Nonlinear Advection Problems and Overlapping Schwarz Waveform Relaxation

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Summary. We analyze the convergence behavior of the overlapping Schwarz waveform relaxation algorithm applied to nonlinear advection problems. We show for Burgers' equation that the algorithm converges super-linearly at a rate which is asymptotically comparable to the rate of the algorithm applied to linear advection problems. The convergence rate depends on the overlap and the length of the time interval. We carefully track dependencies on the viscosity parameter and show the robustness of all estimates with respect to this parameter.

1 Introduction

Overlapping Schwarz waveform relaxation algorithms have been applied successfully to many evolution problems. However, a rigorous error analysis is only available in the case of linear and weakly nonlinear problems. For results covering the heat equation, advection-diffusion equations, and problems with nonlinear source terms, we refer to Gander [1997], Giladi and Keller [2002], Gander [1998], Daoud and Gander [2000], Gander and Zhao [2002].

We present here first convergence results for the algorithm applied to a class of strongly nonlinear problems: scalar parabolic conservation laws with nonlinear fluxes. In Section 2 we present the problem and the necessary analytical background. In particular we focus on conservation laws in the advection dominated case when the problem is singularly perturbed. In Section 3 we introduce the Schwarz waveform relaxation algorithm for parabolic conservation laws. In Section 4 the error analysis for the algorithm is presented for the special case of Burgers' equation. We focus on two topics: the comparison of the results for the linear and the nonlinear case and the influence of the diffusion parameter ε on the convergence rate. The paper concludes with a numerical experiment that confirms the theoretical results.

We note that there is a fundamentally different approach to solve nonlinear conservation laws using domain decomposition. One first discretizes the problem uniformly in time using an implicit scheme, and then applies domain decomposition to the steady problems obtained at each time step, see Dolean et al. [2000] and references therein. For a heterogeneous approach, see also Garbey [1996], Garbey and Kaper [1997].

2 Advection Dominated Conservation Laws

We consider for $T > 0$ and a function $u_0 \in W^{1,\infty}(\mathbb{R})$ the initial boundary value problem

$$\frac{\partial u^\varepsilon}{\partial t} + \frac{\partial}{\partial x} f(u^\varepsilon) = \varepsilon \frac{\partial^2 u^\varepsilon}{\partial x^2} \text{ in } \mathbb{R} \times (0,T), \quad u^\varepsilon(.,0) = u_0 \text{ in } \mathbb{R} \tag{1}$$

for the unknown $u^\varepsilon = u^\varepsilon(x,t) : \mathbb{R} \times (0,T) \to \mathbb{R}$. Here $\varepsilon > 0$ is a constant and $f \in C^2(\mathbb{R})$ denotes the possibly nonlinear flux function. The scalar problem (1) is a simple model for nonlinear systems of conservation laws which arise frequently to describe dynamical processes in continuum mechanics. Important examples are the Navier-Stokes equations in fluid mechanics and the system of thermo-elasticity in solid mechanics. An interesting feature of many applications governed by conservation laws is the fact that they are *advection dominated*. On the level of problem (1) this implies that the diffusion parameter ε is small and we have to consider a singularly perturbed problem. In the limit $\varepsilon = 0$ the parabolic equation in (1) changes type and becomes the hyperbolic equation

$$\frac{\partial u^0}{\partial t} + \frac{\partial}{\partial x} f(u^0) = 0 \quad \text{in } \mathbb{R} \times (0,T) \tag{2}$$

for the unknown $u^0 : \mathbb{R} \times (0,T) \to \mathbb{R}$. It is well-known that classical solutions of the initial value problem for (2) do not exist globally in time for all smooth initial data if f is nonlinear, see for example Dafermos [2000]. Singularities called shock waves occur. In the singularly perturbed case with $\varepsilon > 0$, diffusive layers with $\frac{\partial u^\varepsilon}{\partial x} = \mathcal{O}(\varepsilon^{-1})$ take the role of shock waves. The following theorem reflects the relationships between solutions of (1), (2) and summarizes the results we need later from the theory of conservation laws.

