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V-cycle Multigrid Methods for Wilson Nonconforming Element

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INTRODUCTION

In this paper, we present some optimal V-cycle multigrid algorithms for Wilson's element. Two types of multigrid methods are considered.

The first one uses the bilinear conforming element space as the coarse grid correction space. Similar to [10], here we also consider the effect of numerical integrations. Both the standard and a weak quadrature formula are discussed. As in [2], the quadrature formula on the finest level is also used for all coarse levels. We show that the V-cycle multigrid method for Wilson's element with the standard or the weak quadrature formula needs only one smoothing iteration on each level. Moreover, the result holds for problems without full elliptic regularity (e.g., an L-shaped domain or a domain with a crack boundary). We mention here that in [2] only the standard quadrature formula for the conforming case is studied.

The second type of V-cycle multigrid method for Wilson's element is the so-called nonconforming multigrid, i.e., the both fine mesh and coarse mesh space are the same nonconforming finite element space. Little work has been done in this direction. Many existing nonconforming multigrid methods have been shown to converge only for W-cycle with sufficiently many smoothing steps on each level (cf. [4] for details). One interesting exception is the so-called rotated Q1 nonconforming finite element; see [7], where it is shown that the W-cycle multigrid with any number of smoothing steps converges for Laplace equation. In [6], a V-cycle nonconforming multigrid is

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proposed for the rotated Q1 element and P1 nonconforming element through a modified quadratic form on the coarse mesh space. However, the convergence rate is dependent on the mesh level J. Meanwhile, the numerical experiment [6] shows that the nonconforming multigrid still converges without changing the quadratic form. We do hope that a good choice of an intergrid operator may result in a convergent V-cycle nonconforming multigrid method even without changing the quadratic form on the coarse meshes. It has been done with Wilson's element. Here we define a simple intergrid operator with good stability properties for Wilson's element. Using this operator, an optimal V-cycle nonconforming multigrid with one smoothing step on each level converges for Laplace's equation, and the convergence rate is independent of the mesh level J. It seems to be the first optimal V-cycle nonconforming multigrid method. Similar to the rotated Q1 nonconforming element, the W-cycle nonconforming multigrid, with any number of smoothing steps, is also shown to be convergent for Wilson's element.

A MODEL PROBLEM AND WILSON'S ELEMENT

We consider the second order problem

$$\begin{cases} -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j}) & = f \quad in \quad \Omega, \\ \qquad\qquad\qquad u & = 0 \quad on \quad \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset R^2$ is a bounded polygonal domain, $f \in L^2(\Omega)$, and all function a_{ij} and f are smooth enough. We assume that the differential operator is uniformly elliptic, i.e., there exist positive numbers c, C such that

$$c\xi^t\xi \leq \sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j \leq C\xi^t\xi \quad \forall (x,y) \in \Omega, \xi \in R^2. \tag{2}$$

If Ω is a convex polygon, then the solution u satisfies the so-called full regularity assumption

$$||u||_2 \leq C||f||_0. \tag{3}$$

However, the results of next section are independent of (3).

Here and throughout this paper, c and C (with or without subscript) denote generic positive constants, independent of the mesh parameter J and h_J which will be defined below.

The variational form of (1) is to find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega), \tag{4}$$

where the bilinear form

$$a(u,v) = \int_\Omega \sum_{i,j=1}^{2} a_{ij} \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} dx \quad \forall u, v \in H^1(\Omega).$$

Let h_0 and $\Gamma_{h_0} := \Gamma_0$ be given, where Γ_0 is a partition of Ω into rectangles and h_0 is the maximum diameter of all the rectangles. For each integer, $1 \leq k \leq J$, let $h_k = 2^{-k}h_0$ and $\Gamma_{h_k} := \Gamma_k$ be constructed by connecting the midpoints of two opposite edges of each rectangle in Γ_{k-1}, $k = 1, ..., J$. The finest grid is Γ_J.

On each level k, we define Wilson's nonconforming element space as follows:

First, on the reference rectangle $\hat{K} = [-1, 1] \times [-1, 1]$, the shape functions of Wilson's element is a quadratic polynomial $\hat{p}$, defined by the values of $\hat{p}$ at four vertices of $\hat{K}$ and the mean values of $\frac{\partial^2 \hat{p}}{\partial \xi^2}$ and $\frac{\partial^2 \hat{p}}{\partial \eta^2}$ on $\hat{K}$. We have

$$\begin{aligned} \hat{p} \; &= \frac{(1+\xi)(1+\eta)}{4} p_1 + \frac{(1-\xi)(1+\eta)}{4} p_2 + \frac{(1-\xi)(1-\eta)}{4} p_3 + \frac{(1+\xi)(1-\eta)}{4} p_4 \\ &+ \frac{1}{2}(\xi^2 - 1)\frac{\partial^2 \hat{p}}{\partial \xi^2} + \frac{1}{2}(\eta^2 - 1)\frac{\partial^2 \hat{p}}{\partial \eta^2}, \end{aligned} \tag{5}$$

where p_i is the value of $\hat{p}$ at the vertex $\hat{A}_i$.

