On the Construction of Approximate Boundary Conditions for Solving the Interior Problem of the Acoustic Scattering Transmission Problem

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Summary. The construction of accurate generalized impedance boundary conditions for the three-dimensional acoustic scattering problem by a homogeneous dissipative medium is analyzed. The technique relies on an explicit computation of the symbolic asymptotic expansion of the exact impedance operator in the interior domain. An efficient pseudolocalization of this operator based on Padé approximants is then proposed. The condition can be easily integrated in an iterative finite element solver without modifying its performances since the pseudolocal implementation preserves the sparse structure of the linear system. Numerical results are given to illustrate the method.

1 Introduction

The penetration of an acoustic field into a given medium can be approximately modeled through a Fourier-Robin-type (also called impedance) boundary condition (see for instance Senior and Volakis [1995]). To have both a larger application range and a gain of accuracy of the model, a possible approach consists in designing higher-order *generalized impedance boundary conditions*. These conditions are often defined by a differential operator which describes with some finer informations the behaviour of the transmitted acoustic field. We present some new generalized impedance boundary conditions which extend the validity domain of the usual differential conditions for the scattering problem of an acoustic wave by a three-dimensional homogeneous isotropic scatterer. The proposed conditions have also the interest of not increasing the total cost of a resolution by an iterative finite element solver (or possibly an integral equation procedure). All these points are developed below.

2 The acoustic transmission boundary value problem

Let Ω_1 be a regular bounded domain embedded in $\mathbb{R}^3$ with a $\mathcal{C}^\infty$ boundary Γ. We set Ω_2 as the associated infinite domain defined by $\Omega_1 = \mathbb{R}^3/\overline{\Omega_2}$. We assume that both media Ω_j, $j = 1, 2$, are homogeneous and isotropic. Each one is characterized by two positive real constants: the density ρ_j and the sound velocity c_j. We moreover suppose that Ω_1 may be dissipative. This aspect is modeled by the introduction of a damping parameter $\delta \geq 0$.

Consider now an incident wave u_0 defined in the vicinity of Γ and which satisfies the Helmholtz equation: $\Delta u_0 + k_2^2 u_0 = 0$. We make the assumption that the solution has a time-harmonic dependence of the form $e^{-ik_2 t}$, where $k_2 = \varpi/c_2$ is the wave number in the unbounded domain of propagation, setting ϖ as the frequency of the signal. We can then define the (possibly complex) wave number k_1 in Ω_1 by: $k_1^2 = \varpi^2/c_1^{-2}(1+i\delta/\varpi)$. Two parameters are usually introduced: the complex refraction index $N = c_r^{-1}(1 + i\delta/\varpi)^{1/2}$ and the complex contrast coefficient $\alpha = \rho_r^{-1}(1 + i\delta/\varpi)^{-1}$, where c_r and ρ_r designate respectively the relative velocity and density. Finally, if z is a complex number, we set $z^{1/2}$ as the principal determination of the square root with branch cut along the negative real axis.

We consider now the scattering problem of the wave u_0 by Ω_1 which consists in computing the field v solution to the transmission problem

$$\begin{cases} \Delta v_2 + k_2^2 v_2 = 0, \text{ in } \Omega_2, \\ \Delta v_1 + k_1^2 v_1 = k_2^2(1 - N^2)u_0, \text{ in } \Omega_1, \\ [v] = 0 \text{ and } [\chi \partial_{\mathbf{n}} v] = -[\chi \partial_{\mathbf{n}} u_0], \text{ on } \Gamma, \\ \lim\limits_{|x|\to+\infty} |x|(\nabla v_2 \cdot \dfrac{x}{|x|} - ik_2 v_2) = 0, \end{cases} \tag{1}$$

where χ is the piecewise constant function defined by $\chi = 1$ in Ω_2 and $\chi = \alpha$ in Ω_1. The vector $\mathbf{n}$ stands for the outward unit normal vector to Ω_1. The restriction of the field v to Ω_j, $j = 1, 2$, is denoted by $v_j = v_{|\Omega_j}$; the jump between the exterior and interior traces is given by: $[v] = v_{1|\Gamma} - v_{2|\Gamma}$. The inner product of two complex vector fields $\mathbf{a}$ and $\mathbf{b}$ of $\mathbb{C}^3$ is: $\mathbf{a}\cdot\mathbf{b} = \sum_{\mathbf{j=1}}^{\mathbf{3}} \mathbf{a_j}\overline{\mathbf{b_j}}$. The operator ∇ is the gradient operator of a complex-valued vector field and the Laplacian operator is defined by: $\Delta = \nabla^2$. The last equation of (1) is the so-called Sommerfeld radiation condition at infinity which leads to the uniqueness of the solution to the boundary value problem. We denote by SRC the associated operator. The existence and uniqueness of the solution to (1) can be proved in an adequate functional setting.

