, \phi\right\rangle_\Gamma = \left\langle (z_i + z_j) E_i^{n+1} - \eta_j^n, \phi\right\rangle_\Gamma \quad \forall \phi \in W. \tag{24}$$

(iii) Let $z_i \in L^\infty(\Gamma)$ and $K_{i,j}$ as in (19),(20). Then basically using Lemma (1) we obtain for all $w \in W_h$ that

$$\left|\langle z_i w, E_i^{n+1}\rangle_\Gamma\right| \leq K_{i,1} \; |||E_i^{n+1}|||_{SG,\Omega_i}\|w\|_W, \tag{25}$$

$$\left|\langle \eta_j^{n-1} - z_i E_i^n, w\rangle_\Gamma\right| \leq K_{i,2} \; |||E_i^n|||_{SG,\Omega_i}\|w\|_W, \qquad i \neq j. \tag{26}$$

(iv) Starting from (23), (24), we find after some manipulations

$$A^1_{SG}(E_1^{n+1}, V_1) + A^2_{SG}(E_2^n, V_2) = \left\langle \eta_2^n - z_1 E_1^{n+1}, V_1\right\rangle_\Gamma + \left\langle \eta_1^{n-1} - z_2 E_2^n, V_2\right\rangle_\Gamma$$

$$= \left\langle z_1 (E_2^n - E_1^{n+1}), V_1\right\rangle_\Gamma + \left\langle \eta_1^{n-1} - z_2 E_2^n, V_2 - V_1\right\rangle_\Gamma. \tag{27}$$

Set now $V_1 := E_1^{n+1}$, $V_2 := E_2^n$ in (27). We derive lower and upper bounds using the coercivity of A^i_{SG} and step (iii)

$$\frac{1}{4}\left(|||E_1^{n+1}|||_{1,SG} + |||E_2^n|||_{2,SG}\right)^2 \leq A^1_{SG}(E_1^{n+1}, E_1^{n+1}) + A^2_{SG}(E_2^n, E_2^n)$$

$$\leq \max(K_{1,1}; K_{2,2}) \left(|||E_1^{n+1}|||_{1,SG} + |||E_2^n|||_{2,SG}\right) \|E_2^n - E_1^{n+1}\|_W. \tag{28}$$

Noting that $E_2^n - E_1^{n+1} = U_1^{n+1} - U_2^n$ on Γ this implies the assertion. $\square$

The a-posteriori estimate of Theorem 2 gives some information on a suitable choice of the parameter function z_i. A reasonable upper bound is then

$$||z_1||_{0,\infty,\Gamma} \leq \frac{\sqrt{\epsilon}}{C_I C_{Tr_1}} K_{2,2}, \qquad ||z_2||_{0,\infty,\Gamma} \leq \frac{\sqrt{\epsilon}}{C_I C_{Tr_2}} K_{1,2}. \tag{29}$$

According to Lemma 1, we can rewrite (29) for $i \neq j$ as follows:

$$\begin{aligned} ||z_i||_{0,\infty,\Gamma} &\leq \frac{1}{2} \sup_{t \in I_m} ||\vec{b} \cdot \vec{n}_i||_{0,\infty,\Gamma} + \frac{\epsilon}{\sqrt{H_1 H_2}} \\ &+ \sqrt{\frac{\epsilon}{H_1 H_2}} \left(\left(\frac{1}{\sqrt{\tau_m}} + \sup_{t \in I_m} \sqrt{||c||_{0,\infty,\Omega_j}} \right) H_j + \sqrt{\tau_m} \sup_{t \in I_m} ||\vec{b}||_{0,\infty,\Omega_2} \right). \end{aligned} \tag{30}$$

Finally we select z_i such that the first r.h.s. term in (30) is replaced with $\frac{1}{2}|\vec{b} \cdot \vec{n}_i|$. Observe that time step τ_m is a critical parameter in (30).

Extension to the DG(1)-case

Let us briefly discuss the application to the *time-accurate* DG(1)-variant: We use for the semidiscrete problem (5) ansatz and test functions

$$u^m(t) := \frac{t_{m+1} - t}{\tau_m} u_0^m + \frac{t - t_m}{\tau_m} u_1^m, \quad v^m(t) := \frac{t_{m+1} + t_m - 2t}{\tau_m} v_0^m + v_1^m$$

with $u_s^m, v_s^m \in \mathbf{V_h}$, $s = 0, 1$. Then we arrive at the elliptic system

$$\frac{\tau_m}{6} B_G(u_0^m - u_1^m, v_0^m) + (u_0^m, v_0^m) = \frac{\tau_m}{6}(f^m - f^{m+1}, v) + (u_1^{m-1}, v), \tag{31}$$

$$\frac{\tau_m}{2} B_G(u_0^m + u_1^m, v_1^m) + (u_1^m, v_1^m) = \frac{\tau_m}{2}(f^m + f^{m+1}, v) + (u_1^{m-1}, v). \tag{32}$$

We propose an efficient iterative decoupling of this system starting from the second equation for $u_1^{m;l}$ with an initial guess for $u_0^{m;0}$ and then solving the first equation for $u_0^{m;l}$ using the last approximation $u_1^{m;l}$. As a result, in each time slab we have to solve a sequence of two different stationary problems. This is again done using the full discretization by a stabilized FEM and the non-overlapping DD method.

