Domain Embedding/Controllability Methods for the Conjugate Gradient Solution of Wave Propagation Problems

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Summary. The main goal of this paper is to discuss the numerical simulation of propagation phenomena for time harmonic electromagnetic waves by methods combining controllability and fictitious domain techniques. These methods rely on distributed Lagrangian multipliers, which allow the propagation to be simulated on an obstacle free computational region using regular finite element meshes essentially independent of the geometry of the obstacle and on a controllability formulation which leads to algorithms with good convergence properties to time-periodic solutions. This novel methodology has been validated by the solutions of test cases associated to non trivial geometries, possibly non-convex. The numerical experiments show that the new method performs as well as the method discussed in Bristeau et al. [1998] where obstacle fitted meshes were used.

1 Introduction

Lagrange multiplier based fictitious domain methods have proved to be efficient techniques for the solution of viscous flow problems with moving boundaries (see Glowinski [2003], Chapter 8, Glowinski et al. [2001]). The main goal of this article is to discuss the generalization of this methodology to the simulation of wave propagation phenomena. A motivation for using the fictitious domain approach is that it allows–to some extent–the use of uniform meshes, which is clearly an advantage far from the scatters. In order to capture efficiently time-periodic solutions, the fictitious domain methodology is coupled to exact controllability methods close to those utilized in Bristeau et al. [1998], Glowinski and Lions [1995]. Various formulations of a wave propagation model problem, including a fictitious domain one, will be discussed in Sections 3 and 4. The computation of the gradient of a cost function associated to the control formulation will be briefly addressed in Section 5. The conjugate gradient solution of the control problem will be discussed in Section

6, while the space/time discretization will be discussed in Section 7. Finally, the results of numerical experiments will be presented in Section 8.

2 Formulation of the wave-propagation problem.

Let ω be a bounded domain of $\mathbb{R}^d$ $(d = 2, 3)$; we denote by γ the boundary $\partial\omega$ of ω. Consider now $T(> 0)$. We are looking for the $T-$periodic solutions of the following wave equation:

$$\varphi_{tt} - \Delta\varphi = 0 \text{ in } (\mathbb{R}^d \setminus \overline{\omega}) \times (0, T),\ \varphi = g \text{ on } \gamma \times (0, T), \tag{1}$$

completed by additional conditions such as $\lim_{|x|\to+\infty} \varphi(x, t) = 0$. The time-periodicity conditions take then the following form:

$$\varphi(0) = \varphi(T),\ \varphi_t(0) = \varphi_t(T), \tag{2}$$

where $\varphi(t)$ denotes the function $x \to \varphi(x, t)$. From a computational point of view, we imbed ω in a bounded simple-shape domain Ω of boundary Γ_{ext} (see Figure 1) and consider

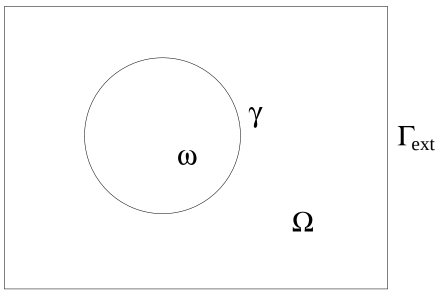

Fig. 1. Imbedding of ω.

the following wave problem:

$$\begin{aligned} &\varphi_{tt} - \Delta\varphi = 0 \text{ in } (\Omega \setminus \overline{\omega}) \times (0, T), \\ &\varphi = g \text{ on } \gamma \times (0, T),\ \frac{\partial\varphi}{\partial n} + \frac{\partial\varphi}{\partial t} = 0 \text{ on } \Gamma_{ext} \times (0, T), \end{aligned} \tag{3}$$

$$\varphi(0) = \varphi(T),\ \varphi_t(0) = \varphi_t(T). \tag{4}$$

3 A fictitious domain formulation of problem (3), (4).

Problem (3), (4) is equivalent to

Find $\{\varphi, \lambda\}$ *verifying:*

$$\int_{\Omega}\varphi_{tt}vdx+\int_{\Omega}\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}vdx+\int_{\Gamma_{ext}}\frac{\partial\varphi}{\partial t}vd\Gamma+\int_{\omega}\lambda vdx=0,\ \forall v\in H^1(\Omega),$$
$$\int_{\omega}\mu(\varphi-\tilde{g})dx=0,\ \forall\mu\in L^2(\omega),\tag{5}$$

$$\varphi(0)=\varphi(T),\varphi_t(0)=\varphi_t(T),\tag{6}$$

$\tilde{g}(t)$ being an ω-extension of $g(t)$ such that $\tilde{g}(t)\in H^1(\Omega)$.

