39

Viscous-Inviscid Interaction: Domain Decomposition Avant la Lettre

A.E.P. VELDMAN[1], C.-H. LAI[2]

THE VISCOUS BOUNDARY LAYER

The paper describes the history of viscous-inviscid interaction methods, and tries to put them in a modern domain-decomposition context. This history started in 1904, when Ludwig Prandtl presented the 'boundary layer' at the Third International Mathematical Congress in Heidelberg. From that moment on, the flow field around a body was divided in two parts: a thin shear layer where viscosity plays a role, and the remaining outer part where the flow can be considered inviscid (Fig. 1). In the boundary layer the flow equations can be simplified by neglecting the viscous streamwise derivatives, which changes the elliptic character of Navier-Stokes to a much easier handled parabolic character. The latter was very relevant in an era where mainly analytical tools were available for solving differential equations.

The boundary layer is driven by the external pressure distribution p_e and (through Bernoulli's law) its related streamwise velocity u_e. In the boundary layer the streamwise velocity component is reduced to zero, herewith effectively thickening and smoothing the shape of the geometry. The resulting effective shape is called the displacement body δ^*, which becomes a streamline for the inviscid flow.

---

1 Department of Mathematics, University of Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands. email: veldman@math.rug.nl
2 School of Computing & Mathematical Sciences, University of Greenwich, Wellington Street, Woolwich, London SE18 6PF, U.K. email: C.H.Lai@grn.ac.uk
Eleventh International Conference on Domain Decomposition Methods
Editors Choi-Hong Lai, Petter E. Bjørstad, Mark Cross and Olof B. Widlund

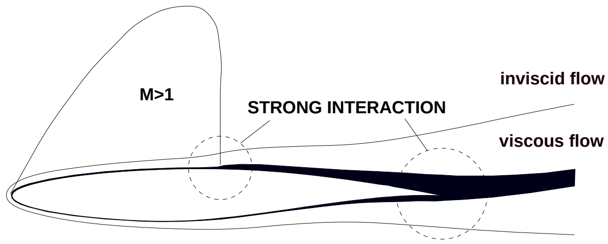


Figure 1 Subdivision of the transonic flow field around an airfoil in an inviscid-flow region and a viscous shear layer (exaggerated in thickness).

The coupled problem can be written as

$$\begin{aligned} \text{external inviscid flow:} \quad & u_e^{(n)} = E[\delta^{*(n-1)}], \\ \text{boundary-layer flow:} \quad & \delta^{*(n)} = B^{-1}[u_e^{(n)}], \end{aligned} \tag{1}$$

where E denotes the external inviscid-flow operator, and B^{-1} is the shear-layer operator; below it will be clear why we denote the latter operator in this way. It is observed that the information exchange between the two regions takes place in terms of the pressure or the streamwise velocity on the one side, and the displacement thickness or the normal velocity on the other side. Introducing a streamfunction ψ, we may correlate

$$u_e \leftrightarrow \frac{\partial \psi}{\partial n}, \qquad \delta^* \leftrightarrow -\psi,$$

hence in domain-decomposition terminology we have a Neumann-Dirichlet coupling.

FLOW SEPARATION AND ALGORITHM BREAK-DOWN

As the convergence rate of the Neumann-Dirichlet problem (1) scales with the thickness of the shear layer [Vel84], a fast convergence is expected. Indeed, for situations with attached flow this fast convergence is found, and the shear layer only provides a small correction to the inviscid-flow solution. However, as soon as the flow wants to separate from the surface the iterative coupling strategy (1) fails, because the shear-layer calculation breaks down, with a solution that tends to infinity.

A number of possible causes can be imagined. Firstly, the assumption of small streamwise derivatives appears to become invalid, and these derivatives would have to be included in the flow equations. Secondly, as the stable parabolic direction of the shear-layer equations is governed by the flow direction, inside separated-flow regions they should have been solved from downstream to upstream. In 1948 Goldstein added another possible cause for the break-down by raising the question “does a singularity always occur except for certain special pressure distributions near separation” [Gol48]. Since then, the singularity at separation bears his name.

It took twenty more years before algorithms and computers were sufficiently powerful to investigate the above options through numerical experiments. One of these was described in 1966 by Catherall and Mangler, who tried to solve the shear-layer equations with prescribed displacement thickness. Indeed, they succeeded to pass the point of flow separation, but ran into difficulties a bit further downstream [CM66]. Not convinced by their success, they stopped further research into this subject.