Theorem 1. *There exists a unique classical solution $u^\varepsilon \in C^1(0,T;C^2(\mathbb{R}))$ of (1) that satisfies*

$$\inf_{x\in\mathbb{R}}\{u_0(x)\} \le u^\varepsilon(x,t) \le \sup_{x\in\mathbb{R}}\{u_0(x)\}, \quad (x,t) \in \mathbb{R} \times [0,T],$$
$$\varepsilon \|u_x^\varepsilon\|_{L^\infty(\mathbb{R}\times[0,T])} \le C,$$

where the positive constant C does not depend on ε. Furthermore there exists a function $u^0 \in L^\infty(\mathbb{R} \times [0,T])$ such that for each compact set $Q \subset \mathbb{R}$ we have

$$\lim_{\varepsilon\to 0} \|u^0 - u^\varepsilon\|_{L^1(Q\times[0,T])} = 0.$$

The analogous statements hold for the initial boundary value problem with Dirichlet boundary conditions.

The proofs of these results are classical and can be found for instance in Dafermos [2000].

3 Overlapping Schwarz Waveform Relaxation

We approximate the solution of (1) using the overlapping Schwarz waveform relaxation algorithm on the two subdomains $\Omega_1 = (-\infty, L)$ and $\Omega_2 = (0, \infty)$ with overlap $L > 0$. The parameter ε is fixed here, so we therefore drop the index ε in this and the next section to simplify the notation. For iteration index $n \in \mathbb{N}$, the overlapping Schwarz waveform relaxation algorithm is defined by

$$\begin{aligned} \frac{\partial u_1^n}{\partial t} + f'(u_1^n)\frac{\partial u_1^n}{\partial x} &= \varepsilon \frac{\partial^2 u_1^n}{\partial x^2} && \text{in } \Omega_1 \times (0,T),\\ u_1^n(\cdot,0) &= u_0 && \text{in } \Omega_1,\\ u_1^n(L,\cdot) &= u_2^{n-1}(L,\cdot) && \text{on } [0,T], \end{aligned} \tag{3}$$

and

$$\begin{aligned} \frac{\partial u_2^n}{\partial t} + f'(u_2^n)\frac{\partial u_2^n}{\partial x} &= \varepsilon \frac{\partial^2 u_2^n}{\partial x^2} && \text{in } \Omega_2 \times (0,T),\\ u_2^n(\cdot,0) &= u_0 && \text{in } \Omega_2,\\ u_2^n(0,\cdot) &= u_1^{n-1}(0,\cdot) && \text{on } [0,T]. \end{aligned} \tag{4}$$

4 Convergence Analysis

We first review results on the iteration (3), (4) for the linear flux $f(u) = cu, \quad c \in \mathbb{R}$. We define the errors in the Schwarz waveform relaxation iteration by $e_1^n := u - u_1^n$ on the left subdomain and $e_2^n := u - u_2^n$ on the right subdomain for $n \in \mathbb{N}_0$. For $n \in \mathbb{N}$, we find that the error e_1^n satisfies

$$\begin{aligned} \frac{\partial e_1^n}{\partial t} + c\frac{\partial e_1^n}{\partial x} &= \varepsilon \frac{\partial^2 e_1^n}{\partial x^2} && \text{in } \Omega_1 \times (0,T),\\ e_1^n(\cdot,0) &= 0 && \text{in } \Omega_1,\\ e_1^n(L,\cdot) &= e_2^{n-1}(L,\cdot) && \text{on } [0,T], \end{aligned} \tag{5}$$

and the analogous equations hold for e_2^n. The error analysis for (5) has been performed in Gander [1997] and independently in Giladi and Keller [2002]. We cite the final result.

Theorem 2 (Linear Advection Diffusion). *The overlapping Schwarz waveform relaxation algorithm (3), (4) for the advection diffusion problem (1) with $f(u) = cu$ converges super-linearly. For each $T > 0$ and $i = 1, 2$ we have*

$$\sup_{x\in\Omega_i,0\le t\le T} |e_i^{2n}(x,t)| \le C_i \mathrm{erfc}\left(\frac{nL}{\sqrt{\varepsilon T}}\right), \tag{6}$$

where $C_1 = \sup_{0\le t\le T} |e_1^0(L,t)|$ *and* $C_2 = \sup_{0\le t\le T} |e_2^0(0,t)|$.

Remark 1. If we apply the expansion $\sqrt{\pi}\mathrm{erfc}(z) = e^{-z^2}(z^{-1} + \mathcal{O}(z^{-3}))$ for large values $z > 0$ in the estimate (6), we obtain

$$\sup_{x\in\Omega_i,0\le t\le T} |e_i^{2n}(x,t)| \approx \frac{C_i}{\sqrt{\pi}} e^{-\frac{n^2L^2}{\varepsilon T}} \frac{\sqrt{\varepsilon T}}{nL}.$$

For fixed $T, L, \varepsilon > 0$ we observe that the algorithm converges super-linearly for $n \to \infty$ and $t \le T$. The error vanishes also for $\varepsilon \to 0$, reflecting the fact that the algorithm applied to the pure advection equation converges in two steps.

