Optimized interface conditions for the compressible Euler equations

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Summary. In this work we design new interface transmission conditions for a domain decomposition Schwarz algorithm for the Euler equations in 2 dimensions. These new interface conditions are designed to improve the convergence properties of the Schwarz algorithm. These conditions depend on a few parameters and they generalize the classical ones. Numerical results illustrate the effectiveness of the new interface conditions.

1 Introduction

In a previous paper [DLN04] we formulated and studied by means of Fourier analysis the convergence of a Schwarz algorithm (interface iteration which relies on the successive solving of the local decomposed problems and the transmission of the result at the interface) involving transmission conditions that are derived naturally from a weak formulation of the underlying boundary value problem. Various works and studies exist when dealing with Schwarz algorithms applied to the scalar problems but to our knowledge, little is known about complex systems. When dealing with systems we can mention some classical works by Quarteroni and al. [Qua90] [QS96] Bjorhus [Bjø95] or Cai et al.[CFS98]. The most related work to our study belongs to Clerc [Cle98] and it describes the principle of building very simple interface conditions for a general hyperbolic system which we will apply and extend to Euler system. In this work we formulate and analyze the convergence of the Schwarz algorithm with new interface conditions inspired by [Cle98], depending on 2 parameters whose value is determined by minimizing the norm of the convergence rate. The paper is organized as follows. In the section 2 we first formulate the Schwarz algorithm for a general linear hyperbolic system of PDEs with general interface conditions built in order to have a well-posed problem. In the section 3 we estimate the convergence rate at the discrete level. We will find the optimal parameters of the interface conditions at the discrete level. In the section 4, we use the new optimal interface conditions in Euler computations which illustrate the improvement over the classical interface conditions (first described in [QS96]).

2 A Schwarz algorithm with general interface conditions

2.1 A well-posed boundary value problem

If we consider here a general non-linear system of conservation laws under the hypothesis that its solution is regular, we can also use a non-conservative (or quasi-linear) equivalent form. Assume that we first proceed to an integration in time using a backward Euler implicit scheme involving a linearization of the flux functions and eventually we symmetrize it (we know that when the system admits an entropy it can be symmetrized by multiplying it by the hessian matrix of this entropy). This operation results in the linearized system:

$$\mathcal{L}(W) \equiv \frac{\mathrm{Id}}{\Delta t}W + \sum_{i=1}^{d} A_i \frac{\partial W}{\partial x_i} = f \tag{1}$$

In the following we will define the boundary conditions that have to be imposed when solving the problem on a domain $\Omega \subset \mathbb{R}^d$. We denote by $A_{\mathbf{n}} = \sum_{i=1}^{d} A_i n_i$, the linear combination of jacobian matrices by the components of the outward normal vector at the boundary of the domain $\partial\Omega$. This matrix is real, symmetric and can be diagonalized $A_{\mathbf{n}} = T\Lambda_{\mathbf{n}}T^{-1}$, $\Lambda_{\mathbf{n}} = diag(\lambda_i)$. It can also be splitted in negative ($A_{\mathbf{n}}^{-}$) and positive ($A_{\mathbf{n}}^{+}$) part using this diagonalization. This corresponds to a decomposition with local characteristic variables. A more general splitting in negative(positive) definite parts, $A_{\mathbf{n}}^{neg}$ and $A_{\mathbf{n}}^{pos}$ of $A_{\mathbf{n}}$ can be done such that these matrices satisfy the following properties:

$$\begin{cases} A_{\mathbf{n}} & = A_{\mathbf{n}}^{neg} + A_{\mathbf{n}}^{pos} \\ rank(A_{\mathbf{n}}^{neg,pos}) & = rank(A_{\mathbf{n}}^{\pm}) \\ A_{-\mathbf{n}}^{pos} & = -A_{\mathbf{n}}^{neg} \end{cases} \tag{2}$$

In the scalar case the only possible choice is $A_{\mathbf{n}}^{neg} = A_{\mathbf{n}}^{-}$. Using the previous formalism we can define the following boundary condition:

$$A_{\mathbf{n}}^{neg}W = A_{\mathbf{n}}^{neg}g, \text{ on } \partial\Omega \tag{3}$$

Within this framework we have a result of well-possedness of the boundary value problem associated to the system (1) with the boundary conditions (3) that can be found in [Cle98]. As the boundary value problem is well-posed, the decomposition (2) enables the design of a domain decomposition method.

