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# 4. Decomposition of Energy Space and Virtual Control for Parabolic Systems

J.L. Lions [1]

## Introduction

Methods of choice for attempting the control of distributed systems (i.e. systems modelled by Partial Differential Equations, PDE's in short) are **decomposition methods**.

Given the state equation -i.e. PDE's containing control (should it be distributed or on the boundary)- one can decompose (i) **the operator**, or (ii) **the geometrical domain**, or (iii) **the spaces describing the domain of the operator**.

Method (i), based on **splitting up** ideas has been used in a paper by A. Bensoussan, J.L. Lions and R. Temam[BLT94]. At the end of this paper, some remarks were made concerning domain decomposition. New methods (also based on virtual control) are given in a note of J.L. Lions and O. Pironneau[LP99a] and in the paper of J.L. Lions[Lio00].

DDM (Domain Decomposition Methods) are now absolutely essential for the **Analysis** of problems (i.e. PDE's without control). As observed by J.E. Lagnese and G. Leugering[LL00] while there is an extensive literature on DDM for direct simulation, the literature is much more scarse concerning DDM and optimal control.

The first contributions were due to B. Despres[Des91], J.D. Benamou and B. Despres[JD97], J.D. Benamou[Ben97, Ben98], and the paper just quoted by J. Lagnese and G. Leugering.

Another set of ideas has been introduced by J.L. Lions and O. Pironneau in 3 notes[LP98a, LP98b, LP99b] where one introduces for **all** problems (i.e. problems with or **without** control functions) so called **virtual controls** with the goal to have **all** problems entering in **one** model.

First numerical results are reported in these notes.

We want here to study the possibility (iii), namely the decomposition of **spaces** describing the domain of the operator. In a (slightly) more precise manner, if $A$ is the main symmetric part of the stationary operator contained in the model, then we consider the "**energy space**" $D(A^{1/2})$ (the domain of $A^{1/2}$) - a space that we denote by $V$. It is this space that we decompose in the present paper.

For stationary problems without control, this technique has been introduced in R. Glowinski, J.L. Lions and O. Pironneau[GLP99] .

We show here how it can be applied for the **control of parabolic systems**.

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## Elliptic regularization of parabolic equations

Let $V$ and $H$ be two real Hilbert spaces, such that

(2.1) $$V \subset H\ , V \text{ dense in } H,\ V \to H \text{ continuous.}$$

We shall identify $H$ with its dual, so that

(2.2) $$V \subset H \subset V'$$

where $V'$ denotes the dual of $V$.

Let $a(\varphi, \hat{\varphi})$ be a continuous bilinear form on $V$, such that

(2.3) $$a(\varphi, \varphi) \geq \alpha \|\varphi\|^2 \qquad \forall\, \varphi \in V\ ,\ \alpha > 0\ ,$$

where $\|\varphi\|$ denotes the norm of $\varphi$ in $V$.

Let $f$ be given such that

(2.4) $$f \in L^2(0, T; V').$$

We are looking for a function $u$ such that

(2.5) $$\left|\begin{array}{l} u \in L^2(0,T;V),\ \dfrac{\partial u}{\partial t} \in L^2(0,T;V'), \\ (\dfrac{\partial u}{\partial t}, \hat{u}) + a(u, \hat{u}) = (f, \hat{u}) \quad \forall\, \hat{u} \in V\ , \\ u|_{t=0} = 0 \end{array}\right.$$

(in (2.5) $(f, \varphi)$ denotes the duality between $V'$ and $V$). It follows from $(2.5)_1$ that (after possible change on a set of measure 0) the function $t \to u(t)$ is continuous from $[0, T] \to H$.

It is known that problem (2.5) **admits a unique solution** (cf. J.L. Lions[Lio61] where considerably more general situations are considered), the mapping $f \to u$ being continuous from $L^2(0, T; V')$ into the space of functions $u$ satisfying to $(2.5)_1$. □

For reasons that will appear later on, we are going to use an **elliptic regularization** (J.L. Lions[Lio63]) **of problem** (2.5).

