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Some Local and Parallel Properties of Finite Element Discretizations

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Introduction

This paper is devoted to the study of some local and parallel properties of finite elements for elliptic boundary value problems of second order. Several local and parallel algorithms are proposed and analyzed by means of two-grid discretizations. The algorithms are motivated from the observation that, for a solution to some elliptic problems, low frequency components can be approximated well by a relatively coarse grid and high frequency components can be computed on a fine grid by some local and parallel procedure. One major technical tool for the analysis is some sharp local a priori estimates for finite element solutions on general shape-regular grids.

This paper can be considered as a sequel of our earlier paper [XZ98] on a similar topic. While in [XZ98] we studied our methods for a rather general class of model partial differential equations, in this paper we focus our attention on a rather specific case, namely, Neumann boundary value problems on a smooth domain and as a result we obtain some better error estimates for higher order elements.

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Preliminaries

Let Ω be a bounded smooth domain in $R^d (d \geq 1)$. We will use the standard notation for Sobolev spaces $W^{s,p}(\Omega)$ and their associated norms and seminorms, see e.g. [Ada75, CL91]. For $p = 2$, we denote $H^s(\Omega) = W^{s,2}(\Omega)$, $\|\cdot\|_{s,\Omega} = \|\cdot\|_{s,2,\Omega}$ and $\|\cdot\|_{\Omega} = \|\cdot\|_{0,2,\Omega}$. (In some places of this paper, $\|\cdot\|_{s,2,\Omega}$ should be viewed as defined piecewise if it is necessary.) For $D \subset G \subset \Omega$, we use the notation $D \subset\subset G$ to mean that $\operatorname{dist}(\partial D \backslash \partial\Omega, \partial G \backslash \partial\Omega) > 0$. In this paper, we will use the letter C (with or without subscripts) to denote a generic positive constant which may stand for different values at its different occurrences. Following [Xu92a], $A \lesssim B$ means that $A \leq CB$ for some constant C is independent of mesh parameters.

Finite element spaces

Assume that $T^h(\Omega) = \{\tau\}$ is a mesh of Ω with mesh-size function $h(x)$ whose value is the diameter h_τ of the element τ containing x. Let $h_\Omega = \max_{x\in\Omega} h(x)$ be the (largest) mesh size of $T^h(\Omega)$. Sometimes, when it is clear from the context, we will drop the subscript in h_Ω and use h for the mesh size on a domain. One basic assumption on the mesh is that it is not exceedingly over-refined locally:

A.0. There exists $\gamma \geq 1$ such that

$$h_\Omega^\gamma \lesssim h(x), \quad x \in \Omega.$$

Associated with a mesh $T^h(\Omega)$, let $S^h(\Omega) \subset H^1(\Omega)$ be a finite dimensional subspace on Ω. Given $G \subset \Omega$, we define $S^h(G)$ and $T^h(G)$ to be the restriction of $S^h(\Omega)$ and $T^h(\Omega)$ to G, respectively, and

$$S_h^0(G) = \{v \in S^h(\Omega) : \operatorname{supp} v \subset\subset G\}.$$

For any $G \subset \Omega$ mentioned here, we assume that it aligns with $T^h(\Omega)$ when it is necessary. We now state our basic assumptions on the finite element spaces.

A.1. *Approximation.* There exists $r \geq 1$ such that for $w \in H^1(\Omega)$,

$$\inf_{v \in S^h(\Omega)} (\|h^{-1}(w - v)\|_{0,\Omega} + \|w - v\|_{1,\Omega}) \lesssim \|h^s w\|_{1+s,\Omega}, \quad 0 \leq s \leq r.$$

A.2. *Inverse Estimate.* For any $v \in S^h(\Omega)$,

$$\|v\|_{1,\Omega} \lesssim \|h^{-1}v\|_{0,\Omega} \text{ and } \|v\|_{0,\Omega} \lesssim \|h^{-s}v\|_{-s,\Omega}.$$

A.3. *Superapproximation.* For $G \subset \Omega_0 \subset \Omega$, let $\omega \in C_0^\infty(\Omega)$ with supp $\omega \subset\subset G$. Then for any $w \in S^h(G)$, there is $v \in S_h^0(G)$ such that

$$\|h^{-1}(\omega w - v)\|_{1,G} \lesssim \|w\|_{1,G}.$$

The assumptions mentioned above are satisfied by most of the finite element spaces used in practice when the elements that meet $\partial\Omega$ are curved to fit $\partial\Omega$ exactly. We refer to [SW77, SW95, Wah91, Wah95, XZ98] for details.