We now consider the iteration (3), (4) for the quadratic flux $f(u) = \frac{u^2}{2}$, that is Burgers' equation. For $n \in \mathbb{N}$, we find that the error $e_1^n := u - u_1^n$ satisfies the equation

$$\begin{aligned} \frac{\partial e_1^n}{\partial t} + u\frac{\partial e_1^n}{\partial x} + \frac{\partial u_1^n}{\partial x} e_1^n &= \varepsilon \frac{\partial^2 e_1^n}{\partial x^2} && \text{in } \Omega_1 \times (0,T), \\ e_1^n(\cdot,0) &= 0 && \text{in } \Omega_1, \\ e_1^n(L,\cdot) &= e_2^{n-1}(L,\cdot) && \text{on } [0,T], \end{aligned} \tag{7}$$

and an analogous problem for e_2^n. We note that in contrast to the linear equation the error equations for the Burgers case contain an additional source term scaled with the spatial derivative of the iterate. Moreover due to Theorem 1 these terms behave like $\mathcal{O}(\varepsilon^{-1})$ (The estimates in Theorem 1 hold mutatis mutandis also for initial boundary value problems).

For our analysis of the non-linear case we require that the iteration starts with the initial guesses

$$u_i^0(x,t) = \inf_{x'\in\Omega_i}\{u_0(x')\}, \quad (x,t) \in \Omega_i \times (0,T),\ i = 1,2. \tag{8}$$

Because of this choice and the comparison principle for parabolic differential equations we have for all iterations $n \in \mathbb{N}_0$

$$e_i^n(x,t) \ge 0, \quad (x,t) \in \Omega_i \times (0,T),\ i = 1,2.$$

It suffices therefore to derive upper bounds for the errors to obtain a bound on the convergence rate of the overlapping Schwarz waveform relaxation algorithm applied to Burgers' equation. The first step of our analysis is to determine *linear advection diffusion problems* that bound the evolution of the errors. We show the derivation of the linear problems in detail, because it is here where the influence of the viscosity parameter ε needs to be traced carefully.

Lemma 1 (Super-Solutions). *For all* $n \in \mathbb{N}$ *we have*

$$0 \leq e_1^n(x,t) \leq \bar{e}_1^n(x,t), \quad \forall\,(x,t) \in \Omega_1 \times (0,T),$$

where the super-solution $\bar{e}_1^n$ *is the solution of the linear, constant coefficient problem*

$$\begin{array}{lll} \dfrac{\partial \bar{e}_1^n}{\partial t} + a_1 \dfrac{\partial \bar{e}_1^n}{\partial x} + b_1 \bar{e}_1^n = \varepsilon \dfrac{\partial^2 \bar{e}_1^n}{\partial x^2} & \text{in } \Omega_1 \times (0,T), & \\ \bar{e}_1^n(\cdot,0) = 0 & \text{in } \Omega_1, & (9) \\ \bar{e}_1^n(L,t) = \exp(\sigma_1 t) \sup\limits_{0\leq\tau\leq t} e_2^{n-1}(L,\tau) & t \in [0,T], & \end{array}$$

with the constants $a_1, b_1, \sigma_1 \in \mathbb{R}$ *given by*

$$\begin{aligned} a_1 &:= \inf_{(x,t)\in\Omega_1\times(0,T)} \{u(x,t)\}, \\ b_1 &:= \inf_{(x,t)\in\Omega_1\times(0,T)} \left\{ \frac{\partial u_1^n}{\partial x}(x,t) + (u(x,t) - a_1)\frac{a_1}{2\varepsilon} \right\}, \\ \sigma_1 &:= \begin{cases} -\frac{a_1^2}{4\varepsilon} - b_1 & \text{if } -\frac{a_1^2}{4\varepsilon} - b_1 \geq 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The number σ_1 *is finite but can be of order* $\mathcal{O}(\varepsilon^{-1})$ *due to (ii) in Theorem 1.*

Remark 2. It is not surprising that the constant coefficient problems contain source terms that are not present in the linear case. Note that the spatial derivatives $\frac{\partial u_i^n}{\partial x}$ are in fact multiplied with the second derivative of the flux $f = f(u)$ which is one for $f(u) = u^2/2$ and vanishes in the linear case.

