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17. Regularized formulations of FETI

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1. Introduction. Our report introduces two regularized formulations of the FETI-1 [2, 3] algorithm. These formulations provide an alternative way for handling the rigid body modes (RBM) associated with floating subdomains. Both formulations start with the FETI-1 Lagrangian but differ in the treatment of the RBMs. They provide coercive bilinear forms on the floating subdomains resulting in symmetric, positive definite finite element linear systems and so pseudoinverse computations can be avoided.

Our report is organized as follows. Section 2 formulates a consistently stabilized variant of FETI-1. This is accomplished by augmenting the FETI-1 Lagrangian with a redundant term that uses a suitable set of solution moments. Section 3 also employs the FETI-1 Lagrangian and the same set of moments but uses them to induce a splitting of the Sobolev space for the floating subdomain. A brief summary of the relevant moments and their properties is given in Section 4.

We quickly review our use of standard notation. Let Ω be a bounded domain in $\mathbb{R}^d$ where $d = 2, 3$ with Lipschitz boundary $\partial\Omega$ and so let $H^1(\Omega)$ denote a Sobolev space of order 1; $H^1(\Omega, \partial\Omega_D)$ denote a subspace of $H^1(\Omega)$ consisting of functions that vanish on $\partial\Omega_D \subset \partial\Omega$. We further suppose that Ω is partitioned into two nonoverlapping subdomains Ω_1 and Ω_2 with interface Γ; let $H^{1/2}(\Gamma)$ denote the trace space of $H^1(\Omega_i)$ on Γ; and let the dual spaces of $H^1(\Omega, \partial\Omega_D)$ and $H^{1/2}(\Gamma)$ be denoted by $H^{-1}(\Omega, \partial\Omega_D)$ and $H^{-1/2}(\Gamma)$, respectively. Let the norms and inner products on $H^1(\Omega)$ be given by $\|\cdot\|_1$ and $(\cdot,\cdot)_1$, respectively; and let $\langle\cdot,\cdot\rangle$ denote the duality pairing between a space and its dual.

Finally, we define the moments $c(\cdot) : H^1(\Omega, \partial\Omega_D) \mapsto \mathbb{R}^p$ for some positive integer p.

2. FETI-CS: A consistently stabilized FETI-1 algorithm. We consider the problem

$$\inf_{v \in H^1(\Omega,\partial\Omega_D)} \frac{1}{2} a(v, v) - \langle f, v\rangle_\Omega \tag{2.1}$$

where $a(v, v)$ is a coercive symmetric bilinear form and $f \in H^{-1}(\Omega, \partial\Omega_D)$. For example, the bilinear form could represent a scalar Poisson or linear elasticity equation in the plane or space. Equivalently, the minimization problem (2.1) may be posed over the subdomains Ω_1 and Ω_2 and recast as: Find a saddle-point $(u_1, u_2, \lambda, \tau, \mu) \in H^1(\Omega_1, \partial\Omega_1) \times H^1(\Omega_2) \times H^{1/2}(\Gamma) \times \mathbb{R}^p \times \mathbb{R}^p$ for the Lagrangian

$$\mathcal{L}(\hat u_1, \hat u_2, \hat\lambda, \hat\tau, \hat\mu) = \sum_{i=1}^{2} \big(\frac{1}{2} a(\hat u_i, \hat u_i)_{\Omega_i} - \langle f, \hat u_i\rangle_{\Omega_i}\big) + \langle\hat\lambda, \hat u_1 - \hat u_2\rangle_\Gamma + \hat\tau^T (c(\hat u_2) - \hat\mu). \tag{2.2}$$

The last term introduces a Lagrange multiplier $\hat\tau$ for the difference of the moments of the Lagrange multiplier $\hat\mu$ representing the (unknown) moment of the minimizer of (2.1) on subdomain Ω_2. Without this term (2.2) is simply the FETI-1 Lagrangian.