4 A virtual control/least squares formulation of problem (5), (6).

A virtual control/least squares formulation of problem (5), (6) reads as follows:

$$\begin{gathered}Find\ \mathbf{e}\in E\ such\ that\\ J(\mathbf{e})\leq J(\mathbf{w}),\ \forall\mathbf{w}\ (=\{w_0,w_1\})\ \in E,\end{gathered}\tag{7}$$

with $E=H^1(\Omega)\times L^2(\Omega)$, and

$$J(\mathbf{w})=\frac{1}{2}\int_{\Omega}[|\nabla(w_0-y(T))|^2+|w_1-y_t(T)|^2]dx,\tag{8}$$

y being the solution for a.e. t of

$$\int_{\Omega}y_{tt}zdx+\int_{\Omega}\nabla y\cdot\nabla zdx+\int_{\Gamma_{ext}}\frac{\partial y}{\partial t}zd\Gamma+\int_{\omega}\lambda zdx=0,\ \forall z\in H^1(\Omega),$$
$$\int_{\omega}\mu(y-\tilde{g})dx=0,\ \forall\mu\in L^2(\omega),\tag{9}$$

$$y(0)=w_0,\ y_t(0)=w_1.\tag{10}$$

Problem (7), being linear-quadratic, can be solved by a conjugate gradient algorithm operating in E. To implement such an algorithm we need to know $J'(\mathbf{w}),\ \forall\mathbf{w}\in E$. The derivation of $J'(\mathbf{w})$ will be addressed in the following section, while the conjugate gradient solution of problem (7)-(10) will be discussed in Section 6.

5 Derivation of $J'(\mathbf{w})$.

It can be shown that if we define p by

$$\begin{gathered}\int_{\Omega}p_{tt}zdx+\int_{\Omega}\nabla p\cdot\nabla zdx-\int_{\Gamma_{ext}}\frac{\partial p}{\partial t}zd\Gamma+\int_{\omega}\lambda^* zdx=0,\\ \forall z\in H^1(\Omega),\\ \int_{\omega}p\mu dx=0,\ \forall\mu\in L^2(\omega),\\ p(T)=y_t(T)-w_1,\\ \int_{\Omega}p_t(T)zdx=\int_{\Gamma_{ext}}p(T)zd\Gamma-\int_{\Omega}\boldsymbol{\nabla}(y(T)-w_0)\cdot\boldsymbol{\nabla}zdx,\ \forall z\in H^1(\Omega),\end{gathered}\tag{11}$$

then we have the following representation for $J'(\mathbf{w})$:

$$< J'(\mathbf{w}), \mathbf{v} > = \int_\Omega \nabla(w_0 - y(T)) \cdot \nabla v_0 dx - \int_\Omega p_t(0) v_0 dx + \int_{\Gamma_{ext}} p(0) v_0 d\Gamma + \int_\Omega (w_1 - y_t(T)) v_1 dx + \int_\Omega p(0) v_1 dx, \ \forall \mathbf{v} = \{v_0, v_1\} \in E. \tag{12}$$

Relations (11) and (12) are largely "formal"; however it is worth mentioning that the discrete variants of them make sense and lead to algorithms with fast convergence properties.

6 Conjugate gradient solution of problem (7).

As in Section 4, we suppose that $E = H^1(\Omega) \times L^2(\Omega)$. A *conjugate gradient algorithm* for the solution of (7) is given by:
Step 0: Initialization

$$\mathbf{e}^0 = \{e_0^0, e_1^0\} \in E \text{ is given.} \tag{13}$$

Solve the following forward wave problem:

$$\begin{gathered} \int_\Omega y_{tt}^0 z dx + \int_\Omega \boldsymbol{\nabla} y^0 \cdot \boldsymbol{\nabla} z dx + \int_{\Gamma_{ext}} \frac{\partial y^0}{\partial t} z d\Gamma + \int_\omega \lambda^0 z dx = 0, \ \forall z \in H^1(\Omega), \\ \int_\omega \mu (y^0 - \tilde{g}) dx = 0, \ \forall \mu \in L^2(\omega), \\ y^0(0) = e_0^0, \ y_t^0(0) = e_1^0. \end{gathered} \tag{14}$$