Proof. (of Lemma 1) We use explicit solutions of the constant-coefficient equation (9) by means of the heat kernel. We define the shifted derivative of the heat kernel by

$$K_{1,x}(x,t) = -\frac{1}{2\sqrt{\pi}} \frac{x-L}{\varepsilon^{1/2} t^{3/2}} \exp\Big(- \frac{(x-L)^2}{4\varepsilon t} \Big). \tag{10}$$

For the linear, constant coefficient problem (9) satisfied by the super-solution, we then have the closed form solution formula

$$\bar{e}_1^n(x,t) = \exp(p_1 x + q_1 t) \int_0^t K_{1,x}(x, t-\tau) g_1(\tau)\, d\tau, \tag{11}$$

where we used the constants

$$p_i = \frac{a_i}{2\varepsilon}, \quad q_i = -\frac{a_i^2}{4\varepsilon} - b_i, \qquad i = 1,2 \tag{12}$$

and the function $g_1 = g_1(t) = \exp(-p_1 L + (\sigma_1 - q_1)t) \sup_{0\leq\tau\leq t} e_2^{n-1}(L,\tau)$. Note that g_1 is nonnegative due to the non-negativity of the errors, and monotonically increasing because of our choice of σ_1. To show that $\bar{e}_1^n$ is indeed a super-solution, we have to show that

$$d_1^n := \bar{e}_1^n - e_1^n \geq 0. \tag{13}$$

Now the difference function d_1^n satisfies the linear advection diffusion equation

$$\frac{\partial d_1^n}{\partial t} + u\frac{\partial d_1^n}{\partial x} + \frac{\partial u_1^n}{\partial x} d_1^n - \varepsilon \frac{\partial^2 d_1^n}{\partial x^2} = Q_1(x,t), \tag{14}$$

where the source term $Q_1(x,t)$ is given by

$$\begin{aligned}
Q_1(x,t) &= (u(x,t) - a_1)\frac{\partial \bar{e}_1^n}{\partial x} + \left(\frac{\partial u_1^n}{\partial x}(x,t) - b_1\right)\bar{e}_1^n(x,t) \\
&= (u(x,t) - a_1)\frac{e^{(p_1 x + q_1 t)}}{2\sqrt{\pi}} \int_0^t \frac{e^{\left(-\frac{(x-L)^2}{4\varepsilon(t-\tau)}\right)}}{\varepsilon^{1/2}(t-\tau)^{3/2}} \left[\frac{(x-L)^2}{2\varepsilon(t-\tau)} - 1\right] g_1(\tau)\, d\tau \\
&\quad -\Big((u(x,t) - a_1)p_1 + \frac{\partial u_1^n}{\partial x}(x,t) - b_1\Big)(x - L) \\
&\quad \times \frac{e^{(p_1 x + q_1 t)}}{2\sqrt{\pi}} \int_0^t \frac{e^{\left(-\frac{(x-L)^2}{4\varepsilon(t-\tau)}\right)}}{\varepsilon^{1/2}(t-\tau)^{3/2}} g_1(\tau)\, d\tau \\
&=: (u(x,t) - a_1)e^{(p_1 x + q_1 t)} Q_{11}(x,t) \\
&\quad + \left((u(x,t) - a_1)p_1 + \frac{\partial u_1^n}{\partial x}(x,t) - b_1\right) e^{(p_1 x + q_1 t)}(L - x) Q_{12}(x,t).
\end{aligned}$$

If we can show that $Q_{11}(x,t)$ and $Q_{12}(x,t)$ are non-negative for all $(x,t) \in \Omega_1 \times (0,T)$, we obtain $Q_1(x,t) \geq 0$ for all $(x,t) \in \Omega_1 \times (0,T)$ by the definition of a_1, b_1, which implies (13) by the maximum principle for (14) with zero initial and boundary data. But Q_{12} is nonnegative since g_1 from (11) is nonnegative. For Q_{11} we observe that it is the x-derivative of the solution w of the heat equation $w_t = \varepsilon w_{xx}$ in $\Omega_1 \times (0,T)$ which satisfies $w(L,.) = g_1$ and $w(.,0) \equiv 0$. Since g_1 is nonnegative and monotonically increasing, Q_{11} must also be nonnegative, which concludes the proof that $\bar{e}_1^n$ is a super-solution of e_1^n.

For the preceding proof we used an explicit solution for the constant coefficient problem (9) which serves to bound the error at a given iteration step. To obtain an upper bound on the error over many iteration steps, one considers then the iterated formula using a similar result on the subdomain Ω_2 for e_2^n. Since the iteration for the bounds is an iteration for linear problems, one can obtain, using similar techniques as the ones used in Gander [1997] or Giladi and Keller [2002], the following result.

