MS09 FETI and Neumann-Neumann Domain Decomposition Methods

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The FETI and Neumann-Neumann families of iterative substructuring methods are among the best known and most severely tested domain decomposition methods for elliptic partial differential equations. The minisymposium will cover new theoretical and computational developments with applications in fluid and structural mechanics and acoustic scattering.

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Charbel Farhat, Jing Li

An Iterative Domain Decomposition Method for the Solution of a Class of Indefinite Problems in Computational Structural Dynamics

Location: Lecture Room, **Time:** Tuesday, 22 July, 11:00

The FETI-DP domain decomposition method (DDM) is extended to address the iterative solution of a class of indefinite problems of the form $(\mathbf{A} - \sigma\mathbf{M})\mathbf{x} = \mathbf{b}$, where $\mathbf{A}$ and $\mathbf{M}$ are two symmetric positive semi-definite matrices arising from the finite element discretization of elastodynamic problems, and σ is a positive number. A key component of this extension is a new coarse problem based on the free-space solutions of Navier's homogeneous displacement equations of motion. These solutions are waves, and therefore the resulting DDM is reminescent of the FETI-H method. For this reason, it is named here the FETI-DPH method. For a given σ, this method is numerically shown to be scalable with respect to all of the global problem size, subdomain problem size, and number of subdomains. Its intrinsic performance for various ranges of σ is illustrated with the solution on an Origin 3800 parallel processor as well as a beowulf cluster of several large-scale structural dynamics and vibration problems.

Olof B. Widlund, Axel Klawonn, Oliver Rheinbach

Selecting Primal Constraints for FETI-DP Algorithms for Linear Elasticity

Location: Lecture Room, **Time:** Tuesday, 22 July, 11:25

The FETI algorithms form one of three families of domain decomposition methods that have been implemented and severely tested on the very largest existing parallel computer systems; the other two are the Balancing Neumann-Neumann methods and the Overlapping Schwarz methods with at least two levels.

For the classical Balancing Neumann-Neumann algorithm, the global space has a dimension of six times the number of floating substructures; it also helps to include six degrees of freedom for the substructures that are constrained by boundary conditions. This count is a direct reflection of the fact that there are six linearly independent rigid body modes; we will only discuss problems in three dimensions. The same count is also typical for the coarse space of a standard Overlapping Schwarz methods which include a second coarse level.

In this contribution, the much more subtle issue of selecting a small but efficient set of primal constraints for Dual-Primal FETI methods will be discussed. The goal is, if possible, to select a smaller or otherwise cheaper coarse problem than those of the main competitors while also guaranteeing a rapid convergence of the FETI-DP algorithm even in the presence of large discontinuities in the Lame parameters. The possibility of using auxiliary computations involving a few substructures at a time in the selection of effective primal constraints will also be discussed.

This research is part of a project conducted jointly with Axel Klawonn and Oliver Rheinbach of the University of Essen, Germany.

Radek Kucera, Jaroslav Haslinger, Zdenek Dostal

The FETI Based Domain Decomposition Method for Solving 3D-Multibody Contact Problems with Coulomb Friction

Location: Lecture Room, **Time:** Tuesday, 22 July, 11:50

The contribution deals with the numerical solution of contact problems with Coulomb friction for 3D bodies. First we introduce auxiliary problems with given friction defining a mapping Φ which associates with a given slip bound the normal contact stress in the equilibrium state. Solutions to contact problems with Coulomb friction are defined as fixed points of Φ and are computed by using the method of successive approximations. The mathematical model of contact problems with given friction leads to a variational inequality of the second kind. Its discretization is based on the so called mixed variational formulation in terms of displacements and stresses on the contact boundary. Therefore the mixed finite element method is used with a piecewise linear approximation of displacements and a piecewise constant approximation of stresses. This discretization reduces the size of the resulting algebraic problem providing a coarse grid for the approximation of contact stresses. In contrast to 2D case, constraints imposed on tangential components of contact stresses are quadratic. After a linear approximation of these constraints, we are able to use our fast algorithm based on the augmented Lagrangian to the solution of quadratic programming problems with simple bounds and equality constraints. The final version of the algorithm for contact problems with Coulomb friction is a variant of the FETI based domain decomposition method.