The optimality system for (2.2) is: Find $(u_1, u_2, \lambda, \tau, \mu) \in H^1(\Omega_1, \partial\Omega_1) \times H^1(\Omega_2) \times$

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$H^{1/2}(\Gamma) \times \mathbb{R}^p \times \mathbb{R}^p$

$$\begin{array}{rcll} a(\hat{u}_1, u_1)_{\Omega_1} + \langle \hat{u}_1, \lambda \rangle_\Gamma &=& \langle f, \hat{u}_1 \rangle_{\Omega_1} & \forall \hat{u}_1 \in H^1(\Omega_1, \partial\Omega_1) \\ a(\hat{u}_2, u_2)_{\Omega_2} - \langle \hat{u}_2, \lambda \rangle_\Gamma + c(\hat{u}_2)^T \tau &=& \langle f, \hat{u}_2 \rangle_{\Omega_2} & \forall \hat{u}_2 \in H^1(\Omega_2) \\ \langle \hat{\lambda}, u_1 - u_2 \rangle_\Gamma &=& 0 & \forall \hat{\lambda} \in H^{1/2}(\Gamma) \\ \hat{\tau}^T (c(u_2) - \mu) &=& 0 & \forall \hat{\tau} \in \mathbb{R}^p \\ \hat{\mu}^T \tau &=& 0 & \forall \hat{\mu} \in \mathbb{R}^p. \end{array} \tag{2.3}$$

The last two equations imply that $\tau = \mathbf{0}$ and $c(u_2) - \mu = \mathbf{0}$. Therefore the last term of the Lagrangian (2.3) is a redundant constraint and we recover the FETI-1 optimality system

$$\begin{array}{rcll} a(\hat{u}_1, u_1)_{\Omega_1} + \langle \hat{u}_1, \lambda \rangle_\Gamma &=& \langle f, \hat{u}_1 \rangle_{\Omega_1} & \forall \hat{u}_1 \in H^1(\Omega_1, \partial\Omega_1) \\ a(\hat{u}_2, u_2)_{\Omega_2} - \langle \hat{u}_2, \lambda \rangle_\Gamma &=& \langle f, \hat{u}_2 \rangle_{\Omega_2} & \forall \hat{u}_2 \in H^1(\Omega_2) \\ \langle \hat{\lambda}, u_1 - u_2 \rangle_\Gamma &=& 0 & \forall \hat{\lambda} \in H^{1/2}(\Gamma). \end{array} \tag{2.4}$$

However, instead of using (2.3) directly, we stabilize the second and third constraints of (2.3) as

$$\begin{array}{ll} \hat{\tau}^T (c(u_2) - \mu) = \hat{\tau}^T \Upsilon^{-1} \tau & \forall \hat{\tau} \in \mathbb{R}^p \\ \hat{\mu}^T \tau = \hat{\mu}^T \Upsilon (c(u_2) - \mu) & \forall \hat{\mu} \in \mathbb{R}^p \end{array} \tag{2.5}$$

where Υ is a diagonal matrix of order p with positive diagonal elements. We can now eliminate τ from (2.3). Equation (2.5) implies

$$c(\hat{u}_2)^T \tau = c(\hat{u}_2)^T \Upsilon (c(u_2) - \mu). \tag{2.6}$$

With these relations, we obtain the optimality system: Find $(u_1, u_2, \lambda, \mu) \in H^1(\Omega_1, \partial\Omega_1) \times H^1(\Omega_2) \times H^{1/2}(\Gamma) \times \mathbb{R}^p$

$$\begin{array}{rcll} a(\hat{u}_1, u_1)_{\Omega_1} + \langle \hat{u}_1, \lambda \rangle_\Gamma &=& \langle f, \hat{u}_1 \rangle_{\Omega_1} & \forall \hat{u}_1 \in H^1(\Omega_1, \partial\Omega_1) \\ \tilde{a}(\hat{u}_2, u_2)_{\Omega_2} - \langle \hat{u}_2, \lambda \rangle_\Gamma - c(\hat{u}_2)^T \Upsilon \mu &=& \langle f, \hat{u}_2 \rangle_{\Omega_2} & \forall \hat{u}_2 \in H^1(\Omega_2) \\ \langle \hat{\lambda}, u_1 - u_2 \rangle_\Gamma &=& 0 & \forall \hat{\lambda} \in H^{1/2}(\Gamma) \\ -\hat{\mu}^T \Upsilon c(u_2) + \hat{\mu}^T \Upsilon \mu &=& 0 & \forall \hat{\mu} \in \mathbb{R}^p \end{array} \tag{2.7}$$

