Direct Schur Complement Method by Hierarchical Matrix Techniques

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Summary. The goal of this paper is the construction of a data-sparse approximation to the Schur complement on the interface corresponding to FEM and BEM approximations of an elliptic equation by domain decomposition. Using the hierarchical ($\mathcal{H}$-matrix) formats we elaborate the *approximate Schur complement inverse* in an explicit form. The required cost $\mathcal{O}(N_\Gamma \log^q N_\Gamma)$ is almost linear in N_Γ – the number of degrees of freedom on the interface. As input, we use the Schur complement matrices corresponding to subdomains and represented in the $\mathcal{H}$-matrix format. In the case of piecewise constant coefficients these matrices can be computed via the BEM representation with the cost $\mathcal{O}(N_\Gamma \log^q N_\Gamma)$, while in the general case the FEM discretisation leads to the complexity $O(N_\Omega \log^q N_\Omega)$.

1 Introduction

In Hackbusch [2003], a direct domain decomposition method was described for rather general elliptic equations based on a traditional FEM. Using $\mathcal{H}$-matrix techniques, almost linear[4] cost in the number N_Ω of degrees of freedom in the computational domain could be achieved. Here we concentrate on the inversion of the Schur complement matrix. We distinguish three approaches to construct and approximate the Schur complement matrix: (a) Methods based on a traditional FEM for rather general variable coefficients (cf. Hackbusch [2003]); (b) Approximation by boundary concentrated FEM for smooth coefficients in subdomains (cf. Khoromskij and Melenk [2003]); (c) BEM based methods for piecewise constant coefficients (cf. Hsiao et al. [2001], Khoromskij and Wittum [2004], Langer and Steinbach [2003]). Below we focus on the cases (a) and (c). In the latter case, which is not covered by Hackbusch [2003], we have the standard advantages of BEM compared to FEM. Furthermore, besides the approximation theory (cf. Theorem 1), we can show (cf. Hackbusch

[4] By "almost linear" we mean $\mathcal{O}(N \log^q N)$ for a fixed q.

et al. [2003, submitted]) the approximability of the Schur complement in the $\mathcal{H}$-matrix format[5]. In both cases we give numerical results.

In a polygonal domain $\Omega \subset \mathbb{R}^2$, we consider the elliptic operator $\mathcal{L} : V \to V'$ for $V = H_0^1(\Omega)$ and $V' = H^{-1}(\Omega)$, with the corresponding V-elliptic bilinear form $a_\Omega : V \times V \to \mathbb{R}$,

$$a_\Omega(u,v) = \int_\Omega (\sum_{i,j=1}^{d} a_{ij}\partial_j u \partial_i v + a_0 uv)dx, \quad a_0 > 0. \tag{1}$$

The corresponding variational equation is: Find $u \in V$ such that

$$a_\Omega(u,v) = \langle f, v\rangle := (f,v)_{L^2(\Omega)} \qquad \text{for all } v \in V, \tag{2}$$

where $f \in H^{-1}(\Omega)$. We suppose the domain Ω to be composed of $M \geq 1$ possibly matching, but non-overlapping polygonal subdomains Ω_i, $\overline{\Omega} = \cup_{i=1}^{M}\overline{\Omega}_i$. We denote the interface (skeleton) of the decomposition of Ω by $\Gamma = \cup\Gamma_i$ ($\Gamma_i := \partial\Omega_i$). Because we focus on the solution of an interface equation, we suppose that the right-hand side f is supported only by the interface, such that

$$\langle f, v\rangle = \sum_{i=1}^{M} \langle \psi_i, v\rangle_{\Gamma_i}, \quad \psi_i \in H^{-1/2}(\Gamma_i). \tag{3}$$

An equation with general f can be reduced to the case (3) by a subtraction of particular solutions in the subdomains which can be performed in parallel.