3 Generalized impedance boundary conditions

When the interior wave number has a sufficiently large modulus $|k_1|$, a reduction of the computational complexity in the practical solution of the boundary value problem (1) can be achieved by approximately modeling the penetration of the wave into the interior domain by a boundary condition set on Γ. This approach is well-known in electromagnetism under the name of *generalized impedance boundary condition* method. The ideas have been introduced during the second world war for modeling the interaction of an electromagnetic field with an irregular terrain (see e.g. Senior and Volakis [1995]). The resulting boundary condition for a given problem takes the form of a generalized mixed boundary condition defined by a differential or a pseudodifferential operator. We give here an outline of the application of the theory of pseudodifferential operators to derive a family of accurate boundary conditions for the transmission problem.

The first point consists in considering the total field formulation of system (1) setting $u = v + u_0$. Therefore, we are led to compute u such that

$$\begin{cases} \Delta u_2 + k_2^2 u_2 = -f, \text{ in } \Omega_2, \\ \Delta u_1 + k_2^2 N^2 u_1 = 0, \text{ in } \Omega_1, \\ [u] = 0 \text{ and } [\chi \partial_{\mathbf{n}} u] = 0, \text{ on } \Gamma, \\ SRC(u_2 - u_0) = 0, \end{cases}$$

for an explicit source term f. Let us now assume that we can construct the Dirichlet-Neumann (DN) operator for the interior problem

$$\begin{cases} \widetilde{\mathcal{Z}^-} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) \\ \qquad u_1 \mapsto \partial_{\mathbf{n}} u_1 = \widetilde{\mathcal{Z}^-} u_1 \end{cases} .$$

This operator, also called the Steklov-Poincaré operator, is a first-order pseudodifferential operator. The determination of this operator yields an *a priori* integro-differential computation of the internal solution from its Cauchy data. Using the transmission conditions at the interface and considering the scattered field formulation, we have to solve the exterior non-standard impedance boundary value problem: find v_2 such that

$$\begin{cases} \Delta v_2 + k_2^2 v_2 = 0, \text{ in } \Omega_2, \\ (\partial_{\mathbf{n}} - \alpha \widetilde{\mathcal{Z}^-}) v_2 = g, \text{on } \Gamma, \\ SRC(v_2) = 0, \end{cases}$$

with $g = -(\partial_{\mathbf{n}} - \alpha \widetilde{\mathcal{Z}^-}) u_0$. In the above system, the operator $\alpha \widetilde{\mathcal{Z}^-}$ is generally called the Exact Impedance Boundary Operator (EIBO).

To achieve an explicit computation of a non-local approximation of the DN operator for an arbitrarily-shaped surface, we rewrite the Helmholtz equation in a generalized coordinates system associated to the surface and next we compute the two first terms of its asymptotic expansion in homogeneous complex symbols. To this end, let us define the wave operator in the interior domain: $L_1 = (\Delta - \partial_t^2)$, where the exponential time dependence of the solution is

$e^{-ik_1 t}$. As a consequence, the multiplication by k_1 must be understood as the action of the first-order time derivative where we take the attenuation effects into account. A calculation in the one-dimensional case in space naturally imposes the choice of k_1. Furthermore, we can notice that we consider the same asymptotic parameter as Senior and Volakis [1995] but without assuming a particular analytical asymptotic form of the interior field. This hypothesis is unnecessary here and yields some more accurate pseudodifferential approximations of the EIBO.