where $\tilde{a}(\cdot,\cdot)_{\Omega_2} \equiv a(\cdot,\cdot)_{\Omega_2} + c(\cdot)^T \Upsilon c(\cdot)$. We remark that this optimality system can also be derived by penalizing a FETI-1 Lagrangian by

$$\frac{1}{2} \| c(\hat{u}_2) - \hat{\mu} \|^2$$

or, equivalently, by replacing the last term of (2.2) with the above least-squares term. In either case, we have the following two results.

Lemma 2.1 *The symmetric bilinear form $\tilde{a}(\cdot,\cdot)_{\Omega_2}$ is coercive on $H^1(\Omega_2) \times H^1(\Omega_2)$.*

Proof. See Bochev and Lehoucq [1] for the proof. ∎

Theorem 2.1 *(u_1, u_2, λ) solves (2.4) if and only if $(u_1, u_2, \lambda, \mu = c(u_2))$ solves (2.7).*

Proof. The theorem is easily established by using the stabilized constraints (2.5) and recalling that $\tau = \mathbf{0}$. ∎

The theorem demonstrates that (2.5) represents a consistent stabilization. The impact of this innocuous sleight of hand is that the resulting coarse grid problem is equivalently stabilized. We now demonstrate this.

A conforming FEM for (2.7) results in the discrete optimality system

$$\begin{bmatrix} \mathbf{K}_1 & \mathbf{0} & \mathbf{B}_1^T & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{K}}_2 & -\mathbf{B}_2^T & -\mathbf{C}_2^T\Upsilon \\ \mathbf{B}_1 & -\mathbf{B}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\Upsilon\mathbf{C}_2 & \mathbf{0} & \Upsilon \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{2.8}$$

where $\tilde{\mathbf{K}}_2 \equiv \mathbf{K}_2 + \mathbf{C}_2^T\Upsilon\mathbf{C}_2$.

Elimination of the primal variables in (2.8) results in the coarse grid problem

$$\begin{bmatrix} \mathbf{B}_1\mathbf{K}_1^{-1}\mathbf{B}_1^T + \mathbf{B}_2\tilde{\mathbf{K}}_2^{-1}\mathbf{B}_2^T & \mathbf{B}_2\tilde{\mathbf{K}}_2^{-1}\mathbf{C}_2^T\Upsilon \\ \Upsilon\mathbf{C}_2\tilde{\mathbf{K}}_2^{-1}\mathbf{B}_2^T & \Upsilon\mathbf{C}_2\tilde{\mathbf{K}}_2^{-1}\mathbf{C}_2^T\Upsilon - \Upsilon \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_1 \end{bmatrix} \tag{2.9}$$

where

$$\begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1\mathbf{K}_1^{-1}\mathbf{f}_1 - \mathbf{B}_2\tilde{\mathbf{K}}_2^{-1}\mathbf{f}_2 \\ -\Upsilon\mathbf{C}_2\tilde{\mathbf{K}}_2^{-1}\mathbf{f}_2 \end{bmatrix}.$$

As compared with FETI-1, the columns of $\tilde{\mathbf{K}}_2^{-1}\mathbf{C}_2^T\Upsilon$ are approximating a basis for the rigid body modes associated with Ω_2, and $\tilde{\mathbf{K}}_2^{-1}$ is an approximation to the pseudoinverse of $\mathbf{K}_2$. Inserting the solution of the coarse grid problem (2.9) into (2.8) results in

$$\mathbf{K}_1\mathbf{u}_1 = \mathbf{f}_1 - \mathbf{B}_1^T\lambda \tag{2.10}$$

$$\tilde{\mathbf{K}}_2\mathbf{u}_2 = \mathbf{f}_2 + \mathbf{B}_2^T\lambda + \mathbf{C}_2^T\Upsilon\mu. \tag{2.11}$$

These two linear systems have symmetric positive definite coefficient matrices and can be solved in parallel.