We may write the bilinear form $a_\Omega(\cdot,\cdot)$ in (1) as a sum of local bilinear forms, $a_\Omega(u,v) = \sum_{i=1}^{M} a_{\Omega_i}(R_i u, R_i v)$, where $R_i : V \to V_i := H^1(\Omega_i)$ is the restriction of functions onto Ω_i and the integration in $a_{\Omega_i} : V_i \times V_i \to \mathbb{R}$ is restricted to Ω_i. Furthermore, a_{Ω_i} is supposed to be V_i-elliptic for $V_i := H_0^1(\Omega_i)$, i.e., there exist $0 < C_1 \leq C_2$ such that (for suitable constants $\mu_i > 0$)

$$C_1\mu_i|u|^2_{H^1(\Omega_i)} \leq a_{\Omega_i}(u,u) \leq C_2\mu_i|u|^2_{H^1(\Omega_i)} \qquad \text{for all } u \in H^1(\Omega_i). \tag{4}$$

We introduce the space $V_\Gamma \subset V$ of piecewise $\mathcal{L}$-harmonic functions by $V_\Gamma := \{v \in V : a_\Omega(v,z) = 0 \ \text{ for all } z \in V_0\}$, with $V_0 := \{v \in V : v(x) = 0 \text{ for all } x \in \Gamma\}$. Note that $V = V_0 + V_\Gamma$ is the orthogonal splitting with respect to scalar product $a_\Omega(\cdot,\cdot)$. The variational equation (2) with f satisfying (3), we next reduce to an *interface equation* (in fact, $u \in V_\Gamma$). To that end, let us introduce the following *trace space* on Γ, $Y_\Gamma := \{u = z_{|\Gamma} : z \in V\}$, $||u||_{Y_\Gamma} = \inf_{z \in V : z_{|\Gamma} = u} ||z||_V$, with the energy norm $||z||_V = \sqrt{a_\Omega(z,z)}$. Next we define the local Poincaré-Steklov operator (Dirichlet-Neumann map) on $\Gamma_i = \partial\Omega_i$, $\mathcal{T}_i : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$ by $\lambda \in H^{1/2}(\Gamma_i)$, $\mathcal{T}_i(\lambda) := \gamma_{1,i}u$. Here $\gamma_{1,i}u$ is the conormal derivative of u on Γ_i and u solves (2) in Ω_i such

[5] Details will be presented in the forthcoming paper (full version).

that $u|_{\Gamma_i} = \lambda$. Now we reduce (2) to the equivalent *interface problem:* Find $z = u_{|\Gamma} \in Y_\Gamma$ such that

$$b_\Gamma(z,v) := \sum_{i=1}^{M} \langle \mathcal{T}_i z_i, v_i \rangle_{\Gamma_i} = \langle \Psi_\Gamma, v \rangle := \sum_{i=1}^{M} \langle \psi_i, v \rangle_{\Gamma_i}, \quad \forall\, v \in Y_\Gamma, \qquad (5)$$

where $b_\Gamma(\cdot,\cdot) : Y_\Gamma \times Y_\Gamma \to \mathbb{R}$ is a continuous bilinear form, $\Psi_\Gamma \in Y'_\Gamma$ and $z_i = z_{|\Gamma_i}$, $v_i = v_{|\Gamma_i}$.

To apply $\mathcal{H}$-matrix approximations to the discrete version of (5), we represent the inverse operator $\mathcal{L}^{-1}$ using the interface map $\mathcal{B}_\Gamma$ defined by $\langle \mathcal{B}_\Gamma u, v \rangle_\Gamma = b_\Gamma(u,v)$ for all $u, v \in Y_\Gamma$. The following statement describes the structure of the inverse $\mathcal{L}^{-1} : Y'_\Gamma \to V$.

Lemma 1. *The representation $\mathcal{L}^{-1} = \mathcal{E}^{\text{harm}}_{\Omega\leftarrow\Gamma}\mathcal{B}_\Gamma^{-1}$ holds, where $\mathcal{E}^{\text{harm}}_{\Omega\leftarrow\Gamma} : Y_\Gamma \to V_\Gamma$ is the $\mathcal{L}$-harmonic extension from Y_Γ to V_Γ.*

Proof. The bilinear form $b_\Gamma(\cdot,\cdot) : Y_\Gamma \times Y_\Gamma \to \mathbb{R}$ is symmetric, continuous and positive definite and thus the same holds for $\mathcal{B}_\Gamma$ and $\mathcal{B}_\Gamma^{-1} : Y'_\Gamma \to Y_\Gamma$. Therefore the operator $\mathcal{L}^{-1} = \mathcal{E}^{\text{harm}}_{\Omega\leftarrow\Gamma}\mathcal{B}_\Gamma^{-1}$ is well-defined. Next, we check that $u = \mathcal{L}^{-1}\Psi_\Gamma$ solves (2). Green's formula yields

$$a_\Omega(u,v) = \sum_{i=1}^{M} a_{\Omega_i}(R_i u, R_i v) = \sum_{i=1}^{M} \langle \mathcal{T}_i u, v_i \rangle_{\Gamma_i} = \sum_{i=1}^{M} \langle \psi_i, v \rangle_{\Gamma_i} \quad \forall v \in V. \qquad (6)$$