A model problem

We consider a Neumann boundary value problem

$$\begin{cases} Lu \equiv -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}(a_{ij}\frac{\partial u}{\partial x_i}) + cu &= f, \text{ in } \Omega, \\ \frac{\partial u}{\partial n_L} &= 0, \text{ on } \partial\Omega. \end{cases} \tag{1}$$

Here $\frac{\partial u}{\partial n_L}$ denotes the co-normal derivative on $\partial\Omega$, a_{ij} and c are smooth, and (a_{ij}) is uniformly positive definite on $\bar{\Omega}$.

The weak form of (1) is as follows: Find $u \equiv L^{-1}f \in H^1(\Omega)$ such that

$$a(u,v) = (f,v), \qquad \forall v \in H^1(\Omega), \tag{2}$$

where $(\cdot,\cdot) \equiv (\cdot,\cdot)_\Omega$ is the standard inner-product of $L^2(\Omega)$ and

$$a(u,v) \equiv a_\Omega(u,v) = \int_\Omega \sum_{i,j=1}^{d} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + cuv.$$

Our basic assumption is that (2) is well-posed, namely (2) is uniquely solvable for any $f \in H^{-1}(\Omega)$. Thus, since $\partial\Omega$ is smooth, if $f \in H^{r-1}(\Omega)$, then $L^{-1}f \in H^{r+1}(\Omega)$.

For $u \in H^1(\Omega)$, we define a finite element solution $u_h : H^1(\Omega) \mapsto S^h(\Omega)$ by

$$a(u_h,v) = (f,v), \quad \forall v \in S^h(\Omega) \tag{3}$$

and it is well-known that

$$\|u_h\|_{1,\Omega} \lesssim \|u\|_{1,\Omega} \quad \text{and} \quad \|u-u_h\|_{-s,\Omega} \lesssim h_\Omega^{s+1}|h^r u|_{r+1,\Omega}, \ -1 \le s \le r-1. \tag{4}$$

Locality of finite element approximations

In this section, we shall study some local properties for finite element discretizations on general shape regular grids. The results presented here generalize local a priori error estimates known in literature (cf. [NS74, SW77, SW95, Wah91, Wah95, ZLSL98]) to more general finite element meshes, which will play a crucial role in our analysis.

We assume that $D \subset\subset \Omega_0 \subset \Omega$ and that $\partial\Omega_0$ is smooth. The proof of the following lemma is similar to a corresponding result in [XZ98] (see also [NS74, SW77, SW95, Wah91, Wah95]).

Lemma 1. *If Assumptions A.0, A.2 and A.3 hold and* $w \in S^h(\Omega_0)$ *satisfies*

$$a(w,v) = 0, \qquad \forall v \in S_h^0(\Omega_0), \tag{5}$$

then

$$\|w\|_{1,D} \lesssim \|w\|_{-l,\Omega_0}, \quad l = 0,1,2,\cdots,r-1. \tag{6}$$

Proof. Let p be an integer such that $p \geq \gamma - 1$ and Ω_j $(j = 1, 2, \cdots, p)$ be subdomains with smooth boundaries which satisfy

$$D \subset\subset \Omega_{2p+1} \subset\subset \Omega_{2p} \subset\subset \cdots \subset\subset \Omega_1 \subset\subset \Omega_0.$$