We remark that (2.11) corresponds to the minimization problem

$$\inf_{v\in H^1(\Omega_2)} \frac{1}{2}\tilde{a}(v,v) - \langle \tilde{f}, v\rangle_{\Omega_2}$$

where $\tilde{f}$ is the continuous load associated with the discrete load of (2.11).

3. FETI-SS: Regularization by space splitting. In this section we introduce a modification of FETI-1 that allows for a wider choice of well-posed primal problems for these domains. In particular, our approach results in nonsingular linear systems with properties that can be easily controlled.

Our starting point is the splitting of $H^1(\Omega_2)$ into the direct sum

$$H^1(\Omega_2) = H^1_c(\Omega_2) \oplus \mathcal{N}_2$$

where $\mathcal{N}_2$ is the RBM space for Ω_2 and

$$H^1_c(\Omega_2) = \{u \in H^1(\Omega_2) \,|\, c_2(u) = 0\},$$

is the complement space with respect to the moments c_2. The report [1] demonstrates that such a splitting exists for any non-degenerate set of moments. As a result, any $u_2 \in H^1(\Omega_2)$ can be uniquely written as $u_{2c} + \mathbf{R}_2\alpha$ where $\mathbf{R}_2$ is a basis for $\mathcal{N}_2$ and $\alpha \in \mathbb{R}^p$. To solve (2.1) we consider the problem of finding the saddle-point $(u_1, u_{2c}, \alpha, \lambda) \in H^1(\Omega_1, \partial\Omega_1) \times H^1_c(\Omega_2) \times \mathbb{R}^p \times H^{1/2}(\Gamma)$ of the Lagrangian

$$\mathcal{L}(\hat{u}_1, \hat{u}_{2c}, \hat{\alpha}, \hat{\lambda}) = \sum_{i=1}^{2} \big(\frac{1}{2}a(\hat{u}_i, \hat{u}_i)_{\Omega_i} - \langle f, \hat{u}_i\rangle_{\Omega_i}\big) + \langle \hat{\lambda}, \hat{u}_1 - (\hat{u}_{2c} + \mathbf{R}_2\hat{\alpha})\rangle_\Gamma. \tag{3.1}$$

This Lagrangian only differs from the FETI-1 Lagrangian by explicitly specifying a particular solution on the floating subdomain. The optimality system for (3.1) is to seek $(u_1, u_{2c}, \alpha, \lambda) \in H^1(\Omega_1, \partial\Omega_1) \times H^1_c(\Omega_2) \times \mathbb{R}^p \times H^{1/2}(\Gamma)$ such that

$$\begin{array}{rcll} a(\hat{u}_1, u_1)_{\Omega_1} + \langle \hat{u}_1, \lambda\rangle_\Gamma & = & \langle f, \hat{u}_1\rangle_{\Omega_1} & \forall \hat{u}_1 \in H^1(\Omega_1, \partial\Omega_1) \\ a(\hat{u}_{2c}, u_{2c})_{\Omega_2} - \langle \hat{u}_{2c}, \lambda\rangle_\Gamma & = & \langle f, \hat{u}_{2c}\rangle_{\Omega_2} & \forall \hat{u}_{2c} \in H^1_c(\Omega_2) \\ -\langle \mathbf{R}_2\hat{\alpha}, \lambda\rangle_\Gamma & = & \langle f, \mathbf{R}_2\hat{\alpha}\rangle_{\Omega_2} & \forall \hat{\alpha} \in \mathbb{R}^p \\ \langle \hat{\lambda}, u_1 - (u_{2c} + \mathbf{R}_2\alpha)\rangle_\Gamma & = & 0 & \forall \hat{\lambda} \in H^{1/2}(\Gamma). \end{array} \tag{3.2}$$

Note that in (3.2) the floating subdomain problem is restricted to finding a particular solution out of the complement space $H^1_c(\Omega_2)$ rather than the space $H^1(\Omega_2)$. This seemingly minor change makes the floating subdomain problem *uniquely solvable*. Therefore, its conforming discretization, that is restriction to a finite element subspace of $H^1_c(\Omega_2)$, would engender a non-singular linear system. However, building a finite element subspace of $H^1_c(\Omega_2)$ may not be a simple matter and discretization by standard finite element subspaces of $H^1(\Omega_2)$ is preferred.