This also provides $\mathcal{B}_\Gamma^{-1}\Psi_\Gamma = u_{|\Gamma}$ completing the proof.

In the general case, we consider a conventional FEM approximation of (2) by piecewise linear elements on a regular triangulation that aligns with Γ. Let $\mathbf{A}_h \in \mathbb{R}^{I_\Omega \times I_\Omega}$ be the Galerkin-FEM stiffness matrix $\mathbf{A}_h = \begin{pmatrix} \mathbf{A}_{II} & \mathbf{A}_{II_\Gamma} \\ \mathbf{A}_{I_\Gamma I} & \mathbf{A}_{I_\Gamma I_\Gamma} \end{pmatrix}$, corresponding to the chosen FE space $V_h \subset V$. Here I_Γ is the index set corresponding to the interface degrees of freedom and $I = I_\Omega \setminus I_\Gamma$ is the complementary one. Eliminating all interior degrees of freedom corresponding to I, we obtain the so-called *FEM Schur complement* matrix $\mathbf{B}_{\Gamma,h} := \mathbf{A}_{I_\Gamma I_\Gamma} - \mathbf{A}_{I_\Gamma I}\mathbf{A}_{II}^{-1}\mathbf{A}_{II_\Gamma} \in \mathbb{R}^{I_\Gamma \times I_\Gamma}$, where $\mathbf{A}_{II} = \mathit{blockdiag}\{\mathbf{A}_1, ..., \mathbf{A}_M\}$ is the stiffness matrix for $\mathcal{L}$ subject to zero Dirichlet conditions on Γ, hence $\mathbf{A}_{II}^{-1} = \mathit{blockdiag}\{\mathbf{A}_1^{-1}, ..., \mathbf{A}_M^{-1}\}$ can be computed in parallel. In a standard way, each of the "substructure" matrices $\mathbf{A}_i^{-1}$ can be represented by the $\mathcal{H}$-matrix format (cf. Hackbusch [2003]).

Using $\mathbf{B}_{\Gamma,h}$, the original FEM system $\mathbf{A}_h U = F$ is reduced to the interface equation

$$\mathbf{B}_{\Gamma,h} U_\Gamma = F_\Gamma, \qquad U_\Gamma, F_\Gamma \in \mathbb{R}^{I_\Gamma}, \quad \text{where} \quad U_\Gamma = U|_{I_\Gamma}. \qquad (7)$$

We construct the approximate direct solver for the Schur complement system (7) focusing on the cases of general and of piecewise constant coefficients. In the latter case, the matrix $\mathbf{B}_{\Gamma,h}$ can be computed by BEM with cost $\mathcal{O}(N_\Gamma \log^q N_\Gamma)$, where $N_\Gamma = card(I_\Gamma)$, while for general coefficients the cost is $O(N_\Omega \log^q N_\Omega)$ (cf. Hackbusch [2003]). Furthermore, $\mathbf{B}_{\Gamma,h}$ is proved to be of almost linear cost in N_Γ concerning operations for storage and for the matrix-by-vector multiplication. Due to the $\mathcal{H}$-matrix arithmetic, our approximate Schur complement inverse matrix $\mathbf{B}_{\Gamma,h}^{-1}$ again needs almost linear complexity $\mathcal{O}(N_\Gamma \log^q N_\Gamma)$.

Notice that our approach can be also viewed as an *approximate direct parallel solver* based on the domain decomposition Schur complement method.