Choose $D_1 \subset \Omega$ satisfying $D \subset\subset D_1 \subset\subset \Omega_{2p+1}$ and $\omega \in C_0^\infty(\Omega)$ such that $\omega \equiv 1$ on $\bar{D}_1$ and supp $\omega \subset\subset \Omega_{2p+1}$. For any $\phi \in H^l(D)(l = 0, 1, \cdots, r-1)$, let $E\phi$ be the universal extension of ϕ on Ω and $\Phi = L^{-1}\phi$. Then $\Phi \in H^{l+2}(\Omega)$ and

$$(w, \phi)_D = (\omega w, \phi) = a(\omega w, \Phi).$$

Note that $||\Phi||_{l+2,\Omega_{2p+1}} \lesssim ||\phi||_{l,D}$ and, for any $v \in S_h^0(\Omega_{2p+1})$,

$$\begin{aligned} & a(\omega w, \Phi) \lesssim a(w, \omega\Phi) + ||w||_{-l-1,\Omega_{2p+1}} ||\Phi||_{l+2,\Omega_{2p+1}} \\ = \quad & a(w, \omega\Phi - v) + ||w||_{-l-1,\Omega_{2p+1}} ||\Phi||_{l+2,\Omega_{2p+1}}, \end{aligned}$$

we get

$$||w||_{-l,D} \lesssim ||w||_{-l-1,\Omega_{2p+1}} + h^{l+1}_{\Omega_{2p+1}} ||w||_{1,\Omega_{2p+1}}. \tag{7}$$

Using $||w||_{1,\Omega_{2p+1}} \lesssim ||w||_{0,\Omega_{2p}}$(see [XZ98]) and taking $l = 0$, we obtain

$$||w||_{0,D} \lesssim ||w||_{-1,\Omega_0} + h_{\Omega_0} ||w||_{0,\Omega_{2p}}.$$

The argument may be repeated for $||w||_{\Omega_{2p}}$ on the right to yield

$$||w||_{0,\Omega_{2p-2j+2}} \lesssim ||w||_{-1,\Omega_0} + h_{\Omega_0} ||w||_{0,\Omega_{2p-2j}}, \quad j = 1, 2, \cdots, p.$$

Thus, we obtain from Assumption A.2 that

$$\begin{aligned} & ||w||_{1,D} \lesssim ||w||_{-1,\Omega_0} + h^{p+1}_{\Omega_0} ||w||_{0,\Omega_0} \\ \lesssim \quad & ||w||_{-1,\Omega_0} + h^{p+1}_{\Omega_0} ||h^{-1} w||_{-1,\Omega_0} \lesssim ||w||_{-1,\Omega_0}. \end{aligned}$$

This proves (5) for $l = 1$ (see [XZ98] for $l = 0$). The argument then proceeds via induction using (7) to complete the proof.

Theorem 2. *If Assumptions A.0, A.1, A.2 and A.3 hold, then*

$$||u - u_h||_{1,D} \lesssim \inf_{v \in S^h(\Omega)} (||u - v||_{1,\Omega_0} + ||u - u_h||_{1-r,\Omega}). \tag{8}$$

Proof. Let $R_h : H^1(\Omega_0) \longrightarrow S^h(\Omega_0)$ be the Galerkin-projection defined by

$$a_{\Omega_0}(w - R_h w, v) = 0, \quad \forall w \in H^1(\Omega_0), \ \forall v \in S^h(\Omega_0). \tag{9}$$

Choose $D_1 \subset \Omega$ satisfying $D \subset\subset D_1 \subset\subset \Omega_0$ and $\omega \in C_0^\infty(\Omega_0)$ such that $\omega \equiv 1$ on $\bar{D}_1$, then for $\tilde{u} = \omega u$,