Theorem 3 (Burgers' Equation). *The overlapping Schwarz waveform relaxation algorithm (3), (4) for the nonlinear advection problem (1) with $f(u) = u^2/2$ and initial guess (8) converges super-linearly. For each $T > 0$ and $i = 1, 2$ we have*

$$\sup_{x\in\Omega_i, 0\leq\tau\leq T} \{e_i^{2n}(x,t)\} \leq C_i e^{\frac{D(T+L)}{\varepsilon} n} \operatorname{erfc}\left(\frac{nL}{\sqrt{\varepsilon T}}\right), \tag{15}$$

where the constants $C_1 = \sup_{0\le t\le T}\{e_1^0(L,t)\}$, $C_2 = \sup_{0\le t\le T}\{e_2^0(0,t)\}$, *and* D *are independent of* ε, L, T *and* n *(but depends on* C *from Theorem 1).*

Remark 3.

(i) If we apply the expansion for the erfc-function for fixed $T, L, \varepsilon > 0$ as in Remark 1, we observe that the algorithm converges super-linearly for $n \to \infty$ and $t \le T$ at the same asymptotic rate as for the linear advection diffusion equation.

(ii) For Burgers' equation, the error estimate contains in addition the factor $e^{\frac{D(T+L)}{\varepsilon}n}$. Thus there exists a $T^* = T^*(n)$ such that (a) the algorithm converges for $\varepsilon \to 0$ on any time interval $[0,T]$ with $T < T^*(n)$, and (b) the estimate for the error e_1^{2n} does not converge to 0 for $\varepsilon \to 0$ on time intervals with $T > T^*(n)$. This scenario does not happen for the linear advection diffusion equation. Even though our estimate might not be sharp, this factor reflects the fact that in the purely hyperbolic case the Schwarz algorithm converges in a finite number of steps. The number of steps however depends on the nonlinearity and the initial data, see Gander and Rohde [2003].

(iii) Theorem 3 can be extended to the case of multiple subdomains, such that the estimate is independent of the number of subdomains, as in the linear case, see Gander and Rohde [2003].

We conclude the paper with a numerical experiment that illustrates the results of Theorem 3. As initial data we take the continuous function

$$u_0(x,t) = \begin{cases} 1 & : \quad x < 0, \\ 1-2x & : \quad 0 \le x < 1, \\ -1 & : \quad x \ge 1. \end{cases}$$

The hyperbolic limit problem with $\varepsilon = 0$ will develop a (standing) shock at $t = 0.5$. Thus for small but positive values of ε the solution will exhibit a sharp layer. For the numerical method we take two bounded subdomains $\Omega_1 = (0, \frac{1}{2} + L)$ and $\Omega_2 = (\frac{1}{2} - L, 1)$ with the overlap parameter $L = 0.1$. We compute the numerical solution up to $T = 0.6$ with a centered finite difference scheme in space, explicit for the nonlinear term and implicit for the Laplacian. The discretization parameters were $\Delta x = 0.01$ and $\Delta t = 0.003$. Figure 1 shows the error on $[0,1]\times[0,0.6]$ in the L^∞-norm versus the number of iterations at even iteration steps. One can clearly see the super-linear convergence behavior of the overlapping Schwarz waveform relaxation algorithm applied to Burgers' equation with the dependence on ε, as predicted by Theorem 3. One can also see that for ε small, the convergence in a finite number of steps of the hyperbolic limit starts to manifest itself.

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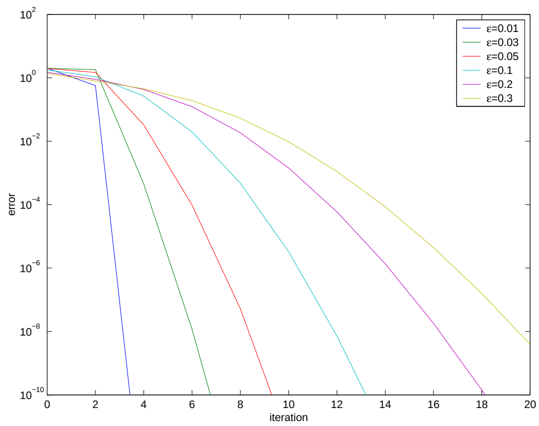

Fig. 1. Convergence rates for various values of the diffusion parameter ε.

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