Then, for each rectangle $\tau_K \in \Gamma_k$, using an affine transformation, we can define a Wilson's element. We denote Wilson's finite element space on Γ_k by V_k. It is known that this element is not continuous on interelement boundaries, so it is nonconforming for a second order problem.

In the following, we also need the bilinear element space on Γ_k, which is denoted by U_k. we note that U_k is a conforming element.

Define the discrete bilinear form over the space V_k by

$$a_k^E(u, v) = \sum_{\tau_K \in \Gamma_k} \int_{\tau_K} \sum_{i,j=1}^{2} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \quad \forall u, v \in V_k, \tag{6}$$

where superscript E indicates that all integrals are computed exactly.

Next we define a discrete semi-norm by

$$|u|_{i,k}^2 = \sum_{\tau_k \in \Gamma_k} |u|_{i,\tau_k}^2 \quad \forall u \in V_k, \quad i = 1, 2. \tag{7}$$

It is known that this semi-norm is also a norm over V_k. Then, the Wilson's nonconforming approximation of (1) on Γ_J is to find $u_J^* \in V_J$ such that

$$a_J^E(u_J^*, v_J) = (f, v_J) \quad \forall v_J \in V_J. \tag{8}$$

It is known that there exists a unique solution of (8); see [5].

We now approximate the integral in $a_J^E(\cdot, \cdot)$ by a quadrature scheme Q_K over each $K \in \Gamma_J$. We first consider the reference rectangle $\hat{K}$ and approximate the integral $\int_{\hat{K}} \hat{\phi}(\hat{x})d\hat{x}$ as follows:

$$\int_{\hat{K}} \hat{\phi}(\hat{x})d\hat{x} \approx \sum_{l=1}^{L} w_l \hat{\phi}(b_l),$$

where w_l are positive weights and $b_l \in \hat{K}$ are quadrature points. We then define the quadrature scheme on $K \in \Gamma_J$ by

$$\int_K \phi(x)dx \approx \sum_{l=1}^{L} w_{K,l}\phi(b_{K,l}) \equiv Q_K[\phi], \tag{9}$$

where $\phi(x) = \hat{\phi}(\hat{x})$, the weights $w_{K,l}$ and quadrature points $b_{K,l}$ are defined in terms of the w_l and b_l by means of the affine mapping from $\hat{K}$ onto K that takes each x in K into $\hat{x}$ in $\hat{K}$.

The quadrature error functional is

$$E_K(\phi) = \int_K \phi(x)dx - Q_K[\phi]. \tag{10}$$

Using the quadrature scheme, we approximate $a_J^E(\cdot,\cdot)$, $(f,\cdot)$ by $a_J(\cdot,\cdot)$, $(f,\cdot)_J$ as follows

$$a_J(u,v) = \sum_{K\in\Gamma_J}\sum_{i,j=1}^{2} Q_K[a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}] \tag{11}$$

and

$$(f,v)_J = \sum_{K\in\Gamma_J} Q_K[fv]. \tag{12}$$

Now the Wilson's and the bilinear element approximation u_J and w_J with the quadrature scheme Q_K are defined, respectively by

$$a_J(u_J,v) = (f,v)_J \quad \forall v \in V_J \tag{13}$$

and

$$a_J(w_J,v) = (f,v)_J \quad \forall v \in U_J. \tag{14}$$

Following [5], a standard quadrature scheme Q_K satisfies the following two assumptions: **(H1)** The quadrature error funtional

$$E_K(\phi) = 0 \quad \forall \phi \in P_2(K),$$

where $P_2(K)$ is the quadratic polynomial space on K.

(H2)

$$\textit{All} \quad \textit{weights} \quad w_{K,l} > 0 \quad \textit{and} \quad \frac{1}{4}\sum_l w_{K,l} = 1.$$

Meanwhile, instead of (H1), there exist other weak assumptions, like

(H3) The union of all quadrature points b_l on $\hat{K}$ contains a $P_1(\hat{K})$ unisolvent subset.

(H4) The quadrature scheme Q_K satisfies:

$$E_K(\phi) = 0 \quad \forall \phi \in Q_1(K),$$

where $Q_1(K)$ is the bilinear polynomial space on K.

Here are two examples.