In the mean time Stewartson formulated the so-called 'triple-deck' theory, describing the flow in the neighbourhood of singular points in the flow field [Ste74]. Near these points the boundary layer is no longer merely providing small corrections to the flow, but instead wants to have an equal say in determining the flow field. In aerodynamical terms, the hierarchy between boundary layer and inviscid flow changes from *weak* interaction into *strong* interaction.

In the late 70-ies – after 75 years of research – it became clear that Goldstein had pointed in the right direction: the problems at flow separation are due to non-existence of a solution of the shear-layer equations at prescribed pressure. This is the reason why we have denoted the boundary layer operator at prescribed pressure by B^{-1}; its existence is not guaranteed, but B does exist.

VISCOUS-INVISCID COUPLING METHODS

From that moment on the path was free to go on, and around 1978 a number of iterative strategies have been brought forward. Two methods have survived [LW87]: the semi-inverse method [LeB78], and the quasi-simultaneous method [Vel79, Vel81].

Semi-inverse The semi-inverse method combines the favourable parts of the direct and the inverse method (Fig. 2, left). It solves both flow domains with prescribed displacement thickness, and could be called a Dirichlet-Dirichlet coupling. In order to obtain convergence some tuning of the relaxation parameter ω is required, and a fair convergence can be obtained:

$$u_e^E = E[\delta^{*(n-1)}], \quad u_e^B = B[\delta^{*(n-1)}], \quad \delta^{*(n)} = \delta^{*(n-1)} + \omega\,(u_e^E - u_e^B). \tag{2}$$

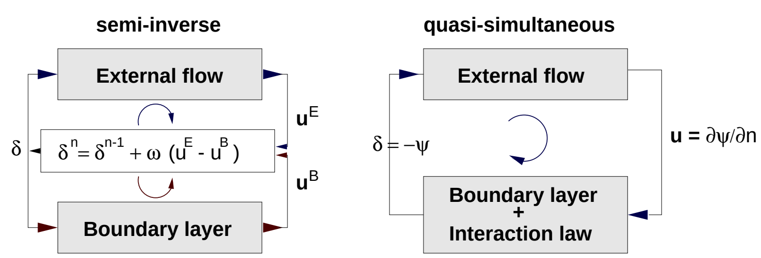


Figure 2 Semi-inverse and quasi-simultaneous VII method.

Quasi-simultaneous The quasi-simultaneous method wants to reflect the lack of hierarchy between the subdomains: in principle, it wants to solve both subdomain problems simultaneously. When the shear layer is modelled by an integral formulation a simultaneous coupling is well feasible, e.g. [HV84]. However, when in both domains a field formulation is chosen, software complexity prevents a practical implementation. Thus the idea was born to solve the shear-layer equations simultaneously with a simple but good *approximation* of the inviscid flow: the *interaction law*. The difference between this approximation and the 'exact' inviscid flow can then be handled iteratively.

Triple-deck theory delivers the flow model that describes the local interaction: thin-airfoil theory. Thus the interaction law

$$\frac{\partial\psi}{\partial n}(s) = -\frac{1}{\pi}\int_\Gamma \frac{\partial\psi}{\partial\sigma}\frac{d\sigma}{s-\sigma} \tag{3}$$

appears naturally, where Γ is the boundary between the subdomains. In this way, for aerodynamical applications the quasi-simultaneous method (Fig. 2, right) can be formulated as

$$\begin{aligned} u_e^{(n)} - I[\delta^{*(n)}] &= E[\delta^{*(n-1)}] - I[\delta^{*(n-1)}], \\ u_e^{(n)} - B[\delta^{*(n)}] &= 0, \end{aligned} \tag{4}$$

where the interaction law reads

$$I[\delta^*] = \frac{1}{\pi}\int_\Gamma \frac{d\delta^*}{d\sigma}\frac{d\sigma}{s-\sigma}. \tag{5}$$

It is observed in (4) that the interaction law is used in defect formulation, i.e. it does not influence the final converged result; it only enhances the rate of convergence!

Discretization The discretization of the thin-airfoil integral (5) leads to a positive-definite matrix I. E.g. the discretization as presented in [Vel84] yields the following expression on a uniform grid with mesh size h

$$I[\delta^*]|_i \approx \frac{1}{\pi}\sum_{j\neq i-1,i}\left\{\frac{1}{h}(\delta^*_{j+1}-\delta^*_j)\ln\left|\frac{i-j}{i-j-1}\right|\right\} - \frac{2}{\pi h}(\delta^*_{i+1}-2\delta^*_i+\delta^*_{i-1}).$$