2.2 Schwarz algorithm with general interface conditions

We consider a decomposition of the domain Ω into N overlapping or non-overlapping subdomains $\bar{\Omega} = \bigcup_{i=1}^{N} \bar{\Omega}_i$. We denote by $\mathbf{n}_{ij}$ the outward normal to the interface Γ_{ij} bewteen Ω_i and a neighboring subdomain Ω_j. Let $W_i^{(0)}$ denote the initial appoximation of the solution in subdomain Ω_i. A general formulation of a Schwarz algorithm for computing $(W_i^{p+1})_{1\le i\le N}$ from $(W_i^p)_{1\le i\le N}$ (where p defines the iteration of the Schwarz algorithm) reads :

$$\begin{cases} \mathcal{L}W_i^{p+1} & = f \qquad \text{in} \Omega_i \\ A_{\mathbf{n}_{ij}}^{neg} W_i^{p+1} & = A_{\mathbf{n}_{ij}}^{neg} W_j^p \text{ on } \Gamma_{ij} = \partial\Omega_i \cap \Omega_j \\ A_{\mathbf{n}_{ij}}^{neg} W_i^{p+1} & = A_{\mathbf{n}_{ij}}^{neg} g \quad \text{on } \partial\Omega \cap \partial\Omega_i \end{cases} \tag{4}$$

where $A_{\mathbf{n}_{ij}}^{neg}$ and $A_{\mathbf{n}_{ij}}^{pos}$ satisfy (2). We have a convergence result of this algorithm in the non-overlapping case, due to ([Cle98]). The convergence rate of the algorithm defined by (4) depends of the choice of the decomposition of $A_{\mathbf{n}_{ij}}$ into $A_{\mathbf{n}_{ij}}^{neg}$ and $A_{\mathbf{n}_{ij}}^{pos}$ satisfying (2). In order to choose the right decomposition we need to relate this choice to the convergence rate of (4).

2.3 Convergence rate of the algorithm with general interface conditions

We consider a two-subdomain non-overlapping or overlapping decomposition of the domain $\Omega = \mathbb{R}^d$, $\Omega_1 =]-\infty, \gamma[\times\mathbb{R}^{d-1}$ and $\Omega_2 =]\beta, \infty[\times\mathbb{R}^{d-1}$ with $\beta \le \gamma$ and study the convergence of the Schwarz algorithm in the subsonic case. A Fourier analysis applied to the linearized equations allows us to derive the convergence rate of the "ξ"-th Fourier component of the error as described in detail in [DLN04]. After having defined in a general frame the well-possedness of the boundary value problem associated to a general equation and the convergence of the Schwarz algorithm applied to this class of problems, we will concentrate ourselves on the conservative Euler equations in two-dimensions:

$$\frac{\partial W}{\partial t} + \nabla.\mathbf{F}(W) = 0,\ W = (\rho,\ \rho\mathbf{V},\ E)^T. \tag{5}$$

In the above expressions, ρ is the density, $\mathbf{V} = (u,\ v)^T$ is the velocity vector, E is the total energy per unit of volume and p is the pressure. In equation (5), $W = W(\mathbf{x}, \mathbf{t})$ is the vector of conservative variables, $\mathbf{x}$ and t respectively denote the space and time variables and $\mathbf{F}(W) = (F_1(W), F_2(W))^T$ is the conservative flux vector whose components are given by

$$F_1(W) = \left(\rho u, \rho u^2 + p, \rho uv, u(E+p)\right)^T,\ F_2(W) = \left(\rho v, \rho uv, \rho v^2 + p, v(E+p)\right)^T.$$

The pressure is deduced from the other variables using the state equation for a perfect gas $p = (\gamma_s - 1)(E - \frac{1}{2}\rho \parallel \mathbf{V} \parallel^2)$ where γ_s is the ratio of the specific heats ($\gamma_s = 1.4$ for the air).