Solve next the following backward wave-problem

$$\begin{gathered} \int_\Omega p_{tt}^0 z dx + \int_\Omega \boldsymbol{\nabla} p^0 \cdot \boldsymbol{\nabla} z dx - \int_{\Gamma_{ext}} \frac{\partial p^0}{\partial t} z d\Gamma + \int_\omega \lambda^{*0} z dx = 0, \ \forall z \in H^1(\Omega), \\ \int_\omega p^0 \mu dx = 0, \ \forall \mu \in L^2(\omega), \\ p^0(T) = y_t^0(T) - e_1^0, \\ \int_\Omega p_t^0(T) z dx = \int_{\Gamma_{ext}} p^0(T) z d\Gamma - \int_\Omega \boldsymbol{\nabla}(y^0(T) - e_0^0) \cdot \boldsymbol{\nabla} z dx, \ \forall z \in H^1(\Omega). \end{gathered} \tag{15}$$

Next, define $\mathbf{g}^0 = \{g_0^0, g_1^0\} \in E \ (= H^1(\Omega) \times L^2(\Omega))$ *by*

$$\begin{gathered} \int_\Omega \boldsymbol{\nabla} g_0^0 \cdot \boldsymbol{\nabla} z dx = \int_\Omega \boldsymbol{\nabla}(e_0^0 - y^0(T)) \cdot \boldsymbol{\nabla} z dx - \int_\Omega p_t^0(0) z dx + \int_{\Gamma_{ext}} p^0(0) z d\Gamma, \ \forall z \in H^1(\Omega), \\ g_1^0 = p^0(0) + e_1^0 - y_t^0(T), \end{gathered} \tag{16}$$

and then

$$\mathbf{w}^0 = \mathbf{g}^0. \quad \square \tag{17}$$

For $n \geq 0$, suppose that $\mathbf{e}^n,\ \mathbf{g}^n,\ \mathbf{w}^n$ are known; we compute their updates $\mathbf{e}^{n+1},\ \mathbf{g}^{n+1},\ \mathbf{w}^{n+1}$ as follows:

Step 1: Descent

Solve

$$\begin{gathered}\int_\Omega \overline{y}^n_{tt} z dx + \int_\Omega \boldsymbol{\nabla}\overline{y}^n \cdot \boldsymbol{\nabla} z dx + \int_{\Gamma_{ext}} \frac{\partial \overline{y}^n}{\partial t} z dx \int_\omega \overline{\lambda}^n z dx = 0,\ \forall z \in H^1(\Omega),\\ \int_\omega \mu \overline{y}^n dx = 0,\ \forall \mu \in L^2(\omega),\\ \overline{y}^n(0) = w_0^n,\ \overline{y}^n_t(0) = w_1^n.\end{gathered} \tag{18}$$

Solve the backward wave problem

$$\begin{gathered}\int_\Omega \overline{p}^n_{tt} z dx + \int_\Omega \boldsymbol{\nabla}\overline{p}^n \cdot \boldsymbol{\nabla} z dx - \int_{\Gamma_{ext}} \frac{\partial \overline{p}^n}{\partial t} z d\Gamma + \int_\omega \overline{\lambda}^{n*} z dx = 0,\ \forall z \in H^1(\Omega),\\ \int_\Omega \overline{p}^n \mu dx = 0,\ \forall \mu \in L^2(\omega),\end{gathered} \tag{19}$$

with $\overline{p}^n(T)$ and $\overline{p}^n_t(T)$ given by

$$\begin{gathered}\overline{p}^n(T) = \overline{y}^n_t - w_1^n,\\ \int_\omega \overline{p}^n_t(T) z dx = \int_\Gamma \overline{p}^n(T) z dx - \int_\Omega \boldsymbol{\nabla}(\overline{y}^n(T) - w_0^n) \cdot \boldsymbol{\nabla} z dx,\ \forall z \in H^1(\Omega),\end{gathered}$$

respectively.

Next define $\overline{\mathbf{g}}^n = \{\overline{g}^n_0, \overline{g}^n_1\} \in H^1(\Omega) \times L^2(\Omega)$ by

$$\begin{gathered}\int_\Omega \boldsymbol{\nabla}\overline{g}^n_0 \cdot \boldsymbol{\nabla} z dx = \int_\Omega \boldsymbol{\nabla}(w_0^n - \overline{y}^n(T)) \cdot \boldsymbol{\nabla} z dx - \int_\Omega \overline{p}^n_t(0) z dx\\ + \int_{\Gamma_{ext}} \overline{p}^n(0) z d\Gamma,\ \forall z \in H^1(\Omega),\end{gathered} \tag{20}$$

$$\overline{g}^n_1 = \overline{p}^n(0) + w_1^n - \overline{y}^n_t(T),$$

and then ρ_n by

$$\rho_n = \int_\Omega [|\boldsymbol{\nabla} g_0^n|^2 + |g_1^n|^2] dx \Big/ \int_\Omega (\boldsymbol{\nabla}\overline{g}^n_0 \cdot \boldsymbol{\nabla} w_0^n + \overline{g}^n_1 w_1^n) dx. \tag{21}$$