To enable the use of standard finite elements the floating subdomain equation is further replaced by a regularized problem in which the bilinear form $a(\cdot,\cdot)_{\Omega_2}$ is augmented by the term $c_2(\hat{u}_2)^T \Upsilon c_2(u_2)$. The regularized optimality system is to seek $(u_1, u_2, \alpha, \lambda) \in H^1(\Omega_1, \partial\Omega_1) \times H^1(\Omega_2) \times \mathbb{R}^p \times H^{1/2}(\Gamma)$ such that

$$\begin{array}{rcll} a(\hat{u}_1, u_1)_{\Omega_1} + \langle \hat{u}_1, \lambda\rangle_\Gamma & = & \langle f, \hat{u}_1\rangle_{\Omega_1} & \forall \hat{u}_1 \in H^1(\Omega_1, \partial\Omega_1) \\ a(\hat{u}_2, u_2)_{\Omega_2} + c_2(\hat{u}_2)^T \Upsilon c_2(u_2) - \langle \hat{u}_2, \lambda\rangle_\Gamma & = & \langle f, \hat{u}_2\rangle_{\Omega_2} & \forall \hat{u}_2 \in H^1(\Omega_2) \\ -\langle \mathbf{R}_2\hat{\alpha}, \lambda\rangle_\Gamma & = & \langle f, \mathbf{R}_2\hat{\alpha}\rangle_{\Omega_2} & \forall \hat{\alpha} \in \mathbb{R}^p \\ \langle \hat{\lambda}, u_1 - (u_2 + \mathbf{R}\alpha)\rangle_\Gamma & = & 0 & \forall \hat{\lambda} \in H^{1/2}(\Gamma). \end{array} \tag{3.3}$$

Theorem 3.1 *Problems (3.2) and (3.3) are equivalent.*

Proof. The only point that needs to be verified is that a solution $(u_1, u_2, \alpha, \lambda)$ of (3.3) has its second component in the complement space $H^1_c(\Omega_2)$. Choosing $\hat{u}_2 = \mathbf{R}_2\hat{\alpha}$ in the second equation in (3.3) combined with the third equation gives

$$c_2(\mathbf{R}_2\hat{\alpha})^T \Upsilon c_2(u_2) = \langle \mathbf{R}_2\hat{\alpha}, \lambda\rangle_\Gamma + \langle f, \mathbf{R}_2\hat{\alpha}\rangle_{\Omega_2} \equiv 0$$

for any $\hat{\alpha} \in \mathbb{R}^p$. Therefore, $c_2(u_2) = 0$ and $u_2 \in H^1_c(\Omega_2)$. ∎

A conforming FEM for (3.3) results in the linear system

$$\begin{bmatrix} \mathbf{K}_1 & \mathbf{0} & \mathbf{B}_1^T & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{K}}_2 & -\mathbf{B}_2^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -(\mathbf{B}_2\mathbf{R}_2)^T & \mathbf{0} \\ \mathbf{B}_1 & -\mathbf{B}_2 & \mathbf{0} & -\mathbf{B}_2\mathbf{R}_2 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \lambda \\ \alpha \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{R}_2^T\mathbf{f}_2 \\ \mathbf{0} \end{bmatrix} \tag{3.4}$$

where $\tilde{\mathbf{K}}_2$ is the same matrix as in (2.8) and we redundantly use $\mathbf{R}_2$ to denote the coefficients associated with the finite element approximants for the RBMs.

We note the close similarity between (3.4) and a FETI-1 discrete problem. In both cases a particular solution for the floating subdomain is generated and a component in $\mathcal{N}_2$ is added to satisfy the interface continuity condition. However, in contrast to a FETI-1, in (3.4) the floating subdomain matrix is non-singular and we have complete control over the choice of the particular solution by virtue of the moments c_2. These moments can be further selected so as to optimize the nonsingular matrix $\tilde{\mathbf{K}}_2$ with respect to a particular solver.

