Thirteenth International Conference on Domain Decomposition Methods
Editors: N. Debit, M.Garbey, R. Hoppe, J. Périaux, D. Keyes, Y. Kuznetsov

29 Efficient Schwarz Methods for Elliptic Mortar Finite Element Problems

P.E. Bjørstad[1], M. Dryja[2], T. Rahman[3]

Introduction

In this paper we investigate an additive and a hybrid Schwarz method for solving systems of algebraic equations resulting from the approximation of second order elliptic boundary value problems with (highly) discontinuous coefficients. The discretization is obtained by using the mortar finite element method on nonmatching meshes, a technique which was first introduced by Bernardi-Maday-Patera [BMP94]. Several efficient iterative methods have thereafter been developed for the mortar element, see for example [CW96, Dry96, Dry97, AMW99, CDS99, BDR00, BDW99, GP00, WK01], and the references therein. The work of this paper is a continuation of the work done in [BDR00], where two variants of the additive Schwarz methods were proposed, the average method and the coarse reformulated average method. The reformulated variant is obtained from the average variant by simply replacing its coarse space by the sum of two special coarse spaces, one associated with the subdomains and the other one defined on the skeleton of the partition of the domain. This results in an algorithm which is very well suited for parallel computation and at the same time retains the necessary convergence behavior of a good scalable additive type Schwarz method. In this paper we improve its parallel feature a step further by splitting the skeleton coarse space into two subspaces, associated with the set of vertices and the set of mortar nodes, respectively. Experiments show that this modification does not change the convergence behavior. In this connection, we also introduce a hybrid version of the method for the problem. Both methods are insensitive to jumps in the coefficients.

The remainder of this paper is organized as follows. In the next section we recall the mortar finite element method for the elliptic problem. Then, in the following two sections, we present our Schwarz methods, and in the last section, we show some preliminary numerical examples.

The Discrete Problem

Let $\overline{\Omega} = \cup_{i=1}^{N} \overline{\Omega}_i$ be the partition of the computational domain in two dimensions, where each Ω_i is a polygonal subregion (subdomain), and the subregions are nonoverlapping. We consider the following differential problem: Find $u^* \in H_0^1(\Omega)$ such that

$$a(u^*, v) = f(v), \; v \in H_0^1(\Omega), \tag{1}$$

[1]Institute for Informatics, University of Bergen, petter@ii.uib.no
[2]Department of Mathematics, University of Warsawa, dryja@mimuw.edu.pl
[3]Institute for Mathematics, University of Augsburg, talal.rahman@math.uni-augsburg.de

where

$$a(u,v) = \sum_{i=1}^{N} a_i(u,v) = \sum_{i=1}^{N} \rho_i(\nabla u, \nabla v)_{L^2(\Omega_i)},$$

and

$$f(v) = \int_{\Omega} fv\, dx = \sum_{i=1}^{N} \int_{\Omega_i} fv\, dx,$$

with ρ_i being positive and constant in each subregion. We remark that the proposed methods can be used as preconditioners for the problem when the coefficients ρ_i depend on x and are discontinuous only across the boundary of Ω_i. In which case, the constant ρ_i can be taken as an average of $\rho_i(x)$ over Ω_i.

We consider only the geometrically conforming case, i.e., the intersection between the closure of two different subdomains is either empty, a vertex, or a whole edge. The subdomains together form a coarse triangulation of the whole domain Ω with the mesh parameter $H = \max_i H_i$, where H_i is the diameter of Ω_i. In each subdomain Ω_i, we use triangular elements. We assume that the triangles touching the subdomain boundary $\partial\Omega_i$ are quasi-uniform, having a mesh size of order h_i. We do not put such restriction on the interior triangles. We also assume that the coarse triangulation of Ω and the fine triangulation in each Ω_i are shape regular in the sense of [Cia78]. The resulting triangulation can be nonmatching across subdomain interfaces.