We define

(2.6) $$W = \{u |\ u \in L^2(0,T;V), \frac{\partial u}{\partial t} \in L^2(0,T;H), u(0) = 0\}\ .$$

For $u, \hat{u} \in W$, we define

(2.7) $$\mathcal{A}_\gamma(u, \hat{u}) = \gamma \int_0^T (\frac{\partial u}{\partial t}\ ,\ \frac{\partial \hat{u}}{\partial t}) dt + \int_0^T [(\frac{\partial u}{\partial t}, \hat{u}) + a(u, \hat{u})] dt\ ,$$

**where $\gamma$ is given** $> 0$.

The bilinear form $u, \hat{u} \to \mathcal{A}_\gamma(u, \hat{u})$ is continuous on $W$. Moreover

(2.8) $$\mathcal{A}_\gamma(u, u) = \gamma \int_0^T\ \|u(t)\|_H^2 dt + \frac{1}{2}\|u(T)\|_H^2 + \int_0^T a(u)\ dt\ ,$$

where $\|u\|_H = (u,u)^{1/2}$ , $a(u) = a(u,u)$.

By virtue of (2.3) it follows that

$$\mathcal{A}_\gamma(u,u) \geq \gamma \int_0^T \|\frac{\partial u}{\partial t}(t)\|_H^2 dt + \alpha \int_0^T \|u\|^2 \, dt \, , \tag{2.9}$$

so that in particular

$$\left| \begin{array}{l} \mathcal{A}_\gamma(u,u) \geq \inf(\gamma,\alpha) \parallel u \parallel_W^2 \qquad \text{where} \\ \parallel u \parallel_W^2 = \int_0^T (\parallel u(t) \parallel^2 + \parallel \frac{\partial u}{\partial t}(t) \parallel_H^2) dt \, . \end{array} \right. \tag{2.10}$$

It then immediately follows (LAX-MILGRAM's Lemma) that there exists **a unique element** $u_\gamma$ solution of

$$\left| \begin{array}{l} \mathcal{A}_\gamma(u_\gamma,\hat{u}) = \int_0^T (f,\hat{u}) dt \quad \forall \, \hat{u} \in W \, , \\ u_\gamma \in W \, . \end{array} \right. \tag{2.11}$$

**Equation** (2.11) **is called an elliptic regularization of** (2.5).

One has the following property (J.L. Lions[Lio63]) :

(2.12) | as $\gamma \to 0$, **the solution** $u_\gamma$ of (2.11) **converges toward the solution** $u$ **of** (2.5) **in the sense that**
$u_\gamma \to u$ in $L^2(0,T;V)$ weakly,
$\frac{\partial u_\gamma}{\partial t} \to \frac{\partial u}{\partial t}$ in $L^2(0,T;V')$ weakly .

Before briefly recalling the (simple) proof of (2.12), a few remarks are in order.

**Remark 2.1.**

Let's give an interpretation of the above equations in **non-**variational terms. We define $A \in \mathcal{L}(V;V')$ by

$$(A\varphi,\psi) = a(\varphi,\psi) \quad \forall \, \varphi,\psi \in V \, .$$

Then (2.5) reads

$$\frac{\partial u}{\partial t} + Au = f \, , \quad u|_{t=0} = 0 \, , \quad u \in L^2(0,T;V) \, , \tag{2.13}$$

and (2.11) is equivalent to

$$\left| \begin{array}{l} -\gamma \, \frac{\partial^2 u_\gamma}{\partial t^2} + \frac{\partial u_\gamma}{\partial t} + Au_\gamma = f \, , \\ u_\gamma|_{t=0} = 0 \, , \quad \frac{\partial u_\gamma}{\partial t}(T) = 0 \, , \\ u_\gamma \in L^2(0,T;V), \ \frac{\partial u_\gamma}{\partial t} \in L^2(0,T;H). \end{array} \right. \tag{2.14}$$

□

**Remark 2.2.**

If $A$ is a second order elliptic operator, then the operator

$$-\gamma \frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial t} + A \tag{2.15}$$

is indeed **an elliptic operator**. We are dealing with elliptic regularization. But if $A$ is, say, a 4th order elliptic operator, then the operator (2.15) is **quasi** elliptic. We nevertheless keep the term of elliptic regularization. □

Let us now sketch the proof of (2.12). It follows from (2.9) that as $\gamma \to 0$ , $u_\gamma(\text{resp.}\sqrt{\gamma}\,\frac{\partial u_\gamma}{\partial t})$ remains in a bounded set of $L^2(0,T;V)$ (resp. $L^2(0,T;H)$). We can therefore extract a subsequence still denoted by $u_\gamma$ such that

$$u_\gamma \to w \text{ in } L^2(0,T;V) \text{ weakly}$$

and $\sqrt{\gamma}\,\frac{\partial u_\gamma}{\partial t} \to \xi$ in $L^2(0,T;H)$ weakly. But $\sqrt{\gamma}\,\frac{\partial u_\gamma}{\partial t} \to 0$ in the space of distributions in $t$ with values in $V$, so that $\xi = 0$.

