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Subspace Correction Methods for Convex Optimization Problems

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Introduction

Domain decomposition and multigrid methods have been intensively studied for linear partial differential equations. Recent research, see for example [Xu92], reveals that domain decomposition and multigrid methods can be analysed using a same framework, see also [BPWX91], [SBG96], [GO95]. The present work uses this framework to analyse the convergence of two algorithms for convex optimization problems. Our emphasis is on nonlinear problems instead of linear problems. The algorithms reduce to the standard additive and multiplicative Schwarz methods when used for linear partial differential equations.

Researches for domain decomposition and multigrid methods have been mostly concentrating on linear elliptic and parabolic partial differential equations. Extension to more difficult problems have been considered by some recent works. In this work, a general nonlinear convex minimization problems is considered. The proposed algorithms can be used for nonlinear partial differential equations, optimal control problems related to partial differential equations and eigenvalue problems [CSar] [Sha97]. The space decomposition can be a domain decomposition method, a multigrid

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method or some other decomposition techniques.

Domain decomposition methods and multigrid methods have been studied for nonlinear partial differential equations by some earlier works, see [AL96], [Ban82], [DH97], [FG97], [HR89], [TE98], [Tai92], [Tai94], [Tai95a], [Tai95b], [Tai95c], [Xu94], [Xu96], etc. In comparison with the existing works, our approach has several features. For example, the proposed algorithms can be used for certain degenerated or singular nonlinear diffusion problems, i.e the nonlinear diffusion coefficient can be zero or infinity and our approach do not need extra assumption on the smoothness of the solutions. The methods work for natural domain decomposition and multigrid meshes. Moreover, only small size nonlinear problems need to be solved on the decomposed subspaces. We also emphasis that our approach is valid for general space decomposition techniques. So the applications is not restricted to domain decomposition and multigrid methods. Other space decomposition techniques can also be considered, see [SBG96], [XZ98].

The two algorithms given in this work were first proposed in [Tai92], see also [Tai94], [Tai95a], [Tai95c] and [TE98], where the qualitative convergence of the algorithms was proved, but the uniform rate of convergence was not given there.

Optimization problems and subspace correction methods

Consider the nonlinear optimization problem

$$\min_{v\in V} F(v) \ . \tag{1}$$

Here V is a reflexive Banach space and $F : V \to R$ is a convex functional. This problem has different applications, see §1

We shall use a space decomposition method to solve (1). A space decomposition method refers to a method that decomposes the space V into a sum of subspaces, i.e. there are subspaces V_i, $i = 1, 2, \cdots, m$, such that

$$V = V_1 + V_2 + \cdots + V_m \ . \tag{2}$$

Following the framework of [Xu92] for linear problems, we consider two types of subspace correction methods based on (2), namely the parallel subspace correction (PSC) method and the successive subspace correction (SSC) method.

Algorithm 1 *[A parallel subspace correction method].*

1. *Choose initial value* $u^0 \in V$ *and relaxation parameters* $\alpha_i > 0$ *such that* $\sum_{i=1}^m \alpha_i \le 1$.
2. *For* $n \ge 0$, *if* $u^n \in V$ *is defined, then find* $e_i^n \in V_i$ *in parallel for* $i = 1, 2, \cdots, m$ *such that*

$$F\left(u^n + e_i^n\right) \le F\left(u^n + v_i\right) \ , \quad \forall v_i \in V_i \ . \tag{3}$$

3. *Set* u^{n+1} *as in (4) and go to the next iteration.*

$$u^{n+1} = u^n + \sum_{i=1}^m \alpha_i e_i^n \ , \tag{4}$$

Algorithm 2 *[A successive subspace correction method].*

1. *Choose initial values* $u_i^0 = u^0 \in V$.
2. *For* $n \geq 0$, *if* $u^n \in V$ *is defined, find* $u^{n+i/m} = u^{n+(i-1)/m} + e_i^n$ *with* $e_i^n \in V_i$ *sequentially for* $i = 1, 2, \cdots, m$ *such that*

$$F\left(u^{n+(i-1)/m} + e_i^n\right) \leq F\left(u^{n+(i-1)/m} + v_i\right) \ , \quad \forall v_i \in V_i \ . \tag{5}$$

3. *Go to the next iteration.*

When $p = q = 2$, some asynchronous algorithms are proposed in [TTed] and [TT98] for nonlinear variational inequalities and corresponding rate of convergence was also analysed in [TT98].