Hyeahyun Kim, Chang-Ock Lee

A FETI-DP Method for the Stokes Problems on Nonmatching Grids

Location: Lecture Room, **Time:** Tuesday, 22 July, 12:15

In this talk, we consider a FETI-DP method for the Stokes problems on nonmatching grids in $2D$. FETI-DP method is a domain decomposition method that uses Lagrange multipliers to match the solutions continuously across the subdomain boundaries in the sense of dual-primal variables. We use the $P_1(h)-P_0(2h)$ inf-sup stable finite elements solving for the Stokes problems and the mortar matching condition on the velocity functions as the constraints for the FETI-DP formulation. Moreover, to satisfy the compatibility condition of local Stokes problems, redundant constraints are introduced. The Lagrange multipliers corresponding to the redundant constraints are treated as primal variables in the FETI-DP formulation. We propose a preconditioner for the FETI-DP operator, which is derived from the dual norm on the Lagrange multiplier space. The dual norm is obtained from the duality pairing between the Lagrange multiplier space and the velocity function space restricted on the slave sides. Then, we show that the condition number of the preconditioned FETI-DP operator is bounded by $C \max_{i=1,\cdots,N} \left\{(1+\log{(H_i/h_i)})^2\right\}$, where H_i and h_i are the sizes of the subdomains and meshes for each subdomain, respectively, and C is a constant independent of H_i's and h_i's.

Dan Stefanica

Parallel FETI Algorithms for Mortars

Location: Lecture Room, **Time:** Wednesday, 23 July, 11:00

We propose a new version of the FETI algorithm which preserves the parallelization properties of the classical FETI algorithms when applied to mortar discretizations. This new version is based on generalized coupling conditions across the interface replacing the mortar conditions.

We present numerical results showing that the new FETI algorithm has the same scalability properties as the classical FETI method. We compare the numerical performance of the algorithm proposed here with that of the FETI and FETI–DP methods for mortars. We also discuss implementation details and storage requirements for the new algorithm, both for general geometrically nonconforming situations and for the case when mortar conditions are required only on a small part of the interface, while continuity is required elsewhere.

We conclude by discussing extensions of our algorithm to problems containing Linear Multipoint Constraints.

Oliver Rheinbach, Axel Klawonn, Olof B. Widlund

Some Computational Results for Dual-Primal FETI Methods for Three Dimensional Elliptic Problems

Location: Lecture Room, **Time:** Wednesday, 23 July, 11:25

Iterative substructuring methods with Lagrange multipliers for elliptic problems are considered. The algorithms belong to the family of dual-primal FETI methods which were introduced for linear elasticty problems in the plane by Farhat, Lesoinne, Le Tallec, Pierson, and Rixen and were later extended to three dimensional elasticity problems by Farhat, Lesoinne, and Pierson.

In a recent paper, the family of algorithms for scalar diffusion problems was extended to three dimensions and was successfully analyzed by Klawonn, Widlund, and Dryja. It has been shown that the condition number of these dual-primal FETI algorithms can be bounded polylogarithmically as a function of the dimension of the individual subregion problems and that the bounds are otherwise independent of the number of subdomains, the mesh size, and jumps in the diffusion coefficients.

In this talk, numerical results for some of these algorithms are presented and their relation to the theoretical bounds is investigated.

The presented results are joint work with Axel Klawonn, Essen, Germany and Olof Widlund, New York, USA.

Christian Rey, Pierre Gosselet

An Hybrid Domain Decomposition Method
Location: Lecture Room, **Time:** Wednesday, 23 July, 11:50

In the field of non overlapping domain decomposition methods, two classical choices are the primal and the dual Schur complement approaches. The first leads to the Balanced Domain Decomposition [1,2] and the second to the Finite Element Tearing and Interconnecting method [3]. Once preconditioned with their most efficient preconditioner, these methods are highly similar: from a mechanical point of view they are based on the same concepts [4], from a mathematical point of view they reach the same theoretical performance levels [5], from a numerical point of view little improvements [6] make them equally efficient, from a computational point of view usually the same solver is used (projected conjugate gradient), from an implementation point of view only few programming tricks differ from one to another.