We rewrite (2.11) as

$$\gamma \int_0^T (\frac{\partial u_\gamma}{\partial t}, \frac{\partial \hat{u}}{\partial t})_H dt - \int_0^T (u_\gamma, \frac{\partial \hat{u}}{\partial t})_H dt + \int_0^T a(u,\hat{u})dt = \int_0^T (f,\hat{u})dt \tag{2.16}$$

where we have taken $\hat{u} \in W$ such that

$$\hat{u}(T) = 0 \,. \tag{2.17}$$

We can pass now to the limit in (2.16). We obtain

$$-\int_0^T (w, \frac{\partial \hat{u}}{\partial t})_H dt + \int_0^T a(w,\hat{u})dt = \int_0^T (f,\hat{u})dt$$

$\forall \hat{u} \in W$ such that (2.17) is satisfied.

Hence $w = u$. □

**Remark 2.3.**

We could also use a different elliptic regularization, namely

$$\int_0^T (\gamma, (\frac{\partial u}{\partial t}, \frac{\partial \hat{u}}{\partial t})_{V'} dt + \int_0^T [(\frac{\partial u}{\partial t}, \hat{u}) + a(v,\hat{u})]dt \tag{2.18}$$

defined on the space of functions $u$ such that $u \in L^2(0,T;V)$ and $\frac{\partial u}{\partial t} \in L^2(0,T;V')$ (instead of $L^2(0,T;H)$). In a sense (2.18) is more natural but (2.7) avoids the use of $V'$. □

**Remark 2.4.**

One has also (cf. J.L. Lions[Lio63])

$$\frac{\partial u_\gamma}{\partial t} \to \frac{\partial u}{\partial t} \qquad \text{in } L^2(0,T;V') \quad \text{weakly} \,. \tag{2.19}$$

□

# A Control problem and its elliptic regularization

We introduce now the **space of controls** $v$

(3.1)
$$\left| \begin{array}{l} v \in L^2(0,T;\mathcal{U}), \\ \mathcal{U} = \text{ real Hilbert space }. \end{array} \right.$$

If $B$ is an operator such that

(3.2)
$$B \in \mathcal{L}(\mathcal{U};V'),$$

**the state equation** is given by

(3.3)
$$\left| \begin{array}{l} (\dfrac{\partial y}{\partial t}, \hat{y}) + a(y,\hat{y}) = (Bv,\hat{y}) \quad \forall\, \hat{y} \in V \ , \\ y \in L^2(0,T;V), \ \dfrac{\partial y}{\partial t} \in L^2(0,T;V'), \\ y|_{t=0} = 0 \ . \end{array} \right.$$

**The cost function** is given by

(3.4)
$$J(v) = \frac{1}{2}\int_0^T \|v\|_{\mathcal{U}}^2 dt + \frac{\beta}{2}\|y(T;v) - y^T\|_H^2$$

where $\beta$ is given $> 0$ and where $y^T$ is a given element of $H$.

**The problem of control is now to find**

(3.5)
$$\inf_{v \in L^2(0,T;\mathcal{U})} J(v)$$

This problem admits a unique solution $v_{\text{opt}}$, i.e. there exists a unique $v_{\text{opt}}$ such that

(3.6)
$$J(v_{\text{opt}}) = \inf_{v \in L^2(0,T;\mathcal{U})} J(v) \ .$$

□

**We consider now the "elliptic regularization" of problem** (3.3) (3.6).
With the notations of previous section , we define the state $y_\gamma \in W$ by

(3.7)
$$\mathcal{A}_\gamma(y_\gamma,\hat{y}) = \int_0^T (Bv,\hat{y})dt \quad \forall \hat{y} \in W \ .$$

This problem admits a unique solution $y_\gamma = y_\gamma(v)$, and we can introduce

(3.8)
$$J_\gamma(v) = \frac{1}{2}\int_0^T \|v\|_{\mathcal{U}}^2 dt + \frac{\beta}{2}\|y_\gamma(T;v) - y^T\|_H^2$$