Let us express the operator L_1 in a tubular neighborhood of Γ. Since Γ is a compact submanifold of $\mathbb{R}^3$, we can choose a local coordinates system at any point x_0 of Γ. Let us designate by $s = (s_1, s_2)$ the tangential variable and by r the radial variable along the unit normal vector $\mathbf{n}$ at x_0. Then, a point x near the surface can be locally rewritten under the form: $x = x_0 + r\mathbf{n}(x_0)$, with $x_0 \in \Gamma$. Let us introduce Γ_r as the surface defined for a fixed value of r and let us choose an orthogonal coordinates system on Γ. The covariant basis $(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ of the tangent plane $T_{x_0}(\Gamma)$ which is compatible with the orientation of $\mathbf{n}(x_0)$ is better known as the principal basis. Vectors $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ are the principal directions of the curvatures to the surface. If we set $\mathcal{R}$ as the curvature tensor of the tangent plane at a given point of the surface, then the diagonalization of $\mathcal{R}$ yields the determination of the principal curvatures κ_1 and κ_2 of Γ which fulfill: $\mathcal{R}\boldsymbol{\tau}_\beta = \kappa_\beta \boldsymbol{\tau}_\beta$ for $\beta = 1, 2$, and the mean curvature $\mathcal{H} = (\kappa_1 + \kappa_2)/2$. Let $h_\beta = 1 + r\kappa_\beta$, $\beta = 1, 2$. After a few calculations, we find the expression of the Helmholtz operator in generalized coordinates

$$L_1 = \partial_r^2 + 2\mathcal{H}_r \partial_r + h_1^{-1} h_2^{-1} \partial_s \cdot (h_2 h_1^{-1} \partial_{s_1}, h_1 h_2^{-1} \partial_{s_2}) - \partial_t^2,$$

setting $\mathcal{H}_r = (h_1^{-2}\kappa_1 + h_2^{-2}\kappa_2)/2$.

To construct the approximation of the EIBO, we have to introduce some tools available from the theory of pseudodifferential operators. Let $A = A(x, D_x)$ be a pseudodifferential operator of OPS^j, $j \in \mathbb{Z}$, $\sigma(A) = \sigma(A)(x, \eta)$ its symbol and $\sigma_j(A)$ its principal symbol. A symbol $\sigma(A)$ admits a symbolic asymptotic expansion in homogeneous symbols if it can be written on the form $\sigma(A) \sim \sum_{m=-j}^{+\infty} \sigma_{-m}(A)$, where functions $\sigma_{-m}(A)$ are some homogeneous functions of degree $-m$ with respect to η, with $m \geq -j$, which continuously depend on x. The above equality holds in the sense of pseudodifferential operators (see Treves [1980]). The partial symbol $\mathcal{L}_1$ of L_1, according to $s = (s_1, s_2)$ and t and their respective covariables $\xi = (\xi_1, \xi_2)$ and $N\omega$, smoothly depends on r. This symbol can be expressed as

$$\mathcal{L}_1 = \partial_r^2 + 2\mathcal{H}_r \partial_r - |\xi|^2 + i h_1^{-1} h_2^{-1} (\partial_{s_1} h_2 h_1^{-1}, \partial_{s_2} h_1 h_2^{-1}) \cdot \xi + N^2 \omega^2,$$

where the length of ξ is defined by: $|\xi| = (\sum_{\beta=1}^{2} h_\beta^{-2} \xi_\beta^2)^{1/2}$. Since N is a complex number, $\mathcal{L}_1$ is a complex symbol. Therefore, the operator L_1 can be factorized since its characteristic equation: $z^2 + N^2\omega^2 - |\xi|^2 = 0$ admits two distinct complex conjugate roots. These two solutions $z_1^{\pm} = \pm i(N^2\omega^2 - |\xi|^2)^{1/2}$ are first-order homogeneous complex functions according to $(\xi, N\omega)$. For a dissipative medium, we remark that: $\Re z_1^- > 0$ and $\Re z_1^+ < 0$.

According to Antoine et al. [2001], the following proposition holds.