Global convergence of the algorithms

In the following, the notation $\langle \cdot, \cdot \rangle$ is used to denote the duality pairing between V and V', here V' is the dual space of V. The functional F is assumed to be Gateaux differentiable (see [ET76]) and there are constants $K, \ L > 0$, $p \geq q > 1$ such that

$$\begin{aligned} \langle F'(w) - F'(v), w - v \rangle \ &\geq K \|w - v\|_V^p, \quad \forall w, v \in V \ , \\ \|F'(w) - F'(v)\|_{V'} \ &\leq L \|w - v\|_V^{q-1}, \quad \forall w, v \in V \ , \end{aligned} \tag{6}$$

and from which it is easy to deduce that

$$F(w) - F(v) \geq \langle F'(v), w - v \rangle + \frac{K}{p} \|w - v\|_V^p \ , \quad \forall v, w \in V \ , \tag{7}$$

$$F(w) - F(v) \leq \langle F'(v), w - v \rangle + \frac{L}{q} \|w - v\|_V^q \ , \quad \forall v, w \in V \ . \tag{8}$$

Under assumption (6), problem (1) and subproblems (3) and (5) have unique solutions, see [ET76]. For some nonlinear problems, the constants K and L depend on v and w. However, just under the condition that F is strictly convex, it has been proved in [Tai92] and [Tai95c] that the iterative solutions of Algorithm 1 and Algorithm 2 converge to the true solution. Thus, one can assume that the computed solutions are in a neighbourhood of the true solution and so the constants K and L can be assumed to be independent of v and w. In case that the functional F is only locally convex in a neighbourhood of the true solution, by choosing the initial value close enough to the true solution, it can be proved that the computed solutions stay always inside the neighbourhood that the functional F is convex (the essential techniques of the proof is contained in the proof of Lemma 4.2 and 4.3 of [KT97]), and so the results given in this work are also applicable to this kind of problems.

For simplicity, we define

$$\sigma = \frac{p}{p - q + 1}, \quad \sigma' = \frac{p}{q - 1}, \quad \text{which satisfy} \quad \frac{1}{\sigma} + \frac{1}{\sigma'} = 1.$$

Note that $\sigma \leq p$. We shall use u to denote the unique solution of (1) which satisfies $\langle F'(u), v \rangle = 0, \quad \forall v \in V$. It is an easy consequence of (7) and (8) that

$$\frac{K}{p} \|v - u\|_V^p \leq F(v) - F(u) \leq \frac{L}{q} \|v - u\|_V^q, \quad \forall v \in V \ . \tag{9}$$

Therefore, in the following, we shall use

$$d_n = F(u^n) - F(u), \ \forall n \geq 0 \ , \tag{10}$$

as a measure of the error between u^n and the true solution u. For the decomposed spaces, we assume that there exits a constant $C_1 > 0$ such that for any $v \in V$, we can find $v_i \in V_i$ to satisfy:

$$v = \sum_{i=1}^{m} v_i \ , \qquad \text{and} \qquad \left(\sum_{i=1}^{m} \|v_i\|_V^\sigma \right)^{\frac{1}{\sigma}} \leq C_1 \|v\|_V \ . \tag{11}$$

Moreover, assume that there is a $C_2 > 0$ such that there holds

$$\begin{aligned} &\sum_{i=1}^{m}\sum_{j=1}^{m} \langle F'(w_{ij} + u_i) - F'(w_{ij}), v_j \rangle \\ \leq \ & C_2 \left(\sum_{i=1}^{m} \|u_i\|_V^p \right)^{\frac{q-1}{p}} \left(\sum_{j=1}^{m} \|v_j\|_V^\sigma \right)^{\frac{1}{\sigma}} , \forall \ w_{ij} \in V, u_i \in V_i \text{ and } v_j \in V_j \ . \end{aligned}$$