Scheme 1 (the standard formula with (H1) and (H2).)

$$\int_{\hat{K}} \hat{\phi}(\hat{x})d\hat{x} \approx \sum_{l=1}^{4} \hat{\phi}(B_l),$$

where $B_1 = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, $B_2 = (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$, $B_3 = (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, $B_4 = (-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$.

Scheme 2 (the weak formula with (H2), (H3) and (H4).)

$$\int_{\hat{K}} \hat{\phi}(\hat{x})d\hat{x} \approx \sum_{l=1}^{4} \hat{\phi}(A_l),$$

where $A_1 = (1,1)$, $A_2 = (1,-1)$, $A_3 = (-1,1)$, $A_4 = (-1,-1)$.

V-CYCLE MULTIGRID WITH NUMERICAL INTEGRATION FOR WILSON'S ELEMENT

In this section, we describe a V-cycle multigrid method with numerical integrations for Wilson's nonconforming element. An optimal V-cycle multigrid with only one smoothing step on each level is derived. In the following, we make use of the bilinear conforming element space U_k, $k = 1, ..., J-1$ as the coarse grid correction space.

It is obvious that

$$U_1 \subset U_2 \subset \cdots \subset U_{J-1} \subset V_J.$$

On each level $k = 1, ..., J$, we introduce the operator $A_{U_k} : U_k \to U_k$, $k = 1, ..., J-1$, by

$$(A_{U_k} v, w) = a_J(v, w) \quad \forall v, w \in U_k \tag{15}$$

and $A_J : V_J \to V_J$ by

$$(A_J v, w) = a_J(v, w) \quad \forall v, w \in V_J. \tag{16}$$

We note that we here apply the quadrature formula on the level J to all coarse levels as in [2].

For $k = 1, ..., J-1$, we define the projection operator $P_k : V_J \to U_k$ by

$$a_J(P_k v, w) = a_J(v, w) \quad \forall v \in V_J, w \in U_k, \tag{17}$$

and $P_k^0 : L^2(\Omega) \to U_k$ by

$$(P_k^0 v, w) = (v, w) \quad \forall v \in V_J, w \in U_k. \tag{18}$$

In order to define a V-cycle multigrid algorithm, we must choose on each level k a smoothing operator $R_{U_k} : U_k \to U_k$, $k = 2, ..., J-1$. We set

$$K_k = I - R_{U_k} A_{U_k}$$

and

$$\bar{R}_k = (I - K_k^* K_k) A_k^{-1},$$

where $R_{U_k}^t$ denotes the adjoint of R_{U_k} with respect to $(\cdot, \cdot)$, and $K_k^* = I - R_{U_k}^t A_{U_k}$ denotes the adjoint of K_k with respect to the $a_J(\cdot, \cdot)$. We set $R_{U_1} = A_{U_1}^{-1}$, i.e., we solve (3.1) on the coarsest space. Finally, set $T_k = R_{U_k} P_k^0 A_J$. Then by (17), (18), it is easy to check that

$$P_k^0 A_J = A_{U_k} P_k. \tag{19}$$

So we have $T_k = R_{U_k} A_{U_k} P_k$. Note that $T_1 = P_1$.

Now we define the following multigrid algorithm.

Multigrid Algorithm 1.

If $J = 1$, set $B_1 = A_1^{-1}$ and if $J > 1$, define $B_J g$ for $g \in V_J$ by

(1) Set $x_J = R_J g$.

(2) Define $B_J g = x_J + q$, where $q \in U_{J-1}$ is given by

$$q = N_{J-1} P_{J-1}^0 (g - A_J x_J).$$

Here N_{J-1} is defined as follows: set $N_1 = A_{U_1}^{-1}$. For $2 < k \le J-1$, assume that N_{k-1} has been defined and define $N_k g$ for $g \in U_k$ by

(i) Set $x_k = R_{U_k} g$.

(ii) Define $N_k g = x_k + q$, where $q \in U_{k-1}$ is given by

$$q = N_{k-1} P_{k-1}^0 (g - A_{U_k} x_k).$$

We can prove that

$$E := I - B_J A_J = (I - T_1)(I - T_2) \cdots (I - T_J). \quad (20)$$

Then, we have

Theorem 1. *For the Wilson's element, there exists a constant $C_R > 1$ dependent of the constants C_0 and M, but independent of h_J and J such that*

$$a_J(Ev, Ev) \le (1 - \frac{1}{C_R}) a_J(v, v) \quad \forall v \in V_J.$$

It is seen that the contraction factor $1 - \frac{1}{C_R} < 1$, is independent of h_J and J, so we have an optimal multigrid algorithm.