Proposition 1. *There exist two classical pseudodifferential operators* Z^- *and* Z^+ *of* OPS^1*, which continuously depend on* r *and such that*

$$L_1 = (\partial_r - Z^+)(\partial_r - Z^-) \quad mod\, \mathcal{C}^\infty,$$

with $\sigma_1(Z^\pm) = z_1^\pm$*. Moreover, the uniqueness of the decomposition is satisfied by the following characterization. Let* $z^\pm$ *be the symbol of* $Z^\pm$*. From the definition of pseudodifferential operators in* OPS^1*, symbols* $z^\pm$ *are some elements of the symbol class* S^1 *and admit the following asymptotic expansion* $z^\pm \sim \sum_{j=-1}^{+\infty} z_{-j}^\pm$*, where* $z_{-j}^\pm$ *are some homogeneous complex valued functions of degree* $-j$ *with respect to* $(\xi, N\omega)$*.*

In the case of a non-dissipative medium with $\Im N = 0$, it can be proved that the factorization is only valid in the cone of propagation $\{(\xi, N\omega), z_1^+ z_1^- > 0\}$.

Using the calculus rules of classical pseudodifferential operators, one can obtain an explicit recursive and constructive algorithm to compute each homogeneous symbol. We refer to Antoine et al. [2001] for further details. We restrict ourselves to the presentation of the effect of the first two terms of the asymptotic expansion ($m = 0$), taking more terms leading to more complicate formulations. The first symbol $z_1^- = -i(N^2\omega^2 - |\xi|^2)^{1/2}$ has already been computed. Concerning the zeroth-order symbol, one gets the explicit expression $z_0^- = -\mathcal{H} - \sum_{l=1}^{2} \kappa_l \xi_l^2 / (2(z_1^-)^2)$. From the analysis developed in Antoine et al. [2001], the EIBO can be suitably approximated by the following generalized Fourier-Robin boundary condition

$$(\partial_{\mathbf{n}} - \alpha \sum_{j=-1}^{m} \widetilde{Z_{-j}^-}) v_2 = \widetilde{g} \equiv -(\partial_{\mathbf{n}} - \alpha \sum_{j=-1}^{m} \widetilde{Z_{-j}^-}) u_0, \tag{2}$$

with the classical pseudodifferential operator: $\widetilde{Z_{-j}^-} = \mathrm{Op}(z^-_{-j|r=0})$.

The resulting approximate boundary condition (2) is not yet completely satisfactory for a numerical treatment. Indeed, the condition is still defined by a non-local pseudodifferential operator. If we approach the numerical solution by a volume finite element method, then we have to consider the following variational formulation: find $v_2 \in H^1(\Omega_b)$ such that

$$\begin{aligned} \int_{\Omega_b} \nabla v_2 \cdot \nabla \varphi - k_2^2 v_2 \varphi d\Omega_b + \int_{\Sigma} \mathcal{M} v_2 \varphi d\Sigma + \alpha \int_{\Gamma} \sum_{j=-1}^{0} \widetilde{Z_{-j}^-} v_2 \varphi d\Gamma \\ = - \int_{\Gamma} \widetilde{g} \varphi d\Gamma. \end{aligned} \tag{3}$$

In the above formulation, the unbounded domain has been truncated by the introduction of a non-reflecting boundary condition of the form: $\partial_{\mathbf{n}} v_2 + \mathcal{M} v_2 = 0$, where $\mathcal{M}$ is a local differential operator defined on a fictitious boundary Σ

enclosing the scatterer. The resulting finite domain of computation is denoted here by Ω_b with a boundary $\partial\Omega_b := \Gamma \cup \Sigma$. Generally, such a linear system is solved by an iterative solver (see e.g. Tezaur et al. [2002]). Therefore, we can assume that v_2 is a given entry at the k-th step of the algorithm and we want to evaluate the action of the operator defined by the left-hand side of Eq.(3). The first two terms are actually classical to compute (Antoine [2001], Tezaur et al. [2002]). This is not the case of the third one which involves two pseudodifferential operators leading to a high computational cost similar to the one involved in an integral equation approach. If we stop at this level, the method is inefficient. However, one can overcome this problem using some suitable Padé approximants. To fix the ideas, let us consider the first-order homogeneous pseudodifferential operator and let us introduce the classical Padé approximants of the square root with branch cut along the negative real line from $z = -1$: $\sqrt{1+z} \approx R_M(z) = c_0 + \sum_{j=1}^{M} a_j z(1+b_j z)^{-1}$. The coefficients c_0 and $(a_j, b_j)_{j=1,\ldots,M}$ are expressed as $c_0 = 1$, $a_j = 2/(2M+1)\sin^2(j\pi/(2M+1))$ and $b_j = \cos^2(j\pi/(2M+1))$, for $j = 1, ..., M$. Then, the evaluation of $\widetilde{Z_1^-}$ applied to a given surface field v_2 is realized by first computing the solution ϕ_j to the surface PDE