Of course, **there exists a unique element** $v_\gamma$ **in** $L^2(0,T;\mathcal{U})$ such that

(3.9)
$$J_\gamma(v_\gamma) = \inf .J_\gamma(v), \quad v \in L^2(0,T;\mathcal{U}) \ .$$

Let us briefly sketch the (easy) proof of

$$(3.10)\qquad \left| \begin{array}{l} \text{as } \gamma \to 0 \;,\; J_\gamma(v_\gamma) \to J(v_{\text{opt}}),\; v_\gamma \to v_{\text{opt}} \text{ in} \\ L^2(0,T;\mathcal{U}) \text{ weakly and } y_\gamma(v_\gamma) \to y(v_{\text{opt}}) \text{ in } L^2(0,T;V) \text{ weakly .} \end{array} \right.$$

For $v$ fixed in $L^2(0,T;\mathcal{U})$, one knows that $y_\gamma(v) \to y(v)$ in $L^2(0,T;V)$ weakly, and (cf. J.L. Lions[Lio63]) $y_\gamma(T;v) \to y(T;v)$ in $H$ strongly. Therefore $J_\gamma(v) \to J(v)$ so that

$$(3.11)\qquad \lim.\sup.J_\gamma(v_\gamma) \leq \inf J(v) \;,\; v \in L^2(0,T;\mathcal{U}).$$

It follows from (3.11) that $v_\gamma$ remains in a bounded subset of $L^2(0,T;\mathcal{U}$. By extracting a subsequence, we can assume that

$$(3.12)\qquad v_\gamma \to w \quad \text{in } L^2(0,T;\mathcal{U}) \quad \text{faible}$$

and one verifies that $y_\gamma(T;v_\gamma) \to y(T;w)$ in $H$ weakly.
Therefore

$$(3.13)\qquad \lim\inf.J_\gamma(v_\gamma) \geq J(w).$$

Comparing (3.11) (3.13) and using (3.12) gives (3.10). □

**Remark 3.1.**
Everything which has been said above readily extends to similar problems with constraints on $v$ :

$$(3.14)\qquad \left| \begin{array}{l} v \in L^2(0,T;\mathcal{U}_{ad}) \\ \mathcal{U}_{ad} \quad \text{closed convex subset of } \mathcal{U} \text{ .} \end{array} \right.$$

□

**Remark 3.2.**
One can write, for all the problems considered, the necessary and sufficient conditions (the so called "**Optimality System**") for $v$ to be optimal. Cf. J.L. Lions[Lio68].

□

**Orientation.**
We want now to "decompose" problem (3.5) based on
(i) a decomposition of the energy space $V$ ;
(ii) the elliptic-regularized problem (3.9). □

# Decomposition of the energy space

We assume that

$$(4.1)\qquad V = V_1 + V_2$$

where

$$(4.2)\qquad V_i = \text{ closed subspace of } V \text{ ,}$$

$$V_1 \cap V_2 = \{0\} \quad \text{or not.} \tag{4.3}$$

In other words, every $\varphi$ in $V$ admits at least a decomposition

$$\varphi = \varphi_1 + \varphi_2$$

and actually an infinite number of them if $V_1 \cap V_2 \neq \{0\}$.

**Remark 4.1.**

Everything we are going to say readily extends to the case when

$$V = V_1 + \cdots + V_m \ , m > 2 \ . \tag{4.4}$$

□

**Remark 4.2.**

Examples (for a stationnary situation without control) are given in R. Glowinski, J.L. Lions and O. Pironneau[GLP99]. □

We introduce now the natural decomposition of $W$ (defined in (2.6) attached to (4.1), namely

$$\left| \begin{array}{l} W = W_1 + W_2 \ , \\ W_i = \{\varphi | \varphi \in L^2(0,T;V_i), \dfrac{\partial \varphi}{\partial t} \in L^2(0,T;H), \varphi(0) = 0\} \ . \end{array} \right. \tag{4.5}$$

Let $s_i$ $(i = 1, 2)$ be a continuous bilinear form on $V$ (or on $V_i$) such that

$$\left| \begin{array}{l} s_i \ \textbf{is symmetric} \text{ and } s_i(\varphi_i, \varphi_i) \geq s_{0i} \|\varphi_i\|^2 \quad \forall \ \varphi_i \in V_i \ , \\ s_{0i} > 0. \end{array} \right. \tag{4.6}$$