The rate of convergence for Algorithm 1 can be estimated as in the following:

Theorem 1 *Assume that the space decomposition satisfies (11), (12) and the functional F satisfies (6). Set*

$$r = \frac{p(p-1)}{q(q-1)}, \quad C^* = \left[C_1 C_2 \left(\alpha_{max}^{\frac{(p-1)(q-1)}{p}} + \alpha_{min}^{-\frac{q-1}{p}} \right) K^{-1} \right]^{\frac{p}{q-1}} pK^{-1} \left(Lq^{-1} \right)^r .$$

Then for Algorithm 1 and d_n given by (10), we have:

1. *If $r = 1$ (namely $p = q$),*

$$d_{n+1} \leq \frac{C^*}{1 + C^*} d_n, \quad \forall n \geq 1 \ . \tag{12}$$

2. *If $r > 1$, then there exists an $\xi_0 = \xi_0(d_0, C^*, r) \in [0,1]$ such that*

$$d_{n+1} \leq \left(\frac{r-1}{C^*}\xi_0 + d_n{}^{1-r} \right)^{\frac{1}{1-r}} \leq \left(\frac{r-1}{C^*}(n+1)\xi_0 + d_0^{1-r} \right)^{\frac{1}{1-r}} , \quad \forall n \geq 1 \ . \tag{13}$$

See [TX98] for the details of the proofs. The estimate implies that when $r = 1$, the convergence is uniform. In case that $r > 1$, the convergence can be slow, i.e. $d_n = O\left((rn)^{-\frac{1}{r-1}} \right)$. Especially, when r is very big, $\frac{1}{1-r} \approx 0$ and the convergence can be very slow. Using that fact that $\sigma \leq p$, we see that it is impossible to have $r < 1$. In order to have $r = 1$, we must require $p = q$. The analysis given in [TE98] and [AL96] was done for $p = q = 2$.

The convergence of Algorithm 2 is similar to Algorithm 1.

Theorem 2 *Let the space decomposition satisfies (11), (12) and the functional F satisfies (6). Define*

$$r = \frac{p(p-1)}{q(q-1)}, \quad C^* = \left[\frac{C_1 C_2}{K}\right]^{\frac{p}{q-1}} \frac{p}{K}\left(\frac{L}{q}\right)^r . \tag{14}$$

1. If $r = 1$, we have

$$d_{n+1} \le \frac{C^*}{1+C^*} d_n, \quad \forall n \ge 1 . \tag{15}$$

2. If $r > 1$, then there exists an $\xi_0 = \xi_0(d_0, C^*, r) \in [0,1]$ such that

$$d_{n+1} \le \left(\frac{r-1}{C^*}\xi_0 + d_n{}^{1-r}\right)^{\frac{1}{1-r}} \le \left(\frac{r-1}{C^*}(n+1)\xi_0 + d_0^{1-r}\right)^{\frac{1}{1-r}}, \quad \forall n \ge 1 . \tag{16}$$

Overlapping domain decomposition for $W^{1,p}(\Omega)$

Let $\{\Omega_i\}_{i=1}^M$ be a shape-regular finite element division, or a coarse mesh, of Ω and Ω_i has diameter of order H. For each Ω_i, we further divide it into smaller simplices with diameter of order h. In case that Ω has a curved boundary, we shall also fill the area between $\partial\Omega$ and $\partial\Omega_H$, here $\bar{\Omega}_H = \cup_{i=1}^M \bar{\Omega}_i$, with finite elements with diameters of order h. We assume that the resulting elements form a shape regular finite element subdivision of Ω, see Ciarlet [Cia78]. We call this the fine mesh or the h-level subdivision of Ω with mesh parameter h. We denote $\Omega_h = \cup\{\mathcal{T} \in \mathcal{T}_h\}$ as the fine mesh subdivision. Let $S_0^H \subset W_0^{1,p}(\Omega_H)$ and $S_0^h \subset W_0^{1,p}(\Omega_h)$ be the continuous, piecewise r^{th} order polynomial finite element spaces, with zero trace on $\partial\Omega_H$ and $\partial\Omega_h$, over the H-level and h-level subdivisions of Ω respectively. More specifically,