2 FEM- and BEM-Galerkin Approximations

Introduce the FE trace space $Y_N := V_h|_\Gamma \subset Y_\Gamma$ with $N = N_\Gamma = dim Y_N$. Based on the representation in Lemma 1 and using the *$\mathcal{H}$-matrix approximation* to the operators involved, we can construct an approximate direct solver of almost linear complexity in N_Γ that realises the action $\mathcal{B}_\Gamma^{-1}\Psi_\Gamma$. For this purpose we split the *numerical realisation* of $\mathcal{L}^{-1} = \mathcal{E}^{\text{harm}}_{\Omega\leftarrow\Gamma}\mathcal{B}_\Gamma^{-1}$ into three independent steps: (i) Computation of a functional $\Psi_{\Gamma,h} \in Y'_\Gamma$ approximating Ψ_Γ; (ii) An $\mathcal{H}$-matrix approximation to the discrete interface operator $\mathcal{B}_\Gamma^{-1}$; (iii) Implementation of a discrete $\mathcal{L}$-harmonic extension operator $\mathcal{E}^{\text{harm}}_{\Omega\leftarrow\Gamma}$.

In Step (i) we define $\Psi_{\Gamma,h} \in Y'_N$ by $\langle \Psi_{\Gamma,h}, v\rangle_\Gamma := \sum_{i=1}^{M} \langle \psi_{ih}, v\rangle_{\Gamma_i}\ \forall v \in Y_N$. Given $\Psi_{\Gamma,h} \in Y'_N$, we consider the Schur complement system approximating the interface equation (5). Let us define the local Schur complement operator $\mathcal{T}_{i,N}$ corresponding to the discrete $\mathcal{L}_i$-harmonic extension based on the FEM Galerkin space $V_{ih} := V_h|_{\Omega_i}$, by $\lambda, v \in Y_N|_{\Gamma_i}$: $\langle \mathcal{T}_{i,N}\lambda, v\rangle_{\Gamma_i} = A_{\Omega_i}(\overline{u}_i, \overline{v})$, where $\overline{u}_i \in V_{ih}$, $A_{\Omega_i}(\overline{u}_i, z) = 0$ for all $z \in V_{ih} \cap H_0^1(\Omega_i)$ and with an arbitrary $\overline{v} \in V_{ih}$ such that $\overline{v}|_{\Gamma_i} = v$. With the aid of the local FEM-Galerkin discretisations $\mathcal{T}_{i,N}$ of the Poincaré-Steklov maps $\mathcal{T}_i$, the discrete operator $\mathcal{B}_{\Gamma,N}$ and the corresponding interface equation are given by

$$z \in Y_N : \quad \langle \mathcal{B}_{\Gamma,N} z, v\rangle_\Gamma := \sum_{i=1}^{M} \langle \mathcal{T}_{i,N} z_i, v_i\rangle_{\Gamma_i} = \langle \Psi_{\Gamma,h}, v\rangle_\Gamma \qquad \text{for all } v \in Y_N,$$

where $v_i := v|_{\Gamma_i}$ and z is a desired approximation to the trace $u|_\Gamma$. The corresponding matrix representation to the interface operator $\mathcal{B}_{\Gamma,N}$ reads as

$$\langle \mathbf{B}_{\Gamma,N} U, Z\rangle_{I_\Gamma} = \sum_{i=1}^{M} \langle \mathbf{T}_{i,N} U_i, Z_i\rangle_{I_{\Gamma_i}} := \langle \mathcal{B}_{\Gamma,N} \mathcal{J} U, \mathcal{J} Z\rangle_\Gamma, \quad \mathbf{B}_{\Gamma,N} \in \mathbb{R}^{I_\Gamma \times I_\Gamma}, \tag{8}$$

where $\mathcal{J} : \mathbb{R}^{I_\Gamma} \to Y_N$ is the natural bijection from the coefficient vectors into the FE functions. $\mathbf{T}_{i,N}$ is the local FEM Schur complement matrix and $U_i, Z_i \in \mathbb{R}^{I_{\Gamma_i}}$, $i = 1, ..., M$, are the local vector components defined by $U_i = \mathbf{R}_{\Gamma_i} U,\ Z_i = \mathbf{R}_{\Gamma_i} Z$, where the connectivity matrix $\mathbf{R}_{\Gamma_i} \in \mathbb{R}^{I_{\Gamma_i} \times I_\Gamma}$ provides the restriction of the vector $Z \in \mathbb{R}^{I_\Gamma}$ onto the index set I_{Γ_i}. Let $\mathbf{A}_i$ be the local stiffness matrix corresponding to $a_{\Omega_i}(\cdot,\cdot)$, $\mathbf{A}_i = \begin{pmatrix} \mathbf{A}_{II} & \mathbf{A}_{I\Gamma_i} \\ \mathbf{A}_{\Gamma_i I} & \mathbf{A}_{\Gamma_i\Gamma_i} \end{pmatrix}$, where I and