We here propose a non overlapping domain decomposition method which provides a general frame which includes traditional primal and dual Schur complement approaches. This so called "hybrid" domain decomposition is based on a partition of the interface degrees of freedom into two sets. DOFs belonging to the first set are processed as primal variables whereas DOFs belonging to the second set are processed as dual variables (note that unlike FETIDP [7] primal DOFs are not suppressed from the resolution process) ; the condensation of the global problem onto the interface requires the computation of a hybrid Schur complement. The resulting interface problem is the sum of local contributions, it is solved with a Krylov iterative solver (GMRes), the preconditioner we propose is composed by a scaled sum of inverses of local contributions. Handling zero energy modes leads to two coarse grid problems (one for the operator, one for the preconditioner) which are managed using projectors ; a third optional coarse problem can also be added.

This new approach has several interests. From a numerical point of view, it leads to performance results comparable to the classical approaches. From an implementation point of view, it proves that primal and dual approaches can be joined into a unique code. From a mechanical point of view, it enables new treatments of multifield problems such as the one involving displacement and pressure fields arising from the study of porous media or incompressible materials. From a mathematical point of view, it offers a framework which may enable to unify theoretical performance results of primal and dual approaches. Last, the open framework provided enables to define more than two sets of DOFs, one can imagine to add a third set of DOFs processed as "direct" variables (like in FETIDP) and a forth set of DOFS processed as mixed variables (Robin conditions), which may lead to a method including most of the non-overlapping domain decomposition approaches. Numerical assessments will be presented on various structural mechanic problems and multiphysics problems.

[1] J. Mandel, *Balancing domain decomposition*, Comm. Appl. Num. Meth. Engrg. 9 (1993) 233-241

[2] P. Le Tallec, *Domain decomposition methods in computational mechanics*, Computational Mechanics Advances 1 (2) (1994) 121-220, north-Holland.

[3] C. Farhat and F.X. Roux, *A method of Finite Element Tearing and Interconnecting and its parallel solution algorithm*, Int. J. Num. Meth. Engrg. 32 (1991) 1205-1227.

[4] D. Rixen, *Encyclopedia of vibration*, Academic Press (2002) Ch. Parallel Computation, 990-1001.

[5] A. Klawon and O. Widlund, *FETI and Neumann-Neumann iterative substructuring mathods: connections and new results*, Com. Pure App. Math. 54 (1) (2001) 57-90.

[6] P. Gosselet, C. Rey and D. Rixen, *On the initial estimate of interface forces in FETI methods*, accepted in Comp. Meth. App. Mech. Engrg.

[7] C. Farhat, M. Lesoinne, P. Le Tallec, K. Pierson and D. Rixen, *FETI-DP: a dual-primal unified FETI method*, Int. JH. Num. Meth. Engrg. 50 (7) (2001) 1523-1544.

Yannis Fragakis, Manolis Papadrakakis

A Family of FETI-Derived Preconditioners for the Primal Substructuring Method: Application to Multiple Right-Hand Side Problems and Implicit Dynamic Analysis
Location: Lecture Room, **Time:** Wednesday, 23 July, 12:15

Over the last decades, DDM have emerged as a powerful computational tool for the static or dynamic analysis of several problems of Computational Mechanics. The last decade, in particular, witnessed the introduction of two major families of DDM, namely the FETI [1] and the Balancing Domain Decomposition (BDD) [2] families of methods. Since their introduction, the two methods evolved fast and were successfully applied to static or dynamic analysis of several problems in Computational Mechanics.

Recently, two new families of DDM, namely a family of FETI-derived primal preconditioners (also called a primal class of FETI methods) and a family of two-level primal substructuring methods were introduced [3,4]. Theoretical aspects and the computational performance of these new methods and the standard FETI and BDD methods were compared in a general study [4]. In that study, it was found that while the introduced two-level primal substructuring methods are equivalent to the BDD methods, the new primal class of FETI methods is superior to the standard FETI methods when applied to heterogeneous problems and/or plate and shell problems.

The first comparison [3,4] of the standard and novel DDM was focused on the basic problem of single right-hand side static structural analysis, while the present work extends this comparison to multiple right-hand side problems and Implicit Dynamics. Furthermore, the discussed methods are equipped with a new technique that accelerates their convergence, while a new method particularly tailored for dynamic problems is proposed. In general, the incorporation of the proposed methods in the family of high-performance DDM brings new possibilities for the efficient analysis of multiple right-hand side and dynamic problems.