Remark 1. From Multigrid Algorithm 1, we see that our V-cycle multigrid method for Wilson's element with the standard or the weak quadrature formula needs only one smoothing iteration on each level. Moreover, the result holds for problems without full elliptic regularity (e.g., an L-shaped domain or a domain with a crack boundary).

V-CYCLE NONCONFORMING MULTIGRID FOR WILSON'S ELEMENT

In this section, we consider a nonconforming multigrid method, where both the coarse and the fine mesh space use the same nonconforming element space. In the following, we construct a suitable intergrid transfer operator I_k for Wilson's element. Using this operator, an optimal V-cycle nonconforming multigrid method can be obtained. For convenience, we do not consider numerical intergration, and simply write the bilinear form $a_k^E(\cdot,\cdot)$ in (6) as $a_k(\cdot,\cdot)$.

We define the operator $A_k : V_k \to V_k$ on each level k by

$$(A_k v, w) = a_k(v, w) \quad \forall v, w \in V_k, k = 1, ..., J.$$

In the nonconforming multigrid method, the intergrid transfer operator plays an important role. For Wilson's element, let π_{k-1} be the bilinear interpolation operator; it is easy to check that for any $v \in V_{k-1}$,

$$\pi_{k-1} v \in U_{k-1} \subset U_k \subset V_k. \quad (21)$$

So we define the intergrid transfer operator $I_k : V_{k-1} \to V_k$ as follows

$$I_k := \pi_{k-1}. \quad (22)$$

We can prove that the operator $I_k : V_{k-1} \to V_k$ satisfies the inequality

$$a_k(I_k v, I_k v) \le a_{k-1}(v, v) \quad \forall v \in V_{k-1}. \quad (23)$$

Now as in above section, we define the projection operator $P'_{k-1} : V_k \to V_{k-1}$ and $Q^0_{k-1} : V_k \to V_{k-1}$ by

$$a_{k-1}(P'_{k-1}v, w) = a_k(v, I_k w) \quad \forall v \in V_k, w \in V_{k-1}, k = 1, ..., J, \tag{24}$$

and

$$(Q^0_{k-1}v, w)_{k-1} = (v, I_k w) \quad \forall v \in V_k, w \in V_{k-1}, k = 1, ..., J. \tag{25}$$

Finally, let $R_k : V_k \to V_k$, for $k = 1, ..., J$, be a linear smoothing operator and let R_k^t be the adjoint of R_k with respect to $(\cdot, \cdot)$. Define

$$R_k^{(l)} = \begin{cases} R_k & l \quad odd, \\ R_k^t & l \quad even. \end{cases}$$

A general multigrid operator $B_k : V_k \to V_k$ can be defined recursively as follows:

Multigrid Algorithm 2

Let $1 \leq k \leq J$ and let p be a positive interger. Set $B_0 = A_0^{-1}$. Assume that B_{k-1} has been defined and define $B_k g$ for $g \in V_k$ by

(1) Set $x_0 = 0$ and $q^0 = 0$.

(2) Define x^l for $l = 1, ..., m(k)$ by

$$x^l = x^{l-1} + R_k^{(l+m(k))}(g - A_k x^{l-1}).$$

(3) Define $y^{m(k)} = x^{m(k)} + I_k q^p$, where q^i for $i = 1, ..., p$ is determined by

$$q^i = q^{i-1} + B_{k-1}(Q^0_{k-1}(g - A_k x^{m(k)}) - A_{k-1}q^{i-1}).$$

(4) Define y^l for $l = m(k) + 1, ..., 2m(k)$ by

$$y^l = y^{l-1} + R_k^{(l+m(k))}(g - A_k y^{l-1}).$$

(5) Set $B_k g = y^{2m(k)}$.

In the Multigrid Algorithm 2, $m(k)$ gives the number of pre- and post-smoothing steps and can vary as a function of k. If $p = 1$, we have a V-cycle method. If $p = 2$, we have a W-cycle method.

The convergence rate for the multigrid algorithm 2 on the kth level is measured by a convergence factor δ_k which satisfies

$$|a_k((I - B_k A_k)v, v)| \leq \delta_k a_k(v, v). \tag{26}$$

We then have

Theorem 2. *For the Wilson's nonconforming element, define B_k by $p = 1$ and $m(k) = m$ for all k in the algorithm 2. Then if $a_{ii} = 1$, $a_{ij} = 0$, $i \neq j$, there exists $C > 0$, independent of k, such that*

$$\delta_k \leq \delta = \frac{C}{C + m}.$$

It is seen that the convergence factor δ_k in (26) is independent of the level l, we then obtain an optimal V-cycle multigrid method with one smoothing step for the Wilson's nonconforming element for Laplace equation.

Remark 2. We can also obtain an optimal convergence order of the W-cycle multigrid method with any number of smoothing steps for Wilson's element.

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