We update then $\mathbf{e}^n$ and $\mathbf{g}^n$ by

$$\mathbf{e}^{n+1} = \mathbf{e}^n - \rho_n \mathbf{w}^n, \tag{22}$$

$$\mathbf{g}^{n+1} = \mathbf{g}^n - \rho_n \overline{\mathbf{g}}^n. \tag{23}$$

Step 2: Test of the convergence and construction of the new descent direction

If $\int_\Omega (|\nabla g_0^{n+1}|^2 + |g_1^{n+1}|^2) dx \Big/ \int_\Omega (|\nabla g_0^n|^2 + |g_1^n|^2) dx \leq \varepsilon$, *take* $\mathbf{e} = \mathbf{e}^{n+1}$; *else, compute*

$$\gamma_n = \int_\Omega (|\nabla g_0^{n+1}|^2 + |g_1^{n+1}|^2) dx \Big/ \int_\Omega (|\nabla g_0^n|^2 + |g_1^n|^2) dx \tag{24}$$

and update $\mathbf{w}^n$ *by*

$$\mathbf{w}^{n+1} = \mathbf{g}^{n+1} + \gamma_n \mathbf{w}^n. \ \square \tag{25}$$

Do $n = n + 1$ *and* g_0 *to (18).*

Algorithm (13)-(25) requires the solution of two waves problems at each iteration and also of an elliptic problem such as (20). For more comments see Bristeau et al. [1998], Glowinski and Lions [1995].

7 Finite difference/finite element implementation.

Compared to what has been done in Bristeau et al. [1998], Glowinski and Lions [1995] the main difficulty is clearly the numerical implementation of the distributed Lagrange multiplier based techniques used to force Dirichlet boundary conditions. We shall consider the forward wave equations only since the backward ones can be treated by similar methods. Dropping the superscript, the forward wave problems to be solved are all of the following type:

$$\int_\Omega y_{tt} z dx + \int_\Omega \nabla y \cdot \nabla z dx + \int_{\Gamma_{ext}} \frac{\partial y}{\partial t} z d\Gamma + \int_\omega \lambda z dx = 0, \ \forall z \in H^1(\Omega), \tag{26}$$

$$\int_\Omega \mu (y - \tilde{g}) dx = 0, \ \forall \mu \in L^2(\omega), \tag{27}$$

$$y(0) = e_0, \ y_t(0) = e_1. \tag{28}$$

Approximating spaces $L^2(\Omega)$ and $H^1(\Omega)$ are pretty classical tasks. Let us suppose that Ω is a bounded polygonal domain of $\mathbb{R}^2$; we introduce a triangulation $\mathcal{T}_h$ of Ω and define a space V_h approximating both $H^1(\Omega)$ and $L^2(\Omega)$ by

$$V_h = \{z_h | z_h \in C^0(\overline{\Omega}), z_h|_T \in P_1, \ \forall T \in \mathcal{T}_h\}. \tag{29}$$

Next, in order to implement the fictitious domain methodology, we proceed as follows: we introduce first a set Σ_h of control points belonging to $\overline{\omega}$ and defined as follows:

$$\Sigma_h = \Sigma_h^\omega \cup \Sigma_h^\gamma, \tag{30}$$

where, in (30), Σ_h^ω is the set of the vertices of $\mathcal{T}_h$ belonging to ω and whose distance at γ is more than Ch, C being a positive constant, and where Σ_h^γ is a set of points of γ. We suppose that $\Sigma_h = \{p_j\}_{j=1}^{N_h}$, where $N_h = Card(\Sigma_h)$.