Γ_i correspond to the interior and boundary index sets in Ω_i, respectively. Then we obtain the FEM Schur complement matrix $\mathbf{T}_{i,N} := \mathbf{A}_{\Gamma_i \Gamma_i} - \mathbf{A}_{\Gamma_i I} \mathbf{A}_{II}^{-1} \mathbf{A}_{I \Gamma_i}$, ($\mathbf{A}_{II}$: stiffness matrix for $\mathcal{L}_i$ subject to zero Dirichlet conditions on Γ_i). Thus, $\mathbf{A}_{II}^{-1}$ can be represented in the $\mathcal{H}$-matrix format (cf. Hackbusch [2003]).

Let us consider the explicit representation of $\mathbf{B}_{\Gamma,N}$ in (8) using the BEM-Galerkin approximation with Lagrange multipliers (cf. Hsiao et al. [2001]). Introduce the classical boundary integral representations involving operators $\mathcal{V}_i$, $\mathcal{D}_i$ and $\mathcal{K}_i$, defined by

$$(\mathcal{V}_i u)(x) = \int_{\Gamma_i} g(x,y) u(y) dy, \qquad (\mathcal{K}_i u)(x) = \int_{\Gamma_i} \frac{\partial}{\partial n_y} g(x,y) u(y) dy,$$
$$(\mathcal{K}_i' u)(x) = \int_{\Gamma_i} \frac{\partial}{\partial n_x} g(x,y) u(y) dy, \quad (\mathcal{D}_i u)(x) = -\frac{\partial}{\partial n_x} \int_{\Gamma_i} \frac{\partial}{\partial n_y} g(x,y) u(y) dy\,,$$

where $g(x,y)$ is the corresponding singularity function (cf. Hackbusch [1995]). In the following, we consider the model case $a_{\Omega_i}(u,v) := \mu_i \int_{\Omega_i} \nabla u \nabla v dx,\ \mu_i > 0$. Introduce the *modified Calderon projection* $\mathcal{C}_{\Gamma_i}$ by

$$C_{\Gamma_i} \begin{pmatrix} u_i \\ \delta_i \end{pmatrix} := \begin{pmatrix} \mu_i \mathcal{D} & \frac{1}{2} I + \mathcal{K}_i' \\ -\frac{1}{2} I - \mathcal{K}_i & \mu_i^{-1} \mathcal{V}_i \end{pmatrix} \begin{pmatrix} u_i \\ \delta_i \end{pmatrix} = \begin{pmatrix} \delta_i \\ 0 \end{pmatrix} \tag{9}$$

(cf. Khoromskij and Wittum [2004] and references therein), applied to the $\mathcal{L}_i$-harmonic function $\overline{u}$ which satisfies $-\Delta \overline{u} = 0$ in Ω_i with $\overline{u}_{|\Gamma_i} = u_i$ and $\delta_i = \mu_i \partial \overline{u} / \partial n$ (see also Costabel [1988], Hackbusch [1995], Wendland [1987]). Note that the Schur complement equation corresponding to (9) reads as

$$\mathcal{T}_i u_i := \mu_i \left(\mathcal{D}_i + \left(\tfrac{1}{2} I + \mathcal{K}_i'\right) \mathcal{V}_i^{-1} \left(\tfrac{1}{2} I + \mathcal{K}_i\right)\right) u_i = \delta_i, \tag{10}$$

providing an explicit symmetric representation to the Poincaré-Steklov map in terms of boundary integral operators.