2.4 A new type of interface conditions

We will apply now the method described previously to the computation of the convergence rate of the Schwarz algorithm applied to the two-dimensional subsonic Euler equations. In the supersonic case there is only one decomposition satisfying (2), that is: $\mathcal{A}^{pos} = A_{\mathbf{n}}$ and $\mathcal{A}^{neg} = 0$ and the convergence follows in 2 steps. Therefore the only case of interest is the subsonic one.
The starting point of our analysis is given by the linearized form of the Euler equations (5) which are of the form (1) to whom we applied a change of variable $\tilde{W} = T^{-1}W$ based on the eigenvector factorization of $A_1 = T\tilde{A}_1T^{-1}$. We denote by $M_n = \dfrac{u}{c}$, $M_t = \dfrac{v}{c}$ respectively the normal and the tangential Mach number. Before estimating the convergence rate we will derive the general transmission conditions at the interface by splitting the matrix A_1 into a positive and negative part.
We have the following general result concerning this decomposition:

Lemma 1. *Let $\lambda_1 = M_n - 1$, $\lambda_2 = M_n + 1$, $\lambda_3 = \lambda_4 = M_n$. Suppose we deal with a subsonic flow: $0 < u < c$ so that $\lambda_1 < 0$, $\lambda_{2,3,4} > 0$. Any decomposition of $A_1 = A_{\mathbf{n}}$, $\mathbf{n} = (1, 0)$ which satisfies (2) has to be of the form:*

$$\begin{aligned}\mathcal{A}^{neg} &= \frac{1}{a_1}\mathbf{u}\cdot\mathbf{u}^t,\ \mathbf{u} = (a_1, a_2, a_3, a_4)^t\\ \mathcal{A}^{pos} &= A_{\mathbf{n}} - \mathcal{A}^{neg}.\end{aligned}$$

where $(a_1, a_2, a_3, a_4) \in \mathbb{R}^4$ satisfies $a_1 \le \lambda_1 < 0$ and $\dfrac{a_1}{\lambda_1} + \dfrac{a_2^2}{a_1\lambda_2} + \dfrac{a_3^2}{a_1\lambda_3} + \dfrac{a_4^2}{a_1\lambda_4} = 1$.

We will proceed now to the estimation of the convergence rate using some results from [DLN04]. Following the technique described here we estimate the convergence rate in the Fourier space in the non-overlapping case. We use the non-dimensioned wave-number $\bar{\xi} = c\Delta t\xi$, we get for the general interface conditions the following:

$$\left\{\begin{aligned}&\rho^2_{2,novr}(\xi) = \left|1 - \frac{4M_n(1-M_n)(1+M_n)R(\xi)a_1^2(a+M_nR(\xi))}{D_1D_2}\right|\\ &D_1 = R(\xi)[a_1(1+M_n) - a_2(1-M_n)] + a[a_1(1+M_n)\\ &\quad +a_2(1-M_n)] - i\sqrt{2}a_3\xi(1-M_n^2)\\ &D_2 = M_na_1[R(\xi)[a_1(1+M_n) - a_2(1-M_n)] + a[a_1(1+M_n) + a_2(1-M_n)]]\\ &\quad +a_3(1-M_n^2)[a_3(R+a) - iM_na_1\xi\sqrt{2}]\end{aligned}\right. \tag{6}$$

In order to simplify our optimization problem we will take $a_3 = 0$, we can thus reduce the number of parameters to 2, a_1 and a_2, as we can see from the lemma that a_4 can be expressed as a function of a_1, a_2 and a_3. In the same time for the optimization purpose only we introduce the parameters:

$b_1 = -a_1/(1-M_n)$ and $b_2 = a_2/(1+M_n)$ which provide a simpler form of the convergence rate. Nevertheless, solving this problem is quite a tedious task even in the non-overlapping case, where we can obtain analytical expression of the parameters only for some values of the Mach number. In the same time, we have to analyze the convergence of the overlapping algorithm. Indeed, standard discretizations of the interface conditions correspond to overlapping decompositions with an overlap of size $\delta = h$, h being the mesh size, as seen in [DLN04]. By applying the Fourier transform technique to the overlapping case we have the following expression of the convergence rate:

$$\begin{cases} \rho^2_{2,ovr} &= \left| A e^{-(\lambda_2(k)-\lambda_1(k))\bar{\delta}} + (B+C) e^{-(\lambda_3(k)-\lambda_1(k))\bar{\delta}} \right| \\ A &= \dfrac{a+M_nR(\xi)}{a-M_nR(\xi)} \cdot \left(\dfrac{b_1(R(\xi)-a)+b_2(R(\xi)+a)}{b_1(R(\xi)+a)+b_2(R(\xi)-a)} \right)^2 \\ B &= -\dfrac{2M_n(b_1(1-M_n)+b_2(1+M_n))R(\xi)(R(\xi)-a)(R(\xi)+a)}{(1-M_n^2)(a-M_nR(\xi))(b_1(R(\xi)+a)+b_2(R(\xi)-a))^2} \\ C &= \dfrac{4((1-M_n)(b_1^2-b_1)-b_2^2(Mn+1))(a+M_nR(\xi))}{(1-M_n^2)(b_1(R(\xi)+a)+b_2(R(\xi)-a))^2} \end{cases} \tag{7}$$

where $\bar{\delta} = \dfrac{\delta}{c\Delta t}$ denotes the non-dimensioned overlap between subdomains. Analytic optimization with respect to b_1 and b_2 seems out of reach. We will have to use numerical procedures of optimization. In order to get closer to the numerical simulations we will estimate the convergence rate for the discretized equations with general transmission conditions, both in the non-overlapping and the overlapping case and then optimize numerically this quantity in order to get the best parameters for the convergence.

3 Optimized interface conditions

In this section we study the convergence of the Schwarz algorithm with general interface conditions applied to the discrete Euler equations as described in [DLN04] for the classical transmission conditions. This BVP is discretized using a finite volume scheme where the flux at the interface of the finite volume cells is computed using a Roe [Roe81] type solver. Afterwards, we formulate a Schwarz algorithm whose convergence rate is estimated in the Fourier space in a discrete context. Optimizing the convergence rate with respect to the 2 parameters is already a very difficult task at the continuous level in the non-overlapping case, we could not carry on such a process and obtaining analytical results at the discrete level in the overlapping case (which is our case of interest). Therefore, we will get the theoretical optimized parameters at the discrete level by means of a numerical algorithm, by calculating the following

$$\begin{aligned} &\rho(b_1, b_2) = \max_{k \in \mathcal{D}_h} \rho_2^2(k, \Delta x, M_n, M_t, b_1, b_2) \\ &\min_{(b_1,b_2)\in \mathcal{I}_h} \rho(b_1, b_2) \end{aligned} \tag{8}$$

where $\mathcal{D}_h$ is a uniform partition of the interval $[0, \pi/\Delta x]$ and $\mathcal{I}_h \subset \mathcal{I}$ a discretization by means of a uniform grid of a subset of the domain of the admissible values of the parameters. This kind of calculations are done once for all for a given pair (M_n, M_t) before the beginning of the Schwarz iterations. An example of such a result is given in the figure 3 Mach number $M_n = 0.2$. The computed parameters from the relation (8) will be further refered to with a superscript *th*. The theoretical estimates are compared afterwards with the

Table 1. Overlapping Schwarz algorithm

M_n	b_1^{th}	b_2^{th}	b_1^{num}	b_2^{num}
0.1	1.6	-0.8	1.6	-0.9
0.2	1.3	-0.5	1.4	-0.6
0.3	1.25	-0.3	1.25	-0.45
0.4	1.08	-0.15	1.08	-0.28
0.5	1.03	-0.08	1.02	-0.23
0.6	1.0	0.0	1.0	0.0
0.7	1.02	0.06	1.01	0.04
0.8	1.03	0.08	1.02	0.06
0.9	1.06	0.08	1.04	0.06

numerical ones obtained by running the Schwarz algorithm with different pairs of parameters which lie in a an interval such that the algorithm is convergent. We are thus able to estimate the optimal values for b_1 and b_2 from these numerical computations. These values will be referred to by a superscript *num*.