Numerical results

We present some results for problem (1)–(2) in $\Omega = (0,1)^2$ showing a reasonable performance of the (discrete) DD method for the time-accurate case. The computations are performed on a triangular mesh with $h = \frac{1}{128}$ with a 2×2 partition for the DD case and using two iterations of decoupling of (31), (32).

Example 1 Consider the heat equation with $\epsilon = 1$, $\vec{b} = \vec{0}$, $c = 0$ and the exact solution $u = \frac{x^2+y^2}{2} \sin(10^3 t)$ which is highly oscillatory in time. In Fig.1 we present the discrete L^2-error norms for the sequential case with different time-steps which shows the better accuracy of the DG(1) method and for the DDM with either fixed

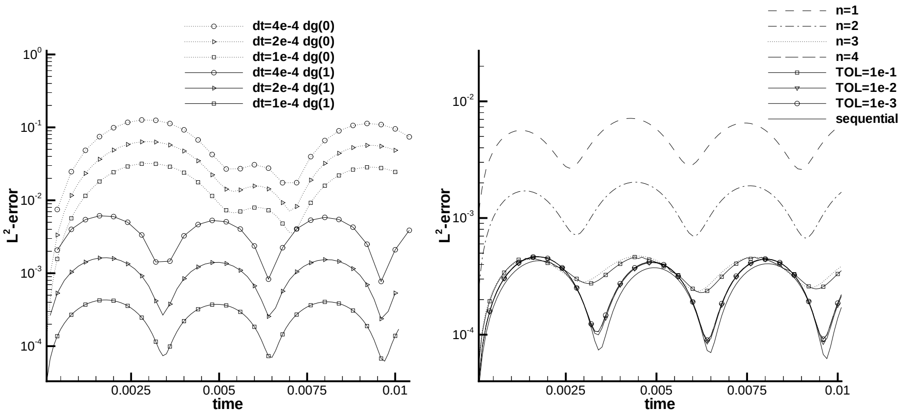

Figure 1 Example 1 (a) sequential (b) DD method

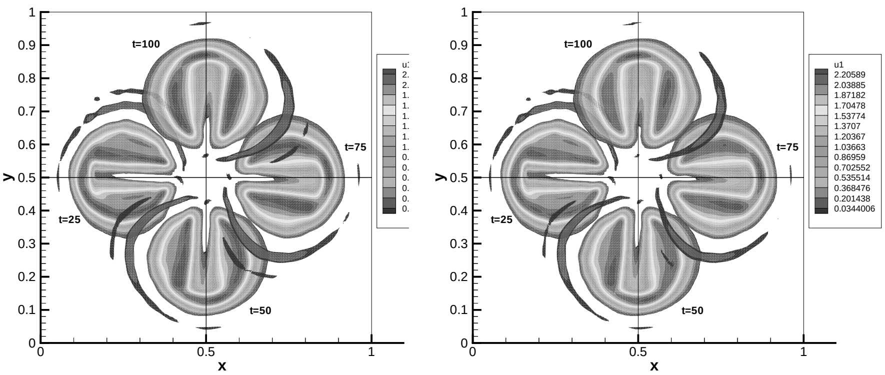

Figure 2 Example 2 (a) sequential (b) DD method

number of DD-steps and the stopping criterion (interface error $\leq$ TOL) arising from the a-posteriori analysis (cf. Thm. 2).

Example 2 Consider an advection-dominated problem with $\epsilon = 10^{-8}$, $c = 0$, $\vec{b} = (-x_2 + 0.5, x_1 - 0.5)^T$. The initial condition is a circular cylinder with a slot being advected with the rotating flow field. In the limit case $\epsilon = 0$, we obtain a periodic solution with period $T = 2\pi$. This is a hard test problem for both the discretization and the interface condition of the DDM. The solution after one rotation without and with domain decomposition is presented in Figure 2. Typical wiggles of the discrete solution around the discontinuity of the initial profile are observed. No reflections can be seen for the DDM thus showing the high quality of the interface condition.

Concluding remarks

The semidiscretization of parabolic problems by the discontinuous Galerkin method results in a sequence of elliptic problems. We propose the solution of these problems using a stabilized FEM and a non-overlapping DDM of Robin/ Robin type. An a-posteriori result allows to control the convergence of the discrete solutions in the subdomains via convergence of the interface data. Furthermore we obtain some information how to design the interface condition depending e.g. on a variable time step. This approach can be easily extended to certain systems of advection-reaction-diffusion equations (for the linear case cf. [Alo98]).
Our main interest is currently the application to coupled models of the incompressible Navier-Stokes equations with scalar advection-diffusion-reaction problems (e.g. from the k/ϵ turbulence model). The research code *Parallel NS* has been implemented on different platforms under a message passing system, fore more detail for the stationary case cf. [Ott98] and [Mue99] for the time-dependent case. Furthermore we refer to [Ott98] for some foundation of the DDM for the linearized Navier-Stokes problem.

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