Let $X_i(\Omega_i)$ be the finite element space of piecewise linear continuous functions defined on the triangulation of Ω_i and vanishing on $\partial\Omega_i \cap \partial\Omega$, and let

$$X^h(\Omega) = X_1(\Omega_1) \times X_2(\Omega_2) \cdots \times X_N(\Omega_N).$$

In order to describe the discrete problem, we need the following auxiliary notations and finite element spaces. Let Γ_{ij} be an open edge common to Ω_i and Ω_j, i.e., $\overline{\Gamma}_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j$, and let $W^{h_i}(\Gamma_{ij})$ and $W^{h_j}(\Gamma_{ij})$ be the restrictions of $X_i(\Omega_i)$ and $X_j(\Omega_j)$ onto Γ_{ij}, respectively. Note that each interface Γ_{ij} inherits two different discretizations from its two sides. We select one side of Γ_{ij} as the master side, called the mortar, and the other side as the slave side, called the nonmortar. Define the skeleton $\mathcal{S} = (\cup\partial\Omega_i) \setminus \partial\Omega$ as follows:

$$\overline{\mathcal{S}} = \cup_m \overline{\gamma}_m, \text{ and } \gamma_m \cap \gamma_n = \emptyset \text{ if } m \neq n,$$

where each γ_m denotes an open mortar edge. We write γ_m as $\gamma_{m(i)}$ if it is an edge of Ω_i, i.e., $\gamma_{m(i)} \subset \partial\Omega_i$. Let $\delta_m = \delta_{m(j)} \subset \partial\Omega_j$ be the corresponding open nonmortar edge of Ω_j that occupies the same geometrical space as $\gamma_{m(i)}$, i.e., $\gamma_{m(i)} = \Gamma_{ij} = \delta_{m(j)}$. See Fig. 1 for illustration, where a thick line is drawn on the mortar side of an interface. The thick dots are used to represent the end points of a mortar or a nonmortar. We say that a function on a mortar is nonzero if the corresponding thick line is black and zero if the edge is light gray. The same applies to the end points.

As a general rule for choosing the mortars and the nonmortars, we let $\gamma_{m(i)}$ be the mortar and $\delta_{m(j)}$ the corresponding nonmortar if $\rho_i \geq \rho_j$. This is necessary for our Schwarz methods to have a rate of convergence which is independent of the jump of the coefficients. We define by ν_i and γ_i respectively the set of vertices and the set of mortar nodes (nodes on open mortar edges) of Ω_i.

Since the triangulations on Ω_i and Ω_j may not match on their interface Γ_{ij}, the functions in $X^h(\Omega)$ can be discontinuous across the interface Γ_{ij}. A weak continuity is therefore imposed across the interface using a condition called the mortar condition. Let $u_h \in X^h$, where $u_h = \{u_i\}_{i=1}^N$. A function $u_h \in X^h$ satisfies the mortar condition on $\delta_{m(j)}$, if, for all functions $\psi \in M^{h_j}(\delta_{m(j)})$ $(\gamma_{m(i)} = \delta_{m(j)} = \Gamma_{ij})$,

$$\int_{\delta_{m(j)}} (u_{i|\gamma_{m(i)}} - u_{j|\delta_{m(j)}})\psi \, ds = 0. \tag{2}$$

Here the space $M^{h_j}(\delta_{m(j)})$ is a subspace of $W^{h_j}(\delta_{m(j)})$, with functions being constants on elements touching $\partial\delta_{m(j)}$. V^h is a subspace of X^h of functions which satisfy the mortar condition for all $\delta_m \subset \mathcal{S}$. The discrete problem has the form: Find $u_h^* = \{u_i\}_{i=1}^N \in V^h$ such that

$$a(u_h^*, v_h) = f(v_h), \quad \forall v_h \in V^h \tag{3}$$

where

$$a(u_h, v_h) = \sum_{i=1}^N a_i(u_i, v_i) = \sum_{i=1}^N \rho_i (\nabla u_i, \nabla v_i)_{L^2(\Omega_i)},$$

and $v_h = \{v_i\}_{i=1}^N \in V^h$. V^h is a Hilbert space with an inner product defined by $a(u_h, v_h)$. This problem has a unique solution and its error bound is known, see [BMP94].

Let $\{\phi_k\}$ be the set of basis functions of V^h so that $V^h = span\{\phi_k\}$. These basis functions are associated with the subdomain interior nodes (Ω_{ih}), the vertices (ν_i) and the mortar nodes $(\gamma_{m(i)h}, \gamma_{m(i)} \subset \partial\Omega_i)$, which are not on the boundary $\partial\Omega$. The values on the nonmortar nodes are determined by the mortar condition. We use $\Pi_m(u_i, u_j)$ to denote the values on the nonmortar side $\delta_{m(j)}$, where the values of u_i on the corresponding mortar side and the values of $u_{j|\partial\delta_{m(j)}}$ are given.