Let us consider the skew-symmetric interface problem for $M > 1$ (see (11) below). Introducing the trace space $\Sigma_\Gamma := Y_\Gamma \times \Lambda_\Gamma$ with $\Lambda_\Gamma := \prod_{i=1}^{M} H^{-1/2}(\Gamma_i)$ and the weighted norm $\|P\|^2_{\Sigma_\Gamma} = \|u\|^2_{Y_\Gamma} + \sum_{j=1}^{M} \mu_j^{-1} \|\lambda_j\|^2_{H^{-1/2}(\Gamma_j)}$, $P = (u, \lambda) \in \Sigma_\Gamma,\ \lambda = (\lambda_1, \ldots, \lambda_M)$, we define the interface bilinear form $c_\Gamma : \Sigma_\Gamma \times \Sigma_\Gamma \to \mathbb{R}$ by $c_\Gamma(P,Q) := \sum_{i=1}^{M} \langle \mathcal{C}_{\Gamma_i} P_i, Q_i \rangle_{\Gamma_i}$ for all $P = (u,\lambda), Q = (v,\eta) \in \Sigma_\Gamma$, with $\mathcal{C}_{\Gamma_i}$ given by (9). Using the representation $\langle \mathcal{C}_{\Gamma_i} P_i, Q_i \rangle_{\Gamma_i} := \mu_i (\mathcal{D}_i u, v) +$ $((\frac{1}{2} I + \mathcal{K}_i')\lambda, v) - ((\frac{1}{2} I + \mathcal{K}_i) u, \eta) + \mu_i^{-1} (\mathcal{V}_i \lambda, \eta)$ in each subdomain, the original equation for u (cf. (2)) will be reduced to the skew-symmetric interface equation: *Given* $\Psi_\Gamma \in Y_\Gamma'$, *find* $P \in \Sigma_\Gamma$ *such that*

$$c_\Gamma(P,Q) \ = \ \langle \Psi_\Gamma, v \rangle_\Gamma \qquad \text{for all } Q = (v,\eta) \in \Sigma_\Gamma. \tag{11}$$

Let $\Lambda_h := \prod_{i=1}^{M} \Lambda_{ih}$, where Λ_{ih} is the FE space of piecewise linear functions. Introducing the FE Galerkin ansatz space $\Sigma_h := Y_N \times \Lambda_h$, we arrive at the corresponding BEM-Galerkin saddle-point system of equations: *Given* $\Psi_\Gamma \in Y'_\Gamma$, *find* $P_h = (u_h, \lambda_h) \in Y_N \times \Lambda_h$ *such that*

$$c_\Gamma(P_h, Q) \;=\; \langle \Psi_\Gamma, v\rangle_\Gamma \qquad \text{for all } Q = (v, \eta) \in Y_N \times \Lambda_h. \tag{12}$$

We further assume $\mathcal{V}_i$, $i = 1, ..., M$, to be positive definite.

Theorem 1. *(i) The bilinear form* $c_\Gamma : \Sigma_\Gamma \times \Sigma_\Gamma \to \mathbb{R}$ *is continuous and* Σ_Γ*-elliptic. (ii) Let* P_h *solve (12), then the optimal error estimate holds:*

$$\|P_h - P\|^2_{\Sigma_\Gamma} \le c \inf_{(w,\mu)\in\Sigma_h} \left[\sum_{i=1}^{M} \mu_i \|u_i - w_i\|^2_{H^{1/2}(\Gamma_i)} + \sum_{i=1}^{M} \mu_i^{-1} \|\lambda_i - \mu_i\|^2_{H^{-1/2}(\Gamma_i)} \right].$$

(iii) Let $\mathbf{T}_{i,BEM}$ *be the local BEM Schur complement given by*

$$\mathbf{T}_{i,BEM} := \mu_i \left(\mathbf{D}_{ih} + \left(\tfrac{1}{2}\mathbf{I}_{ih} + \mathbf{K}'_{ih}\right) \mathbf{V}_{ih}^{-1} \left(\tfrac{1}{2}\mathbf{I}_{ih} + \mathbf{K}_{ih}\right)\right), \tag{13}$$

where $\mathbf{D}_{ih}$*,* $\mathbf{K}_{ih}$*,* $\mathbf{V}_{ih}$ *are the Galerkin stiffness matrices of the boundary integral operators and* $\mathbf{I}_{ih}$ *is the corresponding mass matrix. Then the BEM Schur complement takes the explicit form* $\mathbf{B}_{\Gamma,N} = \sum_{i=1}^{M} \mathbf{R}^\top_{\Gamma_i} \mathbf{T}_{i,BEM} \mathbf{R}_{\Gamma_i} \in \mathbb{R}^{I_\Gamma \times I_\Gamma}$ *due to* $\langle \mathbf{B}_{\Gamma,N} Z, V\rangle_{I_\Gamma} = \sum_{i=1}^{M} \langle \mathbf{T}_{i,BEM} Z_i, V_i\rangle_{I_{\Gamma_i}} = \sum_{i=1}^{M} \langle \mathbf{R}^\top_{\Gamma_i} \mathbf{T}_{i,BEM} \mathbf{R}_{\Gamma_i} Z, V\rangle_{I_\Gamma}$.