$$a(R_h \tilde{u} - u_h, v) = 0, \quad \forall v \in S_h^0(D_1).$$

Thus, Lemma 1 yields

$$||R_h \tilde{u} - u_h||_{1,D} \lesssim ||R_h \tilde{u} - u_h||_{1-r,D_1}.$$

Therefore, estimates similar to (4) lead to

$$\begin{aligned} & \|u-u_h\|_{1,D} \le \|\tilde{u}-R_h\tilde{u}\|_{1,D} + \|R_h\tilde{u}-u_h\|_{1,D} \\ \lesssim\ & \|\tilde{u}-R_h\tilde{u}\|_{1,D} + \|R_h\tilde{u}-u_h\|_{1-r,D_1} \lesssim \|\tilde{u}-R_h\tilde{u}\|_{1,D_1} + \|u-u_h\|_{1-r,D_1} \\ \lesssim\ & \|\tilde{u}-R_h\tilde{u}\|_{1,\Omega_0} + \|u-u_h\|_{1-r,D_1} \lesssim \|u\|_{1,\Omega_0} + \|u-u_h\|_{1-r,\Omega}, \end{aligned}$$

which together with a standard argument produce (8).

Theorem 3. *Under the assumption of Theorem 2, there holds*

$$\|u-u_h\|_{1,D} \lesssim (h^r_{\Omega_0} + h^{2r}_{\Omega})|u|_{r+1,\Omega}. \tag{10}$$

Local and parallel algorithms

In this section, we shall develop some local and parallel finite element algorithms. These algorithms are motivated by the local properties of finite element studied in the previous section. We shall first discuss the local algorithms. The parallelization of the local algorithms is straightforward.

Let $S^h(\Omega) \subset H^1(\Omega)$ be a finite element subspace satisfying Assumptions A.1, A.2 and A.3 associated with a grid $T^h(\Omega)$ which satisfies Assumption A.0. Let $u_h \in S^h(\Omega)$ be the solution of the standard finite element scheme for solving (2). Either locally or globally, with proper regularity assumption, we have the following error estimate:

$$\|u-u_h\|_1 \lesssim h^r.$$

With this type of error estimates in mind, we will, in the rest of this section, only compare the approximate solutions of our algorithms with u_h instead of the exact solution u.

Local algorithms

The main idea for the local algorithm is that the more global component of a finite element solution may be obtained by a relatively coarser grid and, the rest of the computation can then be localized.

Roughly speaking, the algorithms will sometimes be based on a coarse grid of size H and a fine grid of size $h \ll H$, and sometimes on a grid that is fine in a subdomain and coarse on the rest of the domain. The fine grid may be defined locally only. In our analysis, we shall use an auxiliary fine grid, denoted by $T^h(\Omega)$, that is globally defined and coincide with the local fine grid in the subdomain of interest.

Let $T^H(\Omega)$ be a shape-regular coarse grid, of size $H \gg h$, and $T^h(\Omega_0)$ an highly locally refined mesh, where Ω_0 is a slightly larger subdomain with smooth boundary, containing a subdomain $D \subset \Omega$ (namely $D \subset\subset \Omega_0$). More precisely, we let $T^h_H(\Omega)$ denote a locally refined shape-regular mesh that may be viewed as being obtained by refining $T^H(\Omega)$ locally around the subdomain D in such a way that $T^h_H(\Omega_0) = T^h(\Omega_0)$. We are interested in obtaining the approximate solution in the given subdomain D with an accuracy comparable to that from $T^h(\Omega)$. We shall propose two different gridding strategies for obtaining finite element approximations on the subdomain D.

We denote the corresponding finite element space by $S^{H,h}(\Omega) \subset H^1(\Omega)$, which satisfies Assumptions A.1, A.2 and A.3, too.

The first strategy is simply to solve a standard finite element solution in $S^{H,h}(\Omega)$.

Algorithm A0. *Find $u_H^h \in S^{H,h}(\Omega)$ such that*

$$a(u_H^h, v) = (f, v), \quad \forall v \in S^{H,h}(\Omega).$$

Although this algorithm is still a global algorithm since a global problem is solved, it is designed to obtain a local approximation in the subdomain D and it makes use of a mesh that is much coarser away from D.