For the rest of the paper we use the following notations. $x_k^{(i)}$ is the local representation of the node x_k, indicating that the node belongs to $\overline{\Omega}_i$. $\varphi_k^{(i)}$ denotes the standard nodal basis function associated with the node $x_k^{(i)}$.

The Additive Schwarz Method

In this section we introduce the additive Schwarz method for the problem (3). The method is defined using the general framework for the additive Schwarz methods, see [SBG96], i.e., in terms of a decomposition of the global space V^h into subspaces and the bilinear forms defined on these subspaces.

The decomposition of the finite element space V^h takes the form

$$V^h = V^{(-2)} + V^{(-1)} + V^{(0)} + \sum_{i=1}^N V^{(i)}, \tag{4}$$

where $V^{(i)}, i = 1, \cdots, N$, is a subspace of V^h restricted to the subdomain Ω_i with zero values on $\partial\Omega_i$ and the remaining subdomains. The subspaces $V^{(-2)}$, associated with the vertices, and $V^{(-1)}$, associated with the mortar nodes, are defined as follows.

$$\begin{aligned} V^{(-1)} &= \left\{ v \in V^h : v(x) = 0,\ x \in \cup_i (\gamma_i \cup \Omega_{ih}) \right\}, \\ V^{(-2)} &= \left\{ v \in V^h : v(x) = 0,\ x \in \cup_i (\nu_i \cup \Omega_{ih}) \right\}. \end{aligned}$$

The sum $V^{(-2)} + V^{(-1)}$ equals the skeleton coarse space of the reformulated variant (cf. [BDR00]). Note that the basis functions on an interface have nonlocal supports on the non-mortar side, which results in a very dense coupling between the vertices and the mortar nodes in the skeleton coarse stiffness matrix. The idea of the above splitting of the skeleton coarse space is to eliminate the effect of such coupling in the algorithm, and, thereby, improving the computational complexity and the parallel property of the algorithm. The space $V^{(0)}$ is the same as the space $V^{(0)}$ of the reformulated variant. We restate its definition here, but first, some definitions and notations.

Let χ_i, associated with the subdomain Ω_i, be the piecewise linear continuous function on the triangulation of Ω_i, defined by its nodal values at $x \in \overline{\Omega}_{ih}$. For each such node x,

$$\chi_i(x) = \frac{1}{\sum_j \rho_j(x)},$$

where the sum is taken over the subdomains that x is connected to. We say that a node x_k is connected to the subdomain Ω_i if $x_k \in \overline{\Omega}_{ih}$. If the node $x_k \in \overline{\gamma}_{m(i)h}$ ($x_k \in \overline{\delta}_{m(i)h}$) then x_k is said to be connected to both Ω_i and Ω_j if $\gamma_{m(i)} = \delta_{m(j)}$ ($\delta_{m(i)} = \gamma_{m(j)}$). Note that for $\rho_i = \rho_j = 1$, χ_i is 1 at $x \in \Omega_{ih}$, $\frac{1}{2}$ at $x \in (\partial\Omega_{ih} \setminus \nu_i)$ and $\frac{1}{3}$ at $x \in \nu_i$.

We associate with each subdomain Ω_i the sets G_i and Q_i containing the indices of its neighboring subdomains defined as follows. G_i contains the index of a neighbor Ω_j if it shares an edge Γ_{ij} ($\overline{\Gamma}_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j$) with Ω_i. Q_i contains the index of a neighbor Ω_j if $\overline{\Omega}_i \cap \overline{\Omega}_j$ is a crosspoint, there is a subdomain Ω_k such that Γ_{ki} ($\overline{\Gamma}_{ki} = \overline{\Omega}_k \cap \overline{\Omega}_i$) and Γ_{jk} ($\overline{\Gamma}_{jk} = \overline{\Omega}_j \cap \overline{\Omega}_k$) are the two edges of Ω_k which intersect at that crosspoint, and Γ_{ki} is a mortar in Ω_k, cf. Fig. 1(c).

We are now ready to define the coarse space $V^{(0)}$ which is given as the span of its basis functions, $\Phi_i, i = 1, \cdots, N$, i.e.,

$$V^{(0)} = span\left\{\Phi_i : i = 1, \cdots, N\right\}. \tag{5}$$

Each function Φ_i, associated with the subdomain Ω_i, is a function in the finite element space V^h.