Proof. Statements (i), (ii) are proven in Theorems 2, 3 in Hsiao et al. [2001], while (iii) is the direct consequence of the BEM-Galerkin approximation (12).

3 $\mathcal{H}$-Matrix Approximation to $\mathbf{B}^{-1}_{\Gamma,N}$ and Numerics

Now we discuss the $\mathcal{H}$-matrix approximation to $\mathbf{T}_{i,N}$ and $\mathbf{B}^{-1}_{\Gamma,N}$. In the FEM case let $\mathbf{A}_{II}$ be presented in the hierarchical format. Then we need the formatted multiplication and addition to obtain $\mathbf{T}_{i,N} = \mathbf{A}_{\Gamma_i\Gamma_i} - \mathbf{A}_{\Gamma_i I}\mathbf{A}^{-1}_{II}\mathbf{A}_{I\Gamma_i}$, leading to the cost $\mathcal{O}(N_{\Omega_i} \log^q N_{\Omega_i})$. The matrix $\mathbf{T}_{i,BEM}$ can be computed in $\mathcal{O}(N_{\Gamma_i} \log^q N_{\Gamma_i})$ operations. Note that the $\mathcal{H}$-matrix representations of $\mathbf{T}_{i,N}$ and $\mathbf{T}^{-1}_{i,N}$ can be applied within the so-called BETI iterative method Langer and Steinbach [2003].

Our goal is an algorithm of almost linear complexity in $N_\Gamma := \dim Y_N$ realising the matrix-by-vector multiplication by $\mathbf{B}^{-1}_{\Gamma,N}$. Having all local $\mathcal{H}$-matrices $\mathbf{T}_{i,N}$ available, we first compute the $\mathcal{H}$-matrix representation of $\mathbf{B}_{\Gamma,N}$. To that end, we construct an admissible hierarchical partitioning $P_2(I_\Gamma \times I_\Gamma)$ based on the cluster tree T_{I_Γ} of the skeleton index set I_Γ (cf. Figure 1, left). After some levels the clusters correspond to one-dimensional manifolds. Since a lower spatial dimension leads to better constants in the complexity estimates (cf. Grasedyck and Hackbusch [2003], Hackbusch et al. [2003, submitted]), this property makes the algorithm faster.

To calculate a low-rank approximation of blocks in the hierarchical partitioning $P_2(I_\Gamma \times I_\Gamma)$, we propose an SVD recompression of any block $b \in$

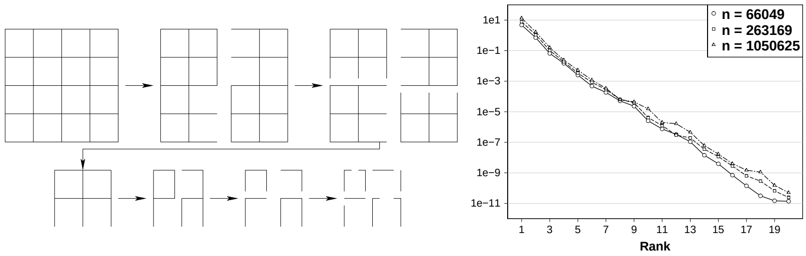

Fig. 1. Clustertree T_{I_Γ} (left); adaptive choice of the local rank (right).