We recognize in the latter term the discretization of derivative $d^2\delta^*/dx^2$, which describes the local contribution from the two intervals adjacent to the i-th grid point. It follows that I is symmetric and diagonally dominant. The discrete form of (4), setting $E = I$ for convenience, reads $u_e - I\delta^* = R_1$, $u_e - B\delta^* = R_2$, where R_1 contains contributions to the integral (5) from the end points of the computational domain. B now stands for the Jacobian of the shear-layer equations (it is lower triangular in attached flow), and in R_2 we can hide effects from their nonlinearity. After elimination of u_e we are left with

$$(-I+B)\,\delta^* = R_1 - R_2. \tag{6}$$

Experience learns that the diagonal of B is negative in regions of attached flow, but it vanishes in a point of separation after which it is slightly positive. This phenomenon is responsible for the break-down of the classical approach (1) for solving the shear-layer equations. But when the interaction law is added to the formulation, the matrix $I - B$ becomes relevant. With I being diagonally dominant, there is some room for subtracting positive contributions from B without $I - B$ becoming singular. Practice shows that a simple Gauss-Seidel procedure (i.e. a number of traditional boundary-layer sweeps) suffices to solve the system (6). For a more detailed discussion of the numerics involved we refer to [Vel84].

Simplification The interaction law (5) can be simplified even further to

$$I[\delta^*] = -\frac{2h}{\pi}\frac{d^2\delta^*}{ds^2}, \tag{7}$$

after which the first equation in (4) simply becomes

$$u_e^{(n)} + \frac{2h}{\pi}\left(\frac{d^2\delta^*}{ds^2}\right)^{(n)} = u_E^{(n-1)} + \frac{2h}{\pi}\left(\frac{d^2\delta^*}{ds^2}\right)^{(n-1)}. \tag{8}$$

Of course, since the description (7) contains less physics than (5) the convergence rate deteriorates [Coe99]. But (8) is easily implementable in existing boundary-layer codes, and it prevents the fatal break-down that occurs when pressure is prescribed.

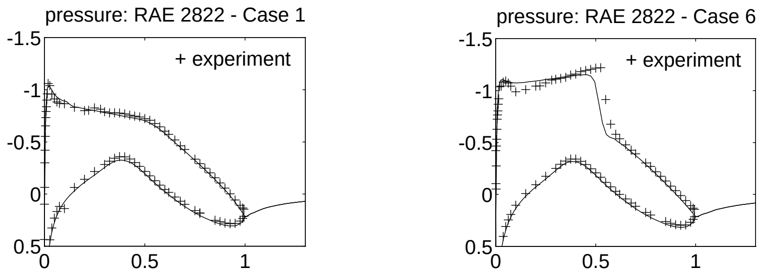
APPLICATIONS

Transonic airfoil flow The performance of the quasi-simultaneous coupling concept will be demonstrated on some calculations of transonic flow past an RAE 2822 airfoil. In these airfoil-flow problems the integral (5) has been used to describe the symmetric displacement effects ('thickness problem'). Its skew-symmetric counterpart has been used to describe the effects of camber ('lift problem'); for details see Veldman *et al.* [VLdBS90].

Fig. 3 (bottom right) shows the inviscid-viscous convergence for a computation of Case 1, subsequently followed by Case 6. It reveals that a handful of quasi-simultaneous iterations suffice to obtain a solution. The convergence is independent of the grids applied in the two subdomains. For completeness we mention that the inviscid flow was computed with an O-type 128x64 grid, whereas the shear layer was covered with a 173x21 C-type grid. Computing time is one minute on a PC. To appreciate the fast convergence even better, one has to realize that the external flow in these examples is transonic, with a significant supersonic flow region in Case 6, whereas the interaction law (5) is based on sub(!)sonic theory.

Also in three dimensions the quasi-simultaneous concept (with the simplified interaction law (7)) has been applied successfully to transonic flow [WM93].

Domain decomposition Viscous-inviscid interaction is an example of a Schwarz non-overlapping domain-decomposition with a Dirichlet-Neumann coupling. Thusfar, VII methods have only been used in situations where the Neumann region is very thin. In [dBV95] the usefullness of the quasi-simultaneous method for differently



	Mach	Reynolds	angle of attack
Case 1	0.676	5.7 mill.	1.93
Case 6	0.725	6.5 mill.	2.44

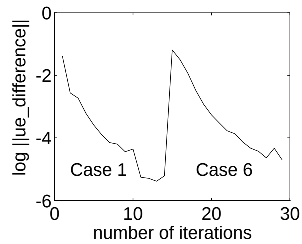


Figure 3 Pressure distributions and convergence for RAE 2822 airfoil.