We have the following theorem

Theorem 4. *If $u_H^h \in S^{H,h}(\Omega)$ is obtained by Algorithm A0, then*

$$\|u_h - u_H^h\|_{1,D} \lesssim H^{2r}|u|_{r+1,\Omega}.$$

Proof. By the definition of Algorithm A0 and our assumption on the auxiliary grid $T^h(\Omega)$ that coincide with $T_H^h(\Omega)$ on Ω_0, we have

$$a(u_H^h - u_h, v) = 0, \quad \forall v \in S_h^0(\Omega_0).$$

By Lemma 1, we get

$$\|u_h - u_H^h\|_{1,D} \lesssim \|u_h - u_H^h\|_{1-r,\Omega_0}$$

and can then finish the proof.

Our second strategy is an improvement of the first strategy by using a residual-correction technique as in [Xu92b, Xu94, Xu96]. In this strategy, we first solve a global problem only on the given coarse grid $T^H(\Omega)$ and then correct the residual locally on the fine mesh $T^h(\Omega_0)(= T_H^h(\Omega_0))$.

A prototype of the local algorithms is as follows.

Algorithm B0.

1. *Find a global coarse grid solution $u_H \in S^H(\Omega)$:*

$$a(u_H, v) = (f, v), \quad \forall v \in S^H(\Omega).$$

2. *Find a local fine grid correction $e_h \in S^h(\Omega_0)$:*

$$a_{\Omega_0}(e_h, v) = (f, v)_{\Omega_0} - a_{\Omega_0}(u_H, v), \quad \forall v \in S^h(\Omega_0).$$

3. *Set $u^h = u_H + e_h$, in Ω_0.*

Theorem 5. *If $u^h \in S^h(\Omega_0)$ is obtained by Algorithm B0, then*

$$\|u_h - u^h\|_{1,D} \lesssim H^{2r}|u|_{r+1,\Omega}.$$

Proof. By the definition of Algorithm B0,

$$a(u^h - u_h, v) = 0, \quad \forall v \in S_h^0(\Omega_0).$$

By Lemma 1, we get

$$\|u_h - u^h\|_{1,D} \lesssim \|u^h - u_h\|_{1-r,\Omega_0} \lesssim \|u_h - u_H\|_{1-r,\Omega_0} + \|e_h\|_{1-r,\Omega_0}.$$

To estimate $\|e_h\|_{1-r,\Omega_0}$, we use the Aubin-Nitsche duality argument. Given any $\phi \in H^{r-1}(\Omega_0)$, there exist $w \in H^{r+1}(\Omega_0)$ and $\tilde{w} \in H^{r+1}(\Omega)$ such that

$$a_{\Omega_0}(v, w) = (\phi, v)_{\Omega_0}, \ \forall v \in H^1(\Omega_0) \ \text{ and } \ a(v, \tilde{w}) = (\phi, v)_{\Omega_0}, \ \forall v \in H^1(\Omega).$$

One sees that there exist $\tilde{w}_H \in S^H(\Omega)$ and $w_h^0 \in S^h(\Omega_0)$ satisfying

$$a(v, \tilde{w}_H) = (\phi, v)_{\Omega_0}, \ \forall v \in S^H(\Omega) \ \text{ and } \ \|\tilde{w} - \tilde{w}_H\|_{1,\Omega} \lesssim H^r \|\phi\|_{r-1,\Omega_0},$$

$$a_{\Omega_0}(v, w_h^0) = (\phi, v)_{\Omega_0}, \ \forall v \in S^h(\Omega_0) \ \text{ and } \ \|w - w_h^0\|_{1,\Omega_0} \lesssim h^r \|\phi\|_{r-1,\Omega_0}.$$

It follows that

$$\begin{aligned} & (e_h, \phi)_{\Omega_0} = a_{\Omega_0}(e_h, w) = a_{\Omega_0}(e_h, w_h^0) = a_{\Omega_0}(u_h - u_H, w_h^0) \\ = \ & a_{\Omega_0}(u_h - u_H, w_h^0 - w) + a(u_h - u_H, \tilde{w} - \tilde{w}_H) \lesssim H^r \|u_h - u_H\|_{1,\Omega} \|\phi\|_{r-1,\Omega_0}, \end{aligned}$$

which implies $\|e_h\|_{1-r,\Omega_0} \lesssim H^r \|u_h - u_H\|_{1,\Omega}$. The desired result then follows.