$$S_0^H = \left\{ v \in W_0^{1,p}(\Omega_H) \middle| \quad v|_{\Omega_i} \in P_r(\Omega_i), \forall i \right\},$$

$$S_0^h = \left\{ v \in W_0^{1,p}(\Omega_h) \middle| \quad v|_{\mathcal{T}} \in P_r(\mathcal{T}), \forall \mathcal{T} \in \mathcal{T}_h \right\}.$$

For each Ω_i, we consider an enlarged subdomain $\Omega_i^\delta = \{\mathcal{T} \in \mathcal{T}_h, dist(\mathcal{T}, \Omega_i) \le \delta\}$. The union of Ω_i^δ covers $\bar{\Omega}_h$ with overlaps of size δ. Let us denote the piecewise r^{th} order polynomial finite element space with zero traces on the boundaries $\partial\Omega_i^\delta$ as $S_0^h(\Omega_i^\delta)$. Then one can show that

$$S_0^h = S_0^H + \sum S_0^h(\Omega_i^\delta) . \tag{17}$$

For the overlapping subdomains, assume that there exist m colors such that each subdomain Ω_i^δ can be marked with one color, and the subdomains with the same color will not intersect with each other. For suitable overlaps, one can always choose $m = 2$ if $d = 1$; $m \le 4$ if $d = 2$; $m \le 8$ if $d = 3$.Let Ω_i' be the union of the subdomains with the i^{th} color, and $V_i = \{v \in S_0^h | \quad v(x) = 0, \quad x \notin \Omega_i'\}$. By denoting subspaces $V_0 = S_0^H$, $V = S_0^h$, we find that decomposition (17) means

$$V = V_0 + \sum_{i=1}^m V_i, \tag{18}$$

and so the two-level method is a way to decompose the finite element space. Similar as in [Wid92], let $\{\theta_i\}_{i=1}^m$ be a partition of unity with respect to $\{\Omega_i'\}_{i=1}^m$, i.e. $\theta_i \in C_0^\infty(\Omega_i' \cap \Omega)$, $\theta_i \geq 0$, $\sum_{i=1}^m \theta_i = 1$ and $|\nabla \theta_i| \leq C/\delta$. Let I_h be an interpolation operator which uses the function values at the h-level nodes. For any $v \in V$, let $v_0 \in V_0$ be the solution of $(v_0, \phi_H) = (v, \phi_H), \forall \phi_H \in V_0$, and $v_i = I_h(\theta_i(v - v_0))$. They satisfy $v = \sum_{i=0}^m v_i$, and

$$\left(\|v_0\|_{1,p}^s + \sum_{i=1}^m \|v_i\|_{1,p}^s \right)^{\frac{1}{s}} \leq C(m+1)^{\frac{1}{s}} \left(1 + \left(\frac{H}{\delta} \right)^{\frac{p-1}{p}} \right) \|v\|_{1,p}, \quad \forall s > 1. \tag{19}$$

See [TX98] for the details of proofs. Using the Cauchy-Schwarz inequality, it is easy to prove by using (6) that:

$$\begin{aligned} \sum_{i=1}^m \sum_{j=1}^m \langle F'(w_{ij} + u_i) - F'(w_{ij}), v_j \rangle &\leq \sum_{i=1}^m \sum_{j=1}^m L \|u_i\|_V^{q-1} \|v_i\|_V \\ &\leq Lm^{\frac{s-1}{s}} \left(\sum_{i=1}^m \|v_i\|_V^s \right)^{\frac{1}{s}} m^{\frac{1}{\sigma}} \left(\sum_{i=1}^m \|u_i\|_V^p \right)^{\frac{q-1}{p}} \\ &\forall u_i \in V_i, \ w_{ij} \in V \text{ and } v_j \in V_j, \ \forall s > 1 \ . \end{aligned} \tag{20}$$