shaped domains or for different governing equations has been investigated. As an example, it has been applied to a Dirichlet Laplace problem on a domain Ω with two subdomains Ω_1 and Ω_2. In this case the multiplicative Dirichlet-Neumann (D-N) domain-decomposition method reads

$$\Delta\psi_1^{(n)} = 0 \text{ in } \Omega_1, \qquad \psi_1^{(n)} = \psi_2^{(n-1)} \text{ on } \Gamma, \tag{9}$$

$$\Delta\psi_2^{(n)} = 0 \text{ in } \Omega_2, \qquad \frac{\partial\psi_2^{(n)}}{\partial n} = \frac{\partial\psi_1^{(n)}}{\partial n} \text{ on } \Gamma. \tag{10}$$

To the Neumann region Ω_2 an interaction law like (3) has been added, after which the Dirichlet-Neumann Interaction (D-NI) method can be formulated as (9) combined with the following modification of (10)

$$\frac{\partial\psi_2^{(n)}}{\partial n} + \frac{1}{\pi}\int_\Gamma \frac{\partial\psi_2^{(n)}}{\partial\sigma}\frac{d\sigma}{s-\sigma} = \frac{\partial\psi_1^{(n)}}{\partial n} + \frac{1}{\pi}\int_\Gamma \frac{\partial\psi_2^{(n-1)}}{\partial\sigma}\frac{d\sigma}{s-\sigma} \text{ on } \Gamma.$$

We emphasize that the added terms describe how subdomain Ω_1 is reacting on changes in subdomain Ω_2, and hence give an immediate response. Heuristically, this is the mechanism that will lead to a speed-up of the inter-subdomain iterations.

A convergence analysis has been made of the D-N method versus the D-NI method for various shapes of the subdomains Ω_1 and Ω_2; some of these shapes are shown in Fig. 4. For each method relaxation has been added, and the optimal relaxation factor ω_{opt} has been determined by trial-and-error. From Fig. 4 we conclude that the D-NI method converges very fast when the Neumann domain is thin (case A). In the opposite case, with a thick Neumann domain (case B), both D-N and D-NI diverge,

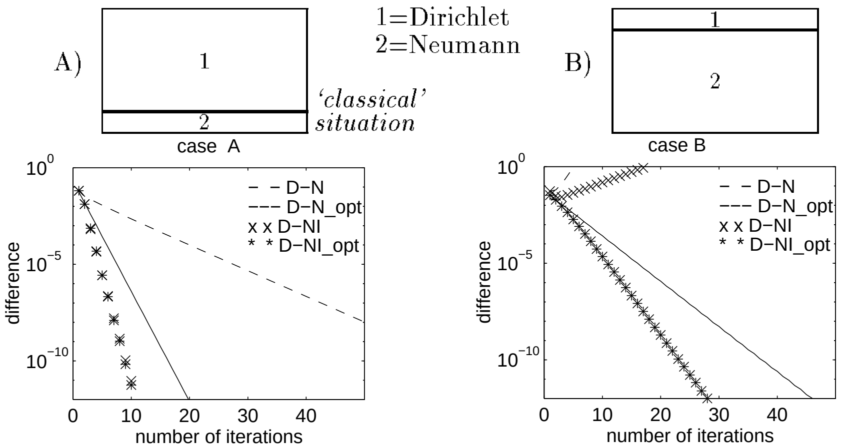


Figure 4 Influence of adding interaction to D-N coupling.

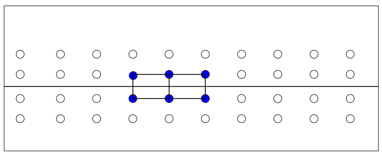


Figure 5 6-point stencil in local coupling by Tan.

but they can be made convergent with appropriate relaxation; again the optimal D-NI method is faster than the optimal D-N method.

In [dBV95] more configurations have been studied. In all of them the convergence behaviour is in line with the above findings, revealing that the (optimum) D-NI method is always faster than the (optimum) D-N method. In many cases the optimum relaxation factor for D-NI is close to $\omega = 1$, whereas the D-N method requires more fine-tuning of the relaxation factor.

Relation with local coupling methods A close relation exists with the class of local coupling methods introduced by Tan [Tan95] which make use of combinations of function values, lateral derivatives and normal derivatives. In fact, his interface condition is a linear combination of ψ, $\frac{\partial \psi}{\partial s}$, $\frac{\partial \psi}{\partial n}$, $\frac{\partial^2 \psi}{\partial s^2}$ and $\frac{\partial^2 \psi}{\partial s \partial n}$, leading to a 6-point stencil as shown in Fig. 5. Tan has optimized the coefficients in this combination based on mathematical arguments. It appears that his optimum for ellipticity-dominated problems is very close to the interactive coupling condition (8) which is based on physical arguments.