We then define $\forall \ \varphi, \hat{\varphi} \in W_i$,

$$\sigma_i(\varphi, \hat{\varphi}) = \gamma \int_0^T (\frac{\partial \varphi}{\partial t}, \frac{\partial \hat{\varphi}}{\partial t}) dt + \int_0^T s_i(\varphi, \hat{\varphi}) dt \ . \tag{4.7}$$

Given the **virtual controls** $\lambda_1, \lambda_2 \in W_1 \times W_2$, we define $y_1, y_2 \in W_1 \times W_2$ as the solution of

$$\left| \begin{array}{l} \sigma_1(y_1 - \lambda_1, \hat{y_1}) + \mathcal{A}_\gamma(\lambda_1 + \lambda_2, \hat{y_1}) = \displaystyle\int_0^T (Bv, \hat{y_1}) dt \quad \forall \ \hat{y_1} \in W_1 \ , \\ \sigma_2(y_2 - \lambda_2, \hat{y_2}) + \mathcal{A}_\gamma(\lambda_1 + \lambda_2, \hat{y_1}) = \displaystyle\int_0^T (Bv, \hat{y_2}) dt \quad \forall \ \hat{y_2} \in W_2 \ . \end{array} \right. \tag{4.8}$$

**Remark 4.3.**

It is obvious that, given $v$ and $\lambda_1, \lambda_2$, the system (4.8) admits a unique solution. For instance $y_1$ is given by the solution of

$$\sigma_1(y_1, \hat{y_1}) = \sigma_1(\lambda_1, \hat{y_1}) - \mathcal{A}_\gamma(\lambda_1 + \lambda_2, \hat{y_1}) + \int_0^T (Bv, \hat{y_1}) dt \quad \forall \ \hat{y_1} \in W_1 \ .$$

□

**Remark 4.4.**

The equations (4.8) can be solved in parallel. □

**Remark 4.5.**
If one can choose $\lambda_1, \lambda_2$ such that

(4.9) $$y_i = \lambda_i$$

then (4.8) is equivalent to

$$\mathcal{A}_\gamma(y_1 + y_2, \hat{y_1}) = \int_0^T (Bv, \hat{y_1})dt$$
$$\mathcal{A}_\gamma(y_1 + y_2, \hat{y_2}) = \int_0^T (Bv, \hat{y_2})dt$$

so that $y_1 + y_2 = y(= y_\gamma)$ the solution of (3.7). □

We now define the new cost function

(4.10) $$\left| \begin{array}{l} \mathcal{J}(v,\lambda) = \frac{1}{2}\int_0^T \|v\|^2_{\mathcal{U}}\, dt + \frac{\beta}{2}\, \|y_1(T) + y_2(T) - y^T\,\|^2_H\ , \\ y_i \text{ solution of (4.8)}, \lambda = \{\lambda_1, \lambda_2\}\ . \end{array} \right.$$

According to Remark 4.5., we have

(4.11) $$\inf_{y_i = \lambda_i} .\mathcal{J}(v,\lambda) = \inf J_\gamma(v)\ .$$

It is therefore natural to introduce a **penalty term** (in order to take care of "$y_i = \lambda_i$") as follows :

(4.12) $$\left| \begin{array}{l} \mathcal{J}_\varepsilon(v,\lambda) = \frac{1}{2}\int_0^T \|v\|^2_{\mathcal{U}}\, dt + \frac{\beta}{2}\, \|y_1(T) + y_2(T) - y^T\,\|^2_H + \\ \qquad + \frac{1}{2\varepsilon}[\sigma_1(y_1 - \lambda_1) + \sigma_2(y_2 - \lambda_2)]\ ,\ \lambda = \{\lambda_1, \lambda_2\} \end{array} \right.$$

and to consider the problem

(4.13) $$\inf .\mathcal{J}_\varepsilon(v,\lambda),$$
$$v \in L^2(0,T;\mathcal{U}),\ \lambda = \{\lambda_1, \lambda_2\} \in W_1 \times W_2\ .$$

We study now (4.13) and **we show it gives an approximation of** (3.9), **which is itself an approximation** (as $\gamma \to 0$) **of** (3.6).

# Approximation results

We are going to show

**Theorem 5.1.** - *We assume that* (2.3), (4.6), (4.1), (4.2), (4.3) *hold true. The elliptic regularization parameter* $\gamma$ *is fixed (arbitrarily small).*

(i) *For* $\varepsilon > 0$ *fixed, problem* (4.13) *admits a unique solution* $v_\varepsilon, y_{i\varepsilon} - \lambda_{i\varepsilon}, i = 1, 2$.