$$\int_\Gamma b_j \nabla_\Gamma \phi_j \cdot \nabla_\Gamma \psi - k_1^2 \phi_j \psi d\Gamma = \int_\Gamma v_2 \psi d\Gamma, \quad \text{for } j = 1, ..., M,$$

and then evaluating

$$\int_\Gamma \widetilde{Z_1^-} v_2 \psi d\Gamma = -ik_1 \int_\Gamma v_2 \psi d\Gamma + ik_1 \sum_{j=1}^{M} a_j \int_\Gamma \nabla_\Gamma \phi_j \cdot \nabla_\Gamma \psi d\Gamma,$$

for any test function ψ in $H^1(\Gamma)$. The operator ∇_Γ is the surfacic gradient operator of a scalar surface field. If the interior medium is weakly dissipative or non-dissipative, the approximation of the square root can require a large number M of PDEs to solve. A modified version of the square root approximation should be preferred as for instance by using the rotating branch cut approximation of Milinazzo et al. [1997]. This new approximation has been introduced within the context of underwater acoustic wave propagation problems resolved by the wide-angle parabolic equations approach. The technique consists of replacing the usual coefficients by the new ones $C_0 = e^{i\theta/2} R_M(e^{-i\theta} - 1)$, $A_j = e^{-i\theta/2} a_j((1 + b_j(e^{-i\theta} - 1))^{-2}$ and $B_j = e^{-i\theta} b_j (1 + b_j(e^{-i\theta} - 1))^{-1}$,for $j = 1, ..., M$. An optimal experimental value for the free rotation angle is $\theta = \pi/4$ and $M = 4$ for the number of equations (to *a priori* choose with respect to the interior frequency).

4 Numerical performance

To evaluate the efficiency of the pseudolocal impedance boundary condition, we represent both the surface field and the far field pattern which is given by

$RCS(\vartheta) = 10\log_{10}(\lim_{r\to+\infty} 2\pi r|v_2(r,\vartheta)|^2)$ (db). We compare it to the local impedance boundary condition developed in Antoine et al. [2001] using some second-order Taylor expansions of the first four symbols. This latter condition has a wider validity domain than the usual Fourier-Robin condition. We consider an incident plane wave of frequency $k_2 = 25$ and with a null incidence angle illuminating the unit circular cylinder. The physical parameters are $\rho_r = 1.3$, $c_r = 1.05$ and $\delta = 5$ ($|N| = 0.96$ and $\Im k_1 = 2.3$). As it can be seen on Fig. 1, the surface field is accurately computed with the new condition compared to the second-order condition. This remark also holds for the bistatic RCS. A more complete analysis shows that it is always preferable to use the Padé approximation than the second-order Taylor expansion without affecting the total computational cost of the iterative procedure.

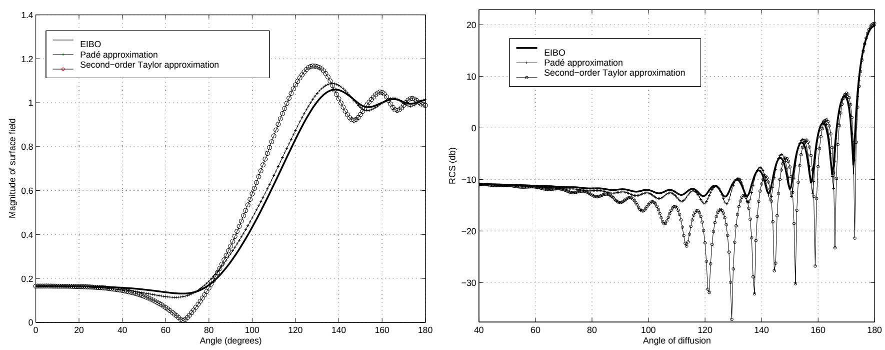

Fig. 1. Surface fields and bistatic RCS computations for the proposed test case.

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