$P_2(I_\Gamma \times I_\Gamma)$ obtained as a sum of fixed number of subblocks extracted as rank-k submatrices in the local Schur complements. This fast algorithm (of almost linear cost) exploits the hierarchical format of the local matrices $\mathbf{T}_{i,N}$ (same for $\mathbf{T}_{i,BEM}$) and will be presented in the next example. The following tables show numerical results for the scaled Laplacian in Ω_i with randomly chosen coefficients $\mu_i \in (0,1]$ (cf. (4)). Presented are the times for computing $\mathbf{T}_{i,N}$, $\mathbf{T}_{i,BEM}$, for the inversion of $B = \mathbf{B}_{\Gamma,N}$ and for its matrix-by-vector multiplication (MV) as well as for the accuracy of this inversion (computed on a SunFire 6800 (900 MHz)). The computing times $\mathbf{T}_{i,BEM}$ for $N_\Omega \approx 4 \cdot 10^6$ and $N_\Omega \approx 16 \cdot 10^6$ are 13.7 s and 36.8 s, respectively[6]. The results correspond to a decomposition of a square into 6×6 subsquares. One can see the almost linear complexity of the inversion algorithm. If we are interested in an efficient preconditioning, the local rank k can be chosen adaptively to achieve the required accuracy ε (cf. Fig. 1 (right) represents ε depending on k).

6×6 domains ($k = 9$)

N_Ω	N_Γ	$t(\mathbf{T}_{i,N})$	$t(\mathbf{T}_{i,BEM})$	$t(\mathbf{B}_{\Gamma,N}^{-1})$	$t(MV)$	$\Vert I - BB_{\mathcal{H}}^{-1}\Vert_2$
16 641	1 245	0.6 s	0.06 s	10.7 s	1.36_{10}-2 s	7.7_{10}-6
66 049	2 525	12.2 s	0.5 s	30.3 s	3.98_{10}-2 s	8.0_{10}-6
263 169	5 085	105.1 s	1.7 s	94.2 s	9.43_{10}-2 s	4.6_{10}-5
1 050 625	10 205	696.2 s	4.9 s	218.1 s	1.85_{10}-1 s	7.1_{10}-5

We present numerical results illustrating an acceleration factor of a *direct multilevel DDM* due to the recursive Schur complement evaluation (see §5.2 in Hackbusch [2003]). To reduce the cost of the local Schur complement matrices $\mathbf{T}_{i,N}$ in each subdomain Ω_i, one can apply the same domain decomposition algorithm as in §3 to the local inverse $\mathbf{A}_i^{-1}$. This leads to a reduction of the computational time. The following table corresponds to a 4×4 decomposition. We consider $q + 1 \geq 1$ grids $N_i = N_0 4^i$ with the problem size $N_i = N_0 4^i$, $i = 0, 1, ..., q$, and with $N_0 = 16641$, so that $N_3 = 1050625$. On each subdomain of level $\ell = 2, ..., q$ one has the matrix size $N_{\ell-2}$, thus one can recursively apply

[6] $t(\mathbf{T}_{i,BEM})$ includes only the dominating cost of two matrix-matrix multiplications and one matrix inversion in the $\mathcal{H}$-matrix format (cf. (13)).

the algorithm on level $\ell-2$ to compute the local inverse matrix $A_{i,\ell}^{-1}$ on level ℓ. The complexity bound satisfies the recursion $W(A_{i,\ell}^{-1}) = 16W(A_{i,\ell-2}^{-1}) + W(B_{\Gamma,\ell-2}^{-1})$, $W(\cdot)$: cost of the corresponding matrix operation. Based on the table below, the simple calculation $W^{ML}(A_{4,\ell}^{-1}) = 16(16 \times 0.1 + 0.8) + 16.9 \approx$ 1 *min* shows an acceleration factor about 33 compared with 2020 *sec* depicted in the last line of our table. Similarly, an extrapolation using the two smaller grids exhibits that our direct solver applied to the problems with $n_\Omega = 4 \cdot 10^6$ and $n_\Omega = 16 \cdot 10^6$ would take about 113 sec. and 1080 sec., respectively, for each subdomain.

4 × 4 domains ($k = 9$)

N_Ω	N_Γ	$t(\mathbf{T}_{i,N})$	$t(\mathbf{B}_{\Gamma,N}^{-1})$	$t(MV)$	$\Vert I - BB_{\mathcal{H}}^{-1}\Vert_2$
16 641	753	3.8 s	3.7 s	3.20_{10}-3 s	4.2_{10}-6
66 049	1 521	43.2 s	16.9 s	9.10_{10}-3 s	7.7_{10}-6
263 169	3 057	317.4 s	48.3 s	4.18_{10}-2 s	1.3_{10}-5
1 050 625	6 129	2 020.1 s	118.8 s	8.92_{10}-1 s	2.1_{10}-5

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