For an interior subdomain Ω_i ($\partial\Omega_i \cap \partial\Omega = \emptyset$), the function Φ_i is constructed in three steps. We define Φ_i first (*i*) on $\overline{\Omega}_i$, then (*ii*) on $\overline{\Omega}_j$ for $i \in G_j$, and then (*iii*) on $\overline{\Omega}_j$ for $i \in Q_j$.

(*i*) Φ_i on $\overline{\Omega}_i$ is given as

$$\Phi_i(x) = \begin{cases} 1, & x \in \Omega_{ih}, \\ \rho_i \chi_i(x), & x \in \gamma_{m(i)h} \cup \nu_i, \\ \rho_i \Pi_m(\chi_j, \chi_i)(x), & x \in \delta_{m(i)h}, \delta_{m(i)} = \gamma_{m(j)}. \end{cases} \tag{6}$$

(*ii*) Φ_i on $\overline{\Omega}_j$, where $i \in G_j$, we have two cases to consider. For the first case, let $\Gamma_{ij} = \delta_{m(j)} = \gamma_{m(i)}$, see Fig. 1(a). Then, on $\overline{\Omega}_j$,

$$\Phi_i(x) = \begin{cases} \rho_i \Pi_m(\chi_i, \chi_j)(x), & x \in \overline{\delta}_{m(j)h},\ \delta_{m(j)} = \gamma_{m(i)}, \\ \Phi(x), & x \in \overline{\delta}_{n(j)h},\ \partial\delta_{\underline{n(j)}} \cap \partial\delta_{m(j)} \neq \emptyset, \\ 0, & \text{at all other } x \text{ in } \overline{\Omega}_{jh}. \end{cases} \tag{7}$$

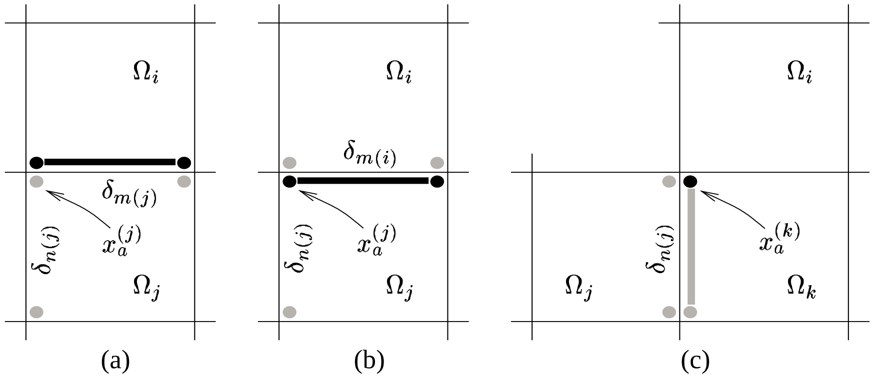

Figure 1: Illustrating Φ_i on $\overline{\Omega}_j$, where $i \in G_j$ ((a) and (b)) and $i \in Q_j$ ((c)). Here Φ_i is the basis function associated with the interior subdomain Ω_i.

For the second case, let $\Gamma_{ij} = \gamma_{m(j)} = \delta_{m(i)}$, see Fig. 1(b). Φ_i on $\overline{\Omega}_j$ is then given as

$$\Phi_i(x) = \begin{cases} \rho_i \chi_j(x), & x \in \overline{\gamma}_{m(j)h},\ \gamma_{m(j)} = \delta_{m(i)}, \\ \Phi(x), & x \in \overline{\delta}_{n(j)h},\ \partial\delta_{n(j)} \cap \partial\gamma_{m(j)} \neq \emptyset, \\ 0, & \text{at all other } x \text{ in } \overline{\Omega}_{jh}. \end{cases} \tag{8}$$

For the function $\Phi(x)$ in (7) and (8), we assume there is no vertex which is a cross point of exactly three subdomains. $\Phi(x)$ is then given as

$$\Phi(x) = \rho_i \chi_j(x_a^{(j)}) \Pi_n(0, \varphi_a^{(j)}), \tag{9}$$

where $x_a^{(j)} \in \nu_j$ (cf. figures 1a-1b).