Relation with quasi-Newton methods The present method has also a close relation with the defect equation approach where an equilibrium or a matching condition [Lai98] is required to satisfy. The idea is to set up such a defect equation along the interface of two subproblems which are most likely governed by two different mathematical models. For the present problem, the typical defect equation is given by (compare (6))

$$D(\lambda) \equiv E\lambda - B\lambda = 0,$$

where E and B are defined as in (1). It is natural to solve the above defect equation using the well-known Newton's method

$$\lambda^{(n+1)} = \lambda^{(n)} - J(\lambda^{(n)})^{-1} D(\lambda^{(n)})$$

which has excellent local convergence properties. However, the viscous and inviscid coupling does not allow the Jacobian matrix being evaluated analytically. Therefore quasi-Newton methods obviously suit the present case. There are a lot of different approximations to the Jacobian and its inverse, such as Broyden's method and Schubert's method, etc. These approximate updating methods are applied in the coupling context [Lai98]. In fact, the equivalent interface condition is a suitable linear combination of the field variables deeply hidden in the two subdomains. In the limiting case when the approximation to the Jacobian is chosen as a diagonal matrix with constant diagonal entries, the method is essentially equivalent to the semi-inverse method (2). One can also easily see that the quasi-simultaneous method (4) is equivalent to choose the approximation of the Jacobian to be $I - B$.

CONCLUSIONS

During the last two decades, it has become clear that interaction laws are very beneficial in a viscous-inviscid coupling strategy. We have described their history, and placed them in a modern domain-decomposition setting. Additionally, it has been shown that they can also be beneficial in non-aerodynamic applications.

REFERENCES

[CM66] Catherall D. and Mangler K. W. (1966) The integration of the two-dimensional laminar boundary-layer equations past the point of vanishing skin friction. *J. Fluid Mech.* 26: 163–182.

[Coe99] Coenen E. G. M. (1999) Quasi-simultaneous coupling for wing and aerofoil flow. In *Proceedings 11th Domain-Decomposition Conference*. July 20-24, 1998, Greenwich.

[dBV95] de Boer B. F. A. and Veldman A. E. P. (1995) Interaction laws and domain decomposition. In Morton K. W. and Baines M. J. (eds) *Numerical Methods for Fluid Dynamics V*, pages 385–390. Oxford University Press.

[Gol48] Goldstein S. (1948) On laminar boundary layers flow near a point of separation. *Quart. J. Mech. Appl. Math.* 1: 43–69.

[HV84] Houwink R. and Veldman A. E. P. (1984) Steady and unsteady separated flow computation for transonic airfoils. AIAA paper 84–1618.

[Lai98] Lai C.-H. (1998) A defect equation approach for the coupling of subdomains in domain decomposition methods. *Computers Math. Applic.* 35: 81–94.

[LeB78] LeBalleur J.-C. (1978) Couplage visqueux-non visceux: méthode numérique et applications aux écoulements bidimensionnels transsoniques et supersoniques. *La Recherche Aérosopatiale* 183: 65–76.

[LW87] Lock R. C. and Williams B. R. (1987) Viscous-inviscid interactions in external aerodynamics. *Prog. Aerospace Science* 24: 51–171.

[Ste74] Stewartson K. (1974) Multi-structured boundary layers on flat plates and related bodies. *Adv. Appl. Mech.* 14: 145–239.

[Tan95] Tan K. H. (1995) *Local coupling in domain decomposition*. PhD thesis, University of Utrecht.

[Vel79] Veldman A. E. P. (1979) A numerical method for the calculation of laminar incompressible bounday layers with strong viscous-inviscid interaction. Technical Report NLR TR 79023, National Aerospace Laboratory.

[Vel81] Veldman A. E. P. (1981) New, quasi-simultaneous method to calculate interacting boundary layers. AIAA J. 19: 79–85.

[Vel84] Veldman A. E. P. (1984) A numerical view on strong viscous-inviscid interaction. In Habashi W. G. (ed) *Computational Methods in Viscous Flows*, pages 343–363. Pineridge Press.

[VLdBS90] Veldman A. E. P., Lindhout J. P. F., de Boer E., and Somers M. A. M. (1990) VISTRAFS: a simulation method for strongly-interacting viscous transonic flow. In Cebeci T. (ed) *Numerical and Physical Aspects of Aerodynamic Flow IV*, pages 37–51. Springer Verlag.

[WM93] Wees A. J. v. d. and Muijden J. v. (1993) A fast and robust viscous-inviscid interaction solver for transonic flow about wing/body configurations on the basis of full potential theory. AIAA paper 93–3026.