(ii) *As* $\varepsilon \to 0$, *one has*

$$\inf \mathcal{J}_\varepsilon(v,\lambda) \to \inf J_\gamma(v) = J_\gamma(v_\gamma)$$

$$v_\varepsilon \to v_\gamma \qquad \text{in} \quad L^2(0,T;\mathcal{U}) \quad \text{weakly}$$

(in fact $v_\varepsilon = v_{\varepsilon,\gamma}$). □

We prove Theorem 5.1. in several steps.

**Step 1.** - The existence of a solution of (4.13) is straightforward, provided we notice that we have informations on $y_i - \lambda_i$ rather that on $y_i$.

**Step 2.** - Given $v$, we compute $y(v) = y_\gamma(v)$ and we decompose $y(v)$ in, say, $y(v) = z_1 + z_2$ , $z_i \in L^2(0,T;V_i)$.

Choosing $\lambda_i = z_i$, it follows that $y_i = z_i = \lambda_i$
so that

$$\inf \mathcal{J}_\varepsilon(v,\lambda) \leq J_\gamma(v) \qquad \forall\, v, i.e.$$

$$\inf \mathcal{J}_\varepsilon(v,\lambda) \leq \inf .J_\gamma(v) \qquad v \in L^2(0,T;\mathcal{U}). \tag{5.1}$$

□

**Step 3.** - It follows from (5.1) that, as $\varepsilon \to 0$,

$$v_\varepsilon \text{ remains in a bounded subset of } L^2(0,T;\mathcal{U}), \tag{5.2}$$

$$\sigma_i(y_{i\varepsilon} - \lambda_{i\varepsilon}) \leq c\sqrt{\varepsilon}. \tag{5.3}$$

**Step 4.** - We use (4.8) with $y_{i\varepsilon}, \lambda_{i\varepsilon}$ but we do not write for a moment the indices "$\varepsilon$". Let $\eta_1 + \eta_2$ be an arbitrary decomposition of $y_1 + y_2$

$$y_1 + y_2 = \eta_1 + \eta_2, \quad \eta_i \in W_i \tag{5.4}$$

(of course $\eta_i = y_i$ if $V_1 \cap V_2 = \{0\}$).

We choose $\hat{y}_i = \eta_i$ in (4.8) and we add up the results.

We obtain

$$\sigma_1(y_1 - \lambda_1, \eta_1) + \sigma_2(y_2 - \lambda_2, \eta_2) + \mathcal{A}(\lambda_1 + \lambda_2, y_2 + y_2) = \int_0^T (Bv, y_1 + y_2)dt \tag{5.5}$$

We observe that (writing $\mathcal{A}_\gamma(\varphi,\varphi) = \mathcal{A}_\gamma(\varphi)$)

$$\left|\begin{array}{l} \mathcal{A}_\gamma(\lambda_1 + \lambda_2, y_1 + y_2) = \dfrac{1}{2}[\mathcal{A}_\gamma(\lambda_1 - y_1 + \lambda_2 - y_2, y_1 + y_2) + \mathcal{A}_\gamma(y_1 + y_2) + \\ \quad +(\mathcal{A}_\gamma(\lambda_1 + \lambda_2, y_1 - \lambda_1 + y_2 - \lambda_2) + \mathcal{A}_\gamma(\lambda_1 + \lambda_2)] \geq \\ \quad \geq c[\ \|y_1 + y_2\|_W^2 + \|\lambda_1 + \lambda_2\|_W^2] - c\sqrt{\varepsilon}\ [\ \|y_1 + y_2\|_W + \|\lambda_1 + \lambda_2\|_W]\ , \end{array}\right. \tag{5.6}$$

(where the c's denote various constants).