(*iii*) Φ_i on $\overline{\Omega}_j$, where $i \in Q_j$, is given as follows. Let Γ_{ki} and Γ_{jk} be the two edges such that $\Gamma_{jk} = \delta_{n(j)} = \gamma_{n(k)}$ and $x_a^{(k)} = \partial\Gamma_{ki} \cap \partial\Gamma_{jk} \in \nu_k$, (cf. Fig. 1(c)). We have then

$$\Phi_i(x) = \begin{cases} \rho_i \chi_k(x_a^{(k)}) \Pi_n(\varphi_a^{(k)}, 0), & x \in \overline{\delta}_{n(j)h}, \\ 0, & \text{at all other } x \text{ in } \overline{\Omega}_{jh}. \end{cases} \tag{10}$$

On the remaining subdomains, $\Phi_i = 0$. This completes the definition of Φ_i for an interior subdomain Ω_i.

If Ω_i is a boundary subdomain ($\partial\Omega_i \cap \partial\Omega \neq \emptyset$) then the function Φ_i is defined as above but by imposing $\chi_j(x) = 0$ at $x \in \partial\Omega_{jh} \cap \partial\Omega_h$ for all $\Omega_j \in N_B$. The values of Φ_i on some nonmortar edges touching $\partial\Omega$ will be different, for details see [BDR00].

A somewhat similar but simpler coarse space defined in terms of discrete harmonic functions in the context of substructuring algorithms for mortar finite element problems can be found in [Dry97].

We use the exact bilinear form for all subproblems, i.e., for $i = -2, \cdots, N$ and $u, v \in V^{(i)}$, we define $b^{(i)}(.,.) : V^{(i)} \times V^{(i)} \to \mathbb{R}$ as $b^{(i)}(u,v) = a(u,v)$. The projection like operators $T^{(i)} : V^h \to V^{(i)}$ are defined in the standard way, i.e., for $i = -2, \cdots, N$ and $u \in V^h, T^{(i)}u \in V^{(i)}$ is the solution of

$$b^{(i)}(T^{(i)}u, v) = a(u, v),\ v \in V^{(i)}.$$

The additive Schwarz operator is then given as $T = \sum_{i=-2}^{N} T^{(i)}$, which can be written implicitly as BA, where B is the additive preconditioner. If we define $B^{(i)}$ as $T^{(i)} = B^{(i)}A$, then the action of B on a function r can be calculated as $v \leftarrow \sum_{i=-2}^{N} B^{(i)}r$. We have the following estimate for $T = BA$, the proof follows from [BDR00].

Theorem 1 *For $u \in V^h$,*

$$c_0 \frac{h}{H} a(u,u) \leq a(Tu,u) \leq c_1 a(u,u), \tag{11}$$

where both c_0 and c_1 are positive constants independent of the mesh parameters $h = \inf_i h_i$ and $H = \max_i H_i$ and the jumps of the coefficients ρ_i.

The Hybrid Schwarz Method

We introduce the hybrid method by replacing the additive preconditioner by the following hybrid preconditioner B. The action of B on r is now calculated in three steps as

$$\begin{aligned} v &\leftarrow (B^{(-2)} + B^{(-1)} + B^{(0)})r \\ v &\leftarrow v + B^{(i)}(r - Av), \; i = 1, \cdots, N \\ v &\leftarrow v + (B^{(-2)} + B^{(-1)} + B^{(0)})(r - Av). \end{aligned}$$

The last step is necessary for symmetrizing the preconditioner. Note that the subdomain solves in the second line can be done completely in parallel since we only have nonoverlapping subdomains. Basically, for this method, in each iteration, we need two extra calculations of the residual, and one extra solving of each coarse problem as compared to the additive method. The residual updates are, however, not expensive since we only need nearest neighbor communication among the subdomains (processors or virtual processors). Due to the special coarse spaces, it is very cheap to calculate the first residual update, and also, in this case, it is possible to avoid communication among the subdomains as only the values at the subdomain interior nodes are needed in the subdomain solves. The analysis of this method can be done using the general theory for Schwarz methods, see [SBG96], resulting in Theorem 1 for $T = BA$ where B is now the hybrid preconditioner.

Numerical Examples

We now present some numerical results using the Schwarz methods of this paper, as preconditioners for the conjugate gradient method. We compare the results with those of the reformulated average method introduced in [BDR00].

For simplicity, we let our model elliptic problem have zero boundary values. The force function f has the form $f(x) = 2\pi^2 \sin(\pi x_1) \sin(\pi x_2)$, and the domain is the unit square. The coefficients ρ_i are picked uniformly from the interval $[10^{-1}, 10^3]$ and then distributed randomly among the subdomains.