Parallel algorithms

Given an initial coarse triangulation $T^H(\Omega)$, let us divide Ω into a number of disjoint subdomains $D_1, \ldots, D_m$ and we then enlarge each D_j to obtain Ω_j that align with $T^H(\Omega)$ and have smooth boundaries. The basic idea of our parallel algorithm is very simple: we just apply the local algorithms in parallel in all Ω_j's.

Let us first discuss the parallel version of Algorithm A0. For each j, we use some adaptive process to obtain a shape-regular mesh $T_j(\Omega)$ and the corresponding finite element solution denoted by u_j. We note that each $T_j(\Omega)$ has a substantially finer mesh inside Ω_j. We note that the $T_j(\Omega)$ provide different triangulations for Ω and that they can be very arbitrary; for simplicity of exposition, we assume that each $T_j(\Omega)$ has the same size h in Ω_j (more precisely, $T_j(\Omega_j) = T^h(\Omega_j)$) and has the size H away from Ω_j. Let $S^{h_j}(\Omega) \subset H^1(\Omega)$ be the corresponding finite element spaces.

Algorithm A1.

1. *Find* $u_j \in S^{h_j}(\Omega)$ $(j = 1, 2, \cdots, m)$ *in parallel:*

$$a(u_j, v) = (f, v), \quad \forall v \in S^{h_j}(\Omega).$$

2. *Set* $u^h = u_j$, *in* D_j $(j = 1, 2, \cdots, m)$.

Define a piecewise norm by $|||u_h - u^h|||_{1,\Omega} = (\sum_{j=1}^m \|u_h - u^h\|_{1,D_j}^2)^{1/2}$. By Theorem 4, we have

$$|||u_h - u^h|||_{1,\Omega} \lesssim H^{2r}|u|_{r+1,\Omega} \text{ and } |||u - u^h|||_{1,\Omega} \lesssim (h^r + H^{2r})|u|_{r+1,\Omega}.$$

We now discuss the parallel version of Algorithm B0. For clarity of exposition, it appears to be most convenient to discuss this method using two globally defined grids: an initial coarse grid $T^H(\Omega)$ and a refinement of $T^H(\Omega)$, $T^h(\Omega)$ with $h \ll H$.

Algorithm B1.

1. Find a global coarse grid solution $u_H \in S^H(\Omega)$*:*

$$a(u_H, v) = (f, v), \quad \forall v \in S^H(\Omega).$$

2. Find local fine grid corrections $e_h^j \in S^h(\Omega_j)$ $(j = 1, 2, \cdots, m)$ *in parallel:*

$$a_{\Omega_j}(e_h^j, v) = (f, v)_{\Omega_j} - a_{\Omega_j}(u_H, v). \quad \forall v \in S^h(\Omega_j),$$

3. Set $u^h = u_H + e_h^j$, *in* D_j $(j = 1, 2, \cdots, m)$.

By Theorem 5, we have the following result for this algorithm.

Theorem 6. *If* u^h *is the solution obtained by Algorithm B1, then*

$$|||u_h - u^h|||_{1,\Omega} \lesssim H^{2r}|u|_{r+1,\Omega} \text{ and } |||u - u^h|||_{1,\Omega} \lesssim (h^r + H^{2r})|u|_{r+1,\Omega}.$$

We note that the approximations obtained by Algorithms A1 and B1 are defined piecewise and they are in general discontinuous and, that $||u_h - u^h||_{0,\Omega}$ does not in general have higher order than $|||u_h - u^h|||_{1,\Omega}$. We point out that some further modifications of these algorithms can achieve: (1) smoothing u^h to obtain a global $H^1(\Omega)$ approximation; (2) improving $||u_h - u^h||_{0,\Omega}$ to be of higher order, cf. [XZ98].

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