We also observe that

$$|\ \sigma_1(y_1 - \lambda_1, \eta_1) + \sigma_2(y_2 - \lambda_2, \eta_2)\ | \leq c\sqrt{\varepsilon}\ (\|\eta_1\|_{W_1} + \|\eta_2\|_{W_2}). \tag{5.7}$$

We can choose $\eta_1, \eta_2$ in such a way that

$$\|\eta_1\|_{W1} + \|\eta_2\|_{W_2} \leq c\|\eta_1 + \eta_2\|_W$$

so that (5.7) implies

$$(5.8) \qquad \mid \sigma_1(y_1 - \lambda_1, \eta_1) + \sigma_2(y_2 - \lambda_2, \eta_2) \mid \leq c\sqrt{\varepsilon}\, \|y_1 + y_2\|_W .$$

It follows from (5.5) (5.6) and (5.8) that, as $\varepsilon \to 0$,

$$(5.9) \qquad \|y_{1\varepsilon} + y_{2\varepsilon}\|_W + \|\lambda_{1\varepsilon} + \lambda_{2\varepsilon}\|_W \leq c .$$

**Step 5.** - One verifies that one can then pass to the limit in $\varepsilon$ in the equations (4.8) (where $\lambda_1 = \lambda_{i\varepsilon}, y_i = y_{i\varepsilon}$). One extracts a subsequence $v_\varepsilon, \lambda_{i\varepsilon}, y_{i\varepsilon}$ such that

$$\begin{array}{lll} v_\varepsilon \to w & \text{in} & L^2(0,T;\mathcal{U}) \quad \text{weakly} , \\ y_{1\varepsilon} + y_{2\varepsilon} \to z & \text{in} & W \quad \text{weakly} , \\ y_{1\varepsilon} - \lambda_{i\varepsilon} \to 0 & \text{in} & W_i \quad (\text{like } \sqrt{\varepsilon}), \end{array}$$

and one obtains

$$\mathcal{A}_\gamma(z, \hat{y}_1) = \int_0^T (Bw, \hat{y}_1) dt \qquad \forall \hat{y}_1 \in W_1,$$

$$\mathcal{A}_\gamma(z, \hat{y}_2) = \int_0^T (Bw, \hat{y}_2) dt \qquad \forall \hat{y}_2 \in W_2,$$

so that $z = y(w) = y_\gamma(w)$.

One can also verify that

$$y_{1\varepsilon}(T) + y_{2\varepsilon}(T) \to z(T) \quad \text{in} \quad H \quad \text{weakly} .$$

Then

$$\mathcal{J}_\varepsilon(v_\varepsilon, \lambda_\varepsilon) \geq \frac{1}{2} \int_0^T \|v_\varepsilon\|_{\mathcal{U}}^2 \, dt + \frac{\beta}{2} \, \|y_{1_\varepsilon}(T) + y_{2_\varepsilon}(T) - y^T \, \|_H^2$$

implies

$$(5.10) \qquad \liminf \mathcal{J}_\varepsilon(v_\varepsilon, \lambda_\varepsilon) \geq J_\gamma(w) .$$

Comparing with (5.1), Theorem 5.1 follows. □

# Algorithms

We proceed with the computation of the 1st variation of $\mathcal{J}_\varepsilon(v, \lambda)$, in fact of $\varepsilon\mathcal{J}_\varepsilon(v, \lambda)$. We have :

(6.1)

$$\left| \begin{array}{l} \delta(\varepsilon\mathcal{J}_\varepsilon(v,\lambda)) = \varepsilon \int_0^T (v, \delta v)_{\mathcal{U}} dt + \varepsilon\, \beta(y_1(T) + y_2(T) - y^T, \delta y_1(T) + \delta y_2(T))_H + \\ \qquad +\sigma_1(y_1 - \lambda_1, \delta y_1 - \delta\lambda_1) + \sigma_2(y_2 - \lambda_2, \delta y_2 - \delta\lambda_2) . \end{array} \right.$$

It follows from (4.8) that

$$(6.2) \qquad \sigma_1(\delta y_1 - \delta\lambda_1, \hat{y}_1) + \mathcal{A}_\gamma(\delta\lambda_1 + \delta\lambda_2, \hat{y}_1) = \int_0^T (B\delta v, \hat{y}_1) dt$$

and the analogous equation for $\sigma_2(\delta y_2 - \delta\lambda_2, \hat{y}_2)$. If we take $\hat{y}_1 = y_1 - \lambda_1$ in (6.2), and the analogous choice with the index "2", we obtain that

(6.3)
$$\left| \begin{array}{l} \sigma_1(y_1 - \lambda_1, \delta y_1 - \delta\lambda_1) + \sigma_2(y_2 - \lambda_2, \delta y_2 - \delta\lambda_2) = \\ \qquad = \int_0^T (B\delta v, y_1 - \lambda_1 + y_2 - \lambda_2)dt - \mathcal{A}_\gamma(\delta\lambda_1 + \delta\lambda_2, y_1 - \lambda_1 + y_2 - \lambda_2)\ . \end{array} \right.$$