The test results are presented in Table 1. Each column of the table corresponds to a method, showing the iteration counts and the condition number estimates (in parentheses) for different partitions of the domain. The ratio $\frac{H}{h}$ remains fixed in all tests.

Subdomains	Additive method		Hybrid method
	Reform. variant	Modified reform.	
4×4	28 (13.36)	31 (16.07)	15 (4.18)
8×8	32 (13.75)	35 (16.19)	17 (4.21)

Table 1: *The number of iterations required to reduce the residual norm by* 10^{-6} *and a condition number estimate for each test.*

The additive Schwarz method of this paper ("Modified reform.") shows condition number estimates (iteration counts) which are close to those of the original reformulated variant ("Reform. variant"). The former method, however, needs less computation per iteration than the latter one. This is due to the splitting of the skeleton coarse space, which, in addition, makes the modified variant simpler and more suitable for parallel computation.

In the third column, we see a very substantial reduction in the condition number for the hybrid method. Thus, the hybrid method needs approximately half the number of iterations compared with the additive methods, but this is partially offset by more computation per iteration. So far, we have not made any comparison between these two methods considering a more detailed model of their computational complexity and parallel performance, this remains to be checked. The results show that the methods are all insensitive to jumps of the coefficient ρ_i across the subdomain boundaries.

We believe that this work extends and complements the work in [BDR00] and that a detailed computational study as well as experiments with realistic applications should follow in the future.

Acknowledgement

The second author was supported in part by the National Science Foundation under the Grant NSF-CCR-9732208 and in part by the Polish Science Foundation under the Grant 2 P03A 021 16.

References

[AMW99]Yves Achdou, Yvon Maday, and Olof B. Widlund. Iterative substructuring preconditioners for mortar element methods in two dimensions. *SIAM J. Numer. Anal.*, 36(2):551–580, 1999.

[BDR00]Petter E. Bjørstad, Maksymilian Dryja, and Talal Rahman. Additive average Schwarz methods for elliptic mortar finite element problems. Reports in Informatics 197, Institute for Informatics, University of Bergen, 2000. Submitted to Numer. Math.

[BDW99]Dietrich Braess, Wolfgang Dahmen, and Christian Wieners. A multigrid algorithm for the mortar finite element method. *SIAM J. Numer. Anal.*, 37:48–69, 1999.

[BMP94]Christine Bernardi, Yvon Maday, and Anthony T. Patera. A new non conforming approach to domain decomposition: The mortar element method. In Haim Brezis and Jacques-Louis Lions, editors, *Collège de France Seminar*. Pitman, 1994. This paper appeared as a technical report about five years earlier.

[CDS99]X.-C. Cai, M. Dryja, and M. V. Sarkis. Overlapping nonmatching grid mortar element methods for elliptic problems. *SIAM J. Numer. Anal.*, 36:581–606, 1999.

[Cia78]Philippe G. Ciarlet. *The Finite Element Method for Elliptic Problems*. North-Holland, Amsterdam, 1978.

[CW96]Mario A. Casarin and Olof B. Widlund. A hierarchical preconditioner for the mortar finite element method. *ETNA*, 4:75–88, June 1996.

[Dry96]Maksymilian Dryja. Additive Schwarz methods for elliptic mortar finite element problems. In K. Malanowski, Z. Nahorski, and M. Peszynska, editors, *Modeling and Optimization of Distributed Parameter Systems with Applications to Engineering*. IFIP, Chapman & Hall, London, 1996. To appear.

[Dry97]Maksymilian Dryja. An iterative substructuring method for elliptic mortar finite element problems with a new coarse space. *East-West J. Numer. Math.*, 5(2):79–98, 1997.

[GP00]Jayadeep Gopalakrishnan and Joseph E. Pasciak. Multigrid for the mortar finite element method. *SIAM J. Numer. Anal.*, 37(3):1029–1052, 2000.

[SBG96]Barry F. Smith, Petter E. Bjørstad, and William Gropp. *Domain Decomposition: Parallel Multilevel Methods for Elliptic Partial Differential Equations*. Cambridge University Press, 1996.

[WK01]Barbara Wohlmuth and Rolf Krause. Multigrid methods based on the unconstrained product space arising from mortar finite element discretizations. *SIAM J. Numer. Anal.*, 39:192–213, 2001.