Following Glowinski [2003], Glowinski et al. [2001], we shall use as “multiplier” space, Λ_h defined by

$$\Lambda_h = \{\mu_h | \mu_h = \Sigma_{j=1}^{N_h} \mu_j \delta(x - p_j),\ \mu_j \in \mathbb{R}\}. \tag{31}$$

Collecting the above results leads to the following *collocation based* approximation of problem (26)-(28):

$$\int_\Omega \frac{y_h^{n+1} + y_h^{n-1} - 2y_h^n}{\tau^2} z_h dx + \int_\Omega \boldsymbol{\nabla} y_h^n \cdot \boldsymbol{\nabla} z_h dx + \int_{\Gamma_{ext}} \frac{y_h^{n+1} - y_h^{n-1}}{2\tau} z_h d\Gamma$$
$$+ \sum_{j=1}^{N_h} \lambda_j^{n+1} z_h(p_j) = 0,\ \forall z_h \in V_h, \tag{32}$$

$$y_h^{n+1}(p_j) - \tilde{g}_h(p_j, (n+1)\tau) = 0,\ \forall j = 1, ..., N_h, \tag{33}$$

$$y_h^0 = e_{0h},\ y_h^1 - y_h^{-1} = 2\tau e_{1h}; \tag{34}$$

in (32)-(34), $\tilde{g}_h,\ e_{0h},\ e_{1h}$ are approximations–all belonging to V_h–of $\tilde{g},\ e_0\ e_1$, respectively.
The *finite dimensional linear variational problem* (32), (33) is of the form

$$\begin{aligned} Ax + B^t \lambda &= b, \\ Bx &= c, \end{aligned} \tag{35}$$

where matrix A is symmetric and positive definite. To solve *the saddle point problem* (35), we can use for example the *Uzawa/conjugate gradient algorithms* discussed in, e.g., Glowinski and Lions [1995], Fortin and Glowinski [1982]. Suppose, for simplicity, that functions g and $\tilde{g}$ are time independent. Taking $z_h = (y_h^{n+1} - y_h^{n-1})/2\tau$ in (32) we can easily show that scheme (7.7)-(7.9) is stable if τ verifies a stability condition such as

$$\tau \leq c^{-1} h, \tag{36}$$

where c (which has the dimension of a velocity) is a positive constant *independent* of ω. Related distributed Lagrange multiplier based fictitious domain methods for the solution of wave propagation problems with obstacles are discussed in Bokil [2004].

8 Numerical experiments.

In order to validate the methods discussed in the above sections, we will address the solution of three test problems already solved in Bristeau et al. [1998] and Glowinski and Lions [1995] using controllability and obstacle fitted finite element meshes. These problems concern the scattering of planar incident waves by a disk, a convex ogive, and a non-convex reflector (air-intake like).

First Test Problem: For this problem, ω is a disk of radius .25m. This disk is illuminated by an incident planar wave of wavelength .125m (which corresponds to a $2.4 \times GHz$ frequency) coming from the right, the incidence angle with horizontal being zero. The artificial boundary Γ_{ext} is located at a 3 wave length distance from ω. On Figure 2, we have visualized the uniform finite element triangulation used over Ω to define the discrete spaces V_h and Λ_h. It consists of 19,881 vertices and 39,200 triangles; the number of control points used to define Λ_h is 19,881. The value of Δt corresponds to 80 time steps per period, the space discretization $h_x = h_y = 8.928571 \cdot 10^{-3}$ and $\Delta s = 1.6071 \cdot 10^{-2}$ the length between two adjacent point on the disk. The decay of the discrete cost functional is a function of the number of iterations of the discrete analogue of algorithm (13)-(25) is shown on Figure 3. Comparing to the results reported in Glowinski and Lions [1995] shows that the convergence performances are not modified by the addition of the fictitious domain procedure. The computed scattered field has been visualized on Figure 4.

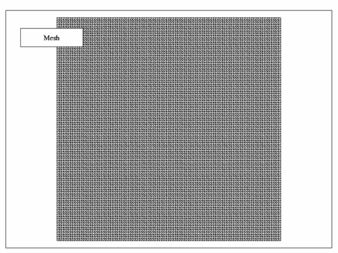

Fig. 2. Uniform finite element triangulation used for the fictitious domain method

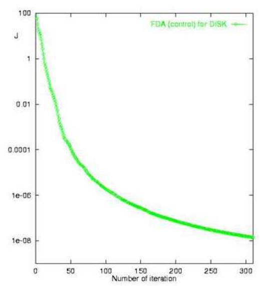

Fig. 3. Disk: convergence of the fictitious domain algorithm

Second Test Problem: For this problem, ω is an ogive-like obstacle of length .6m and thickness .16m, respectively. The wavelength of the incident wave is 0.1m. The artificial boundary is located at a 3 wave-length distance from ω. The finite element triangulation is uniform and has 11,571 vertices and 22,704 triangles. The convergence for a zero degree of incidence monochromatic wave is shown on Figure 5. The imaginary component of the scattered field is shown on Figure 6.