[1] C. Farhat, F.X. Roux, *A method of finite element tearing and interconnecting and its parallel solution algorithm*, Int. J. Numer. Meth. Engng., 32, 1205-1227 (1991).

[2] J. Mandel, *Balancing domain decomposition*, Commun. Appl. Numer. Meth., 9, 233-241 (1993).

[3] Y. Fragakis, M. Papadrakakis, *A Family of New FETI Methods for Structural Mechanics*, Proceedings of the Fifth World Congress on Computational Mechanics (WCCM-V), Vienna, Austria, July 7-12, 2002, http://wccm.tuwien.ac.at

[4] Y. Fragakis, M. Papadrakakis, *The mosaic of high performance Domain Decomposition Methods for Structural Mechanics: Formulation, interrelation and numerical efficiency of primal and dual methods*, Comput. Methods Appl. Mech. Engrg., 2003 (Accepted).

Max Dryja, Olof B. Widlund

A Dual-Primal FETI Method with Face Constraints for Mortar Discretization of Elliptic Problems

Location: Lecture Room, **Time:** Friday, 25 July, 9:00

A discretization of elliptic problems on nomatching triangulations is considered. The discrete problem is formulated using the finite element mortar method. For solving this problem a Dual-Primal FETI method, with face constraints only, is designed and analyzed and it is proved that its rate of convergence is almost optimal. The algorithm is well suited for parallel computations

Wlodek Proskurowski, Max Dryja

A FETI-DP Method for the Mortar Discretization of Elliptic Problems with Discontinuous Coefficients

Location: Lecture Room, **Time:** Friday, 25 July, 9:25

We consider an elliptic problem with discontinuous coefficients. The problem is discretized on non-matching triangulation determined by the jump of coefficients. The resulting discrete problem is obtained employing a mortar technique.

For this discrete problem we design and analyze a FETI-DP method. We extend the results obtained in [1] for problems with continuous coefficients on many subregions and in [2] for problems with discontinuous coefficients on two subdomains.

We prove that the convergence rate of the method is almost optimal and independent of the jump of the discontinuity. Theoretical results are confirmed by numerical experiments.

[1] M. Dryja and O. Widlund, *A generalized FETI-DP method for the mortar discretization of elliptic problems*, in I. Herrera et al. (eds.), Fourteenth Conference on Domain Decomposition Methods, 2003.

[2] M. Dryja and W. Proskurowski, *On preconditioners for the mortar discretization of elliptic problems*, Numer. Linear Algebra with Applications, vol.10, 65-82, 2003.

Andrea Toselli, Xavier Vasseur

FETI and Neumann-Neumann Preconditioners for hp Finite Element Approximations on Anisotropic Meshes: Algorithms and Theory

Location: Lecture Room, **Time:** Friday, 25 July, 9:50

We consider Neumann-Neumann and a class of FETI methods for hp finite element approximations of scalar elliptic problems on geometrically refined boundary layer meshes in two and three dimensions. These are meshes that are highly anisotropic: the aspect ratio typically grows exponentially with the polynomial degree. The condition number of our preconditioners is independent of arbitrarily large aspect ratios of the mesh and of potentially large jumps of the coefficients.

In addition, it only grows polylogarithmically with the polynomial degree, as in the case of p approximations on shape-regular meshes.

Xavier Vasseur, Andrea Toselli

FETI and Neumann-Neumann Preconditioners for hp Finite Element Approximations on Anisotropic Meshes: Numerical Validation

Location: Lecture Room, **Time:** Friday, 25 July, 10:15

We investigate numerically certain Balancing Neumann-Neumann and one–level FETI domain decomposition methods for the solution of algebraic linear systems arising from *hp* finite element approximations of scalar elliptic problems on geometrically refined boundary layer meshes. These are meshes that are highly anisotropic where the aspect ratio grows exponentially with the polynomial degree. Various two- and three-dimensional applications will be presented. The numerical results are found to be in good agreement with the theoretical bounds for the condition numbers of the preconditioned operators derived in Andrea Toselli's talk ("FETI and Neumann-Neumann preconditioners for hp finite element approximations on anisotropic meshes: Algorithms and theory"). They confirm that the condition numbers are independent of the aspect ratio of the mesh and of potentially large jumps of the coefficients. In addition, they only grow polylogarithmically with the polynomial degree, as in the case of p approximations on shape-regular meshes.