Let us introduce the adjoint $\mathcal{A}_\gamma^*$ of $\mathcal{A}_\gamma$ :

$$\mathcal{A}_\gamma^*(\varphi, \hat{\varphi}) = \mathcal{A}_\gamma(\hat{\varphi}, \varphi)$$

and let us define $p_1, p_2 \in W_1 \times W_2$ by

$$(6.4) \quad \left| \begin{array}{l} \sigma_1(p_1, \hat{p}_1) = \mathcal{A}_\gamma^*(y_1 - \lambda_1 + y_2 - \lambda_2, \hat{p}_1) - \varepsilon\ \beta(y_1(T) + y_2(T) - y^T, p_1(T)) \\ \qquad\qquad \forall\ \hat{p}_1 \in W_1 \\ \sigma_2(p_2, \hat{p}_2) = \mathcal{A}_\gamma^*(y_1 - \lambda_1 + y_2 - \lambda_2, \hat{p}_2) - \varepsilon\ \beta(y_1(T) + y_2(T) - y^T, p_2(T)) \\ \qquad\qquad \forall\ \hat{p}_2 \in W_2\ . \end{array} \right.$$

Then using (6.3) and (6.4) one obtains

$$(6.5) \quad \left| \begin{array}{l} \delta(\varepsilon\mathcal{J}_\varepsilon(v, \lambda)) = \displaystyle\int_0^T (\varepsilon v + B^*(y_1 - \lambda_1 + y_2 - \lambda_2), \delta v)_{\mathcal{U}}\, dt - \\ \qquad -\sigma_1(p_1, \delta\lambda_1) - \sigma_2(p_2, \delta\lambda_2)\ , \end{array} \right.$$

where $B^*$ is the adjoint of $B$ defined by

$$(6.6) \quad (B^*\ f, v)_{\mathcal{U}} = (f, Bv) \qquad \forall\ f \in V'\ , \quad \forall v \in \mathcal{U}\ .$$

The simplest (if not the most efficient) algorithm one can deduce from (6.5) is then the following. Assuming that $v^n, \lambda_1^n, \lambda_2^n, y_1^n, y_2^n$ have been computed, define

$$(6.7) \quad \left| \begin{array}{l} v^{n+1} = v^n - \rho(\varepsilon v^n + B^*(y_1^n - \lambda_1^n + y_2^n - \lambda_2^n)), \\ \lambda_1^{n+1} = \lambda_1^n + \rho\ p_1^n\ , \\ \lambda_2^{n+1} = \lambda_2^n + \rho\ p_2^n\ , \end{array} \right.$$

where $\rho > 0$ is chosen small enough.

Compute $y_1^{n+1}, y_2^{n+1}$ (in parallel) by (4.8), where one uses $v^{n+1}, \lambda_i^{n+1}$. Then compute $p_1^{n+1}, p_2^{n+1}$ (in parallel by (6.4) and proceed. □

**Remark 6.1.**

More powerful algorithms (conjugate gradients) are given, for similar situations, without control, in R. Glowinski, J.L. Lions and O. Pironneau[GLP99]. □

**Remark 6.2.**

As we already said, everything extends to the situation when

$$V = V_1 + \cdots + V_m, m > 2.$$

□

**Remark 6.3.**

The "elliptic regularization parameter" $\gamma$ is **fixed** ("small"). What happens to the above algorithms when $\gamma \to 0$ is an open question. □

# Remarks and extensions

**Remark 7.1.**

In principle all the methods introduced here apply to problems **without control**.

But one is led to 2-points Boundary Value Problems (BVP) **in time**, not a wise thing to do. Of course the situation is different when effective control is present, since there 2 points BVP are needed anyway (one way or the other). □

**Remark 7.2.**

In case there are constraints on $v$ then, of course, the algorithms in previous section should be modified accordingly. □

**Remark 7.3.**

All the methods presented here can apply, with suitable modifications, for systems modelled by

non linear PDE

or

hyperbolic (or Petrowsky type) models

or

Schroedinger models

or coupled models. We shall return to these questions on other occasions. □

**Remark 7.4.**

A systematic presentation of other decomposition methods for the control of distributed systems is given in the paper of J.L. Lions[Lio00]. □

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