Elimination of the primal variables in (3.4) results in the coarse grid problem

$$\begin{bmatrix} \mathbf{B}_1\mathbf{K}_1^{-1}\mathbf{B}_1^T + \mathbf{B}_2\tilde{\mathbf{K}}_2^{-1}\mathbf{B}_2^T & -\mathbf{B}_2\mathbf{R} \\ -(\mathbf{B}_2\mathbf{R})^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \lambda \\ \alpha \end{bmatrix} = \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_1 \end{bmatrix} \tag{3.5}$$

where

$$\begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1\mathbf{K}_1^{-1}\mathbf{f}_1 - \mathbf{B}_2\tilde{\mathbf{K}}_2^{-1}\mathbf{f}_2 \\ \mathbf{R}^T\mathbf{f}_2 \end{bmatrix}.$$

Inserting the solution of the coarse grid problem (3.5) into (3.4) results in

$$\mathbf{K}_1\mathbf{u}_1 = \mathbf{f}_1 - \mathbf{B}_1^T\lambda \tag{3.6}$$

$$\tilde{\mathbf{K}}_2\mathbf{u}_2 = \mathbf{f}_2 + \mathbf{B}_2^T\lambda. \tag{3.7}$$

This primal system and the FETI-1 primal system only differ in the coefficient matrix for $\mathbf{u}_2$. Here $\tilde{\mathbf{K}}_2$ is symmetric positive definite whereas FETI-1 uses the singular $\mathbf{K}_2$. Therefore a computation of a pseudoinverse is avoided.

4. The moments $c(\cdot)$**.** Suppose that we have a floating subdomain Ω, a RBM subspace $\mathcal{N}$ and resulting basis $\mathbf{R}$ (discrete or continuous). The moments $c(\cdot)$ play a central role in our regularization strategy. Both of the FETI formulations introduced in this report rely upon these moments to regularize the floating subdomain problems. The purpose of the moments is to provide an “energy” measure for the RBMs that otherwise have zero strain energy $a(\cdot,\cdot)$.

Therefore, the guiding principle in their choice is to ensure that they form a non-degenerate set. By non-degenerate here we mean that the matrix $c(\mathbf{R})$ of order p is non-singular. For linear elasticity [1] one such set of moments is given by the functional

$$c(v) \equiv \begin{bmatrix} \displaystyle\int_\Omega \Theta_1 v \\ \displaystyle\int_\Omega \Theta_2 \nabla \times v \end{bmatrix} \tag{4.1}$$

where the diagonal elements of

$$\Theta_1 = \mathrm{diag}(\theta_{1,1}, \theta_{1,2}, \theta_{1,3}) \quad \text{and} \quad \Theta_2 = \mathrm{diag}(\theta_{2,1}, \theta_{2,2}, \theta_{2,3}) \tag{4.2}$$

are elements of $H^{-1}(\Omega)$ satisfying the hypothesis

$$\int_\Omega \theta_{1,i} \neq 0 \quad \text{and} \quad \int_\Omega \theta_{2,i} \neq 0$$

for $i = 1, 2, 3$. These dual functions serve the useful purpose of allowing us to enforce the mean and mean of the curl of the displacement along a portion of Ω.

When the moments (4.1) are restricted to finite element subspaces they generate full rank matrices with p columns, where p is the dimension of $\mathcal{N}$. The regularizing term added to the singular stiffness matrix on a floating subdomain is simply a rank-p correction to this matrix. When the dual functions in (4.2) have small supports the rank-p correction is a sparse matrix and the regularized problem is amenable to a direct solver methods. Larger supports generally improve the condition number of the regularized matrix but they also lead to formally dense systems. Therefore, regularization via moments is useful for iterative solution methods where it is only necessary to compute the product of the rank-p correction matrix with a direction vector.

5. Conclusions. Our report introduced two regularized formulations of the FETI- 1 [2, 3] algorithm. These formulations provide an alternative way for handling the rigid body modes (RBM) associated with floating subdomains. Both formulations start with the FETI-1 Lagrangian but differ in the treatment of the RBMs. They provide coercive bilinear forms on the floating subdomains resulting in symmetric, positive definite finite element linear systems and so pseudoinverse computations can be avoided.

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