Third Test Problem: Denote by λ the wavelength of the propagation phenomena. For this problem ω is an idealized air intake; it has a semiopen cavity geometry defined by two horizontal plates (length 4λ and thickness equal 0.2λ) and a vertical one (length 1.4λ and thickness $\lambda/5$). We have $f = 1.2GHz$ implying a .25m wavelength. The artificial boundary is located again at a distance of 3λ from ω. The uniform finite element triangulation has 20,202 points and 39,820 triangles. The convergence to the solution for

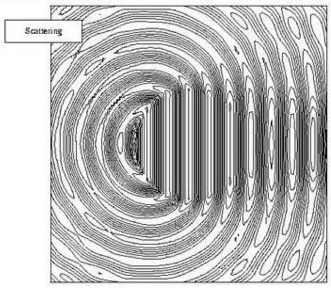

Fig. 4. Disk: visualization of the scattered field

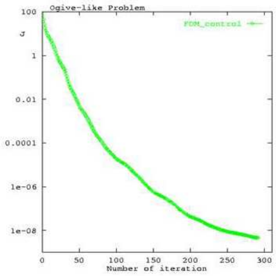

Fig. 5. Ogive: convergence of the fictitious domain algorithm

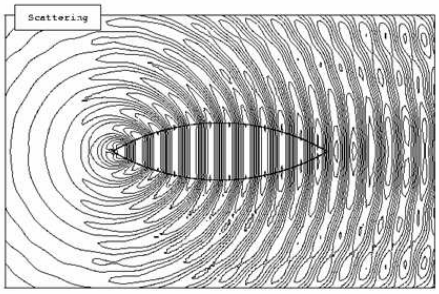

Fig. 6. Ogive: imaginary component of the scattered field

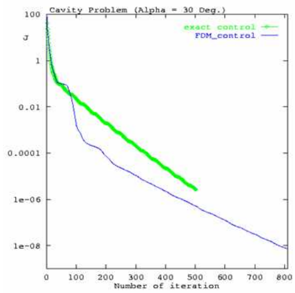

Fig. 7. Idealized air intake: convergence of the fictitious domain algorithm

an illuminating monochromatic wave of incidence $\alpha = 30°$ is shown on Figure 7. We observe from Figure 7 that the convergence of the method combining controllability and fictitious domain is faster than the one of the "pure" controllability method discussed in Glowinski and Lions [1995]. In order to compare the first method with those discussed in Glowinski and Lions [1995], we have visualized on Figures 8 and 9 the scattered field obtained with the methods of Glowinski and Lions [1995] and of this article on Figures 4, 6, and 8. We can observe the extension of the scattered field inside the scatters.

9 Conclusion and Future.

The fictitious domain based methods discussed in this article appear to be competitive with the boundary fitted one discussed in Glowinski and Lions [1995]. One of the main advantages of the fictitious domain approach is that it is well-suited to those shape optimization problems with several scatters where we have the shape and position of the obstacles in order to minimize

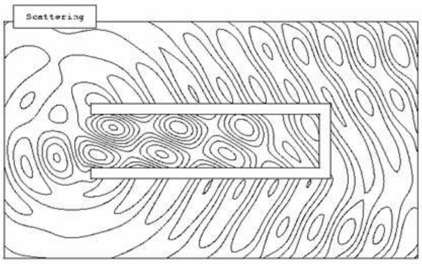

Fig. 8. Idealized air intake: scattered field obtained by the method of reference

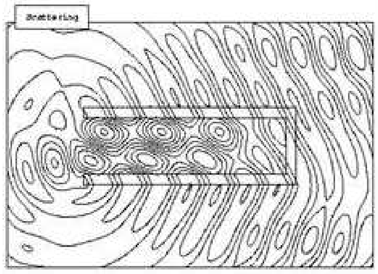

Fig. 9. Idealized air intake: scattered field obtained by the fictitious domain method

for example a Radar Cross section. Only the acoustic wave equation has been considered in this investigation, but we consider generalizing the methods discussed here to Maxwell equations in two and three dimensions.

Acknowledgement. The authors gratefully acknowledge Giovanna Guidoboni for her kind assistance in preparing the paper while visiting University of Houston. We would like also to thank the Association Franco Chinoise pour la Science, la Recherche et la Technologie (AFCSRT) for its support. This work was performed while the first and third author had a PRA SI-O1-O6 from AFCSRT.

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