Estimates (19) and (20) show that for the constants in (11) and (12) only depend on the number of colors m and the ratio H/δ, i.e.

$$C_1 = C(m) \left(1 + \left(\frac{H}{\delta} \right)^{\frac{p-1}{p}} \right), \qquad C_2 = C(m).$$

Multilevel decomposition for $W^{1,p}(\Omega)$

From the space decomposition point of view, a multigrid algorithm is built upon the subspaces that are defined on a nested sequence of finite element partitions. We assume that the finite element partition $\mathcal{T}$ is constructed by a successive refinement process. More precisely, $\mathcal{T} = \mathcal{T}_J$ for some $J > 1$, and $\mathcal{T}_j$ for $j \leq J$ are a nested sequence of quasi-uniform finite element partitions, i.e. $\mathcal{T}_j$ consist of finite elements $\mathcal{T}_j = \{\tau_j^i\}$ of size h_j such that $\Omega = \cup_i \tau_j^i$ for which the quasi-uniformity constants are independent of j (cf. [Cia78]) and τ_{j-1}^l is a union of elements of $\{\tau_j^i\}$. We further assume that there is a constant $\gamma < 1$, independent of j, such that h_j is proportional to γ^{2j}.

Corresponding to each finite element partition $\mathcal{T}_j$, a finite element space $\mathcal{M}_j$ can be defined by

$$\mathcal{M}_j = \{ v \in W^{1,p}(\Omega) : v|_\tau \in \mathcal{P}_1(\tau), \quad \forall\, \tau \in \mathcal{T}_j \}.$$

Each finite element space $\mathcal{M}_j$ is associated with a nodal basis, denoted by $\{\phi_j^i\}_{i=1}^{n_j}$ satisfying $\phi_j^i(x_j^k) = \delta_{ik}$, where $\{x_j^k\}_{k=1}^{n_j}$ is the set of all nodes of the elements of $\mathcal{T}_j$. Associated with each such a nodal basis function, we define a one dimensional subspace as follows

$$\mathcal{M}_j^i = \text{span } (\phi_j^i).$$

On each level, the nodes can be colored so that the neighboring nodes are always of different colors. The number of colors needed for a regular mesh is always a bounded constant; call it m_c. Let $V_j^k,\ k=1,2,\cdots m_c$ be the sum of the subspaces $\mathcal{M}_j^i$ associated with nodes of the k^{th} color on level j. Letting $V=\mathcal{M}_J$, we have the following trivial space decomposition:

$$V=\sum_{j=1}^{J}\sum_{k=1}^{m_c} V_j^k. \tag{21}$$

Each subspace V_j^k contains some orthogonal one dimensional subspaces $\mathcal{M}_j^i$ and so the minimization problems (3) and (5) for each V_j^k can be done in parallel over the one dimensional subspaces $\mathcal{M}_j^i$. Under the assumption that

$$\langle F'(w+u)-F'(w),v\rangle \le L\|u\|^{q-1}_{1,p,supp(u)\cap supp(v)}\ \|v\|_{1,p,supp(u)\cap supp(v)}, \tag{22}$$

it was proved in [TX98] that

$$C_1 \cong J^{\frac{1}{\sigma}} \cong |\log h|^{\frac{1}{\sigma}}, \qquad C_2=C.$$

1 Some Applications

The algorithms can be used for linear second order equation

$$-\nabla\cdot(a\nabla u)=f \text{ in } \Omega\subset R^d\ , \qquad u=0 \text{ on } \partial\Omega\ ,$$

and linear fourth order equation

$$\Delta(a\Delta u)=f \text{ in } \Omega\subset R^d\ , \qquad u=0,\ \frac{\partial u}{\partial n}=0, \text{ on } \partial\Omega\ .$$

If we use Algorithm 2 for a general symmetric positive define linear problem

$$a(u,v)=(f,v), \qquad \forall v\in V.$$

then the implementation can be divided into the following steps:

Algorithm 3 *(Application to linear problems)*

1. *Choose initial values $u^0\in V$ and compute the initial residual r^0 such that $(r^0,v)=(f,v)-a(u^0,v),\forall v\in V$.*
2. *For $i=1,2,\cdots,m$, if $r^{n+\frac{(i-1)}{m}}$ is known, compute $e_i^n\in V_i$ such that*

$$a(e_i^n,v_i)=(r^{n+\frac{(i-1)}{m}},v_i)\ , \quad \forall v_i\in V_i\ . \tag{23}$$

3. *Update the residual $r^{n+\frac{i}{m}}$ such that*

$$(r^{n+\frac{i}{m}},v)=(r^{n+\frac{(i-1)}{m}},v)-a(e_i^n,v), \quad \forall v\in V\ . \tag{24}$$

4. Update the solution as in (25) and go to the next iteration.

$$u^{n+\frac{i}{m}} = u^{n+\frac{(i-1)}{m}} + e_i^n \ . \tag{25}$$

The implementation for Algorithm 1 is similar. If the subspaces V_i are associated with the overlapping domain decomposition, then equation (23) is the solving of the subdomain problems. Equations (24) and (25) are just the simple updatings of the residual and the solution in the subdomains. If the subspaces V_i are associated with the multigrid method, then equation (23) is to compute the correction value for the nodal bases at different levels. Equations (24) and (25) are the updatings for the residual and solution corresponding to the nodal bases.

For applications to nonlinear problems, consider

$$-\nabla\cdot(|\nabla u|^{s-2}\nabla u) = f \text{ in } \Omega\subset R^d \ (1<s<\infty)\ , \qquad u=0 \text{ on } \partial\Omega\ . \tag{26}$$

We assume $f\in W^{-1,s'}(\Omega)$, $\frac{1}{s}+\frac{1}{s'}=1$. By standard techniques, it can be shown, see [ET76], that (26) possesses a unique solution which is the minimizer of

$$\min_{v\in W_0^{1,s}(\Omega)}\left[\frac{1}{s}\int_\Omega |\nabla v|^s - \langle f, v\rangle\right] \ .$$

For the functional F associated with (26), it is true that conditions (6) and (22) are valid with

$$\begin{aligned} p&=s, \quad q=2 \qquad \text{if } s\geq 2;\\ p&=2, \quad q=s \qquad \text{if } 1<s\leq 2. \end{aligned}$$

See Ciarlet [Cia78], Glowinski and Marrocco [GM75] and [TX98] for the details. The full potential equation considered in [CGKT94] is of a similar type to equation (26).

For more general problem

$$\min_{v\in W_0^{1,s}(\Omega)}\int_\Omega \frac{1}{2}a(|\nabla v|^2)+f(v)\ , \tag{27}$$

we assume that a is strictly convex and f is convex and both are differentiable. If we use Algorithm 2 for (27), then we obtain

Algorithm 4 *(Application to nonlinear problems)*

1. *Choose initial values* $u^0\in V$.
2. *For* $i=1,2,\cdots,m$, *if* $u^{n+\frac{(i-1)}{m}}$ *is known, compute* $e_i^n\in V_i$ *such that*

$$\int_\Omega\Big[a'(|\nabla(u^{n+\frac{(i-1)}{m}}+e_i^n)|^2)\nabla(u^{n+\frac{(i-1)}{m}}+e_i^n)\cdot\nabla v_i + f'(u^{n+\frac{(i-1)}{m}}+e_i^n)v_i\Big]dx = 0\ , \forall v_i\in V_i\ . \tag{28}$$

3. *Update the solution as in (29) and go to the next iteration.*

$$u^{n+\frac{i}{m}} = u^{n+\frac{(i-1)}{m}} + e_i^n \ . \tag{29}$$

If V_i are the domain decomposition subspaces, then problem (28) is a nonlinear problem in each subdomain, which has a smaller size than the original problem. For some minimization methods, the convergence and the computing time depend on the size of the problem. Thus by first reducing the problem into smaller size problems and then minimize, we may gain efficiency. If V_i are the multigrid nodal basis subspaces, then (28) is equivalent to some one dimensional nonlinear problems and we can use efficient minimization routines to solve the one dimensional problems.

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