4 Implementation and numerical results

We present here a set of results of numerical experiments that are concerned with the evaluation of the influence of the interface conditions on the convergence of the non-overlapping Schwarz algorithm of the form. The computational domain is given by the rectangle $[0, 1] \times [0, 1]$. The numerical investigation is limited to the resolution of the linear system resulting from the first implicit time step using a Courant number CFL=100. In all these calculations we considered a model problem: a flow normal to the interface (that is when $M_t = 0$). In figures 3 we can see an example of a theoretical and numerical estimation of the reduction factor of the error. We illustrate here the level curves which represent the log of the precision after 20 iterations for different

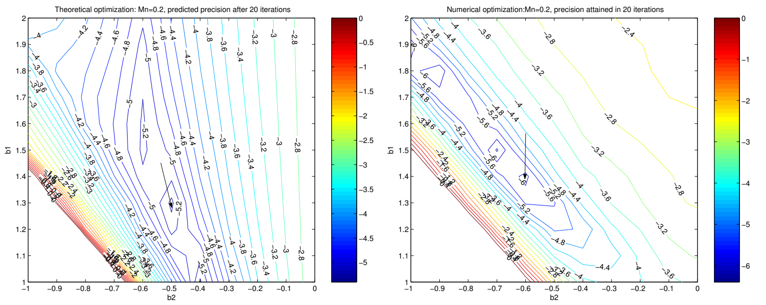

Fig. 1. Isovalues of the predicted (theoretical via formula (8)) and numerical(FV code) reduction factor of the error after 20 iterations

values of the parameters (b_1, b_2), the minimum being attained in this case for $b_1^{th} = 1.3$ and $b_2^{th} = -0.5$, $b_1^{num} = 1.4$ and $b_2^{num} = -0.6$. We can see that we have good theoretical estimates of these parameters we can therefore use them in the interface conditions of the Schwarz algorithm. Table 2 summarizes the

Table 2. Overlapping Schwarz algorithm

Classical vs. optimized counts for different values of M_n

M_n	IT_0^{num}	IT_{op}^{num}	M_n	IT_0^{num}	IT_{op}^{num}
0.1	48	19	0.5	22	18
0.2	41	20	0.7	20	16
0.3	32	20	0.8	22	15
0.4	26	19	0.9	18	12

number of Schwarz iterations required to reduce the initial linear residual by a factor 10^{-6} for different values of the reference Mach number with the optimal parameters b_1^{num} and b_1^{num}. Here we denoted by IT_0^{num} and IT_{op}^{num} the observed (numerical) iteration number for classical and optimized interface conditions in order to achieve a convergence with a threshlod $\varepsilon = 10^{-6}$. The same results are presented in second picture of figure 2. In the first picture of figure 2 we compare the theoretical estimated iteration number in the classical and optimized case. Comparing the 2 pictures of figure 2 we can see that the theoretical prediction are very close to the numerical tests. The conclusion of these numerical tests is, on one hand, that the theoretical prediction is very close to the numerical results: we can get by a numerical optimization (8) a very good estimate of optimal parameters (b_1, b_2)). On the other hand, the gain, in number of iterations, provided by the optimized interface conditions, is very promising for low Mach numbers, where the classical algorithm doesn't

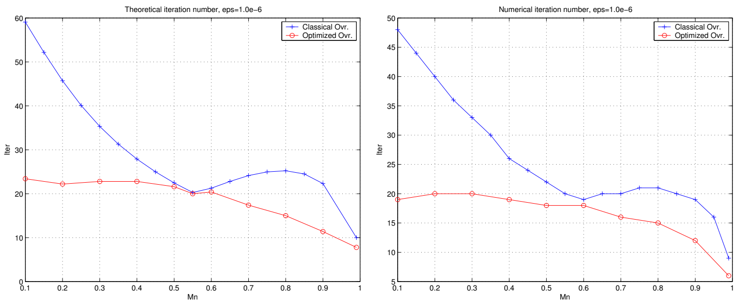

Fig. 2. Theoretical and numerical iteration number: classical vs. optimized conditions

give optimal results. For bigger Mach numbers, for instance, those who are close to 1, the classical algorithm already has a very good behaviour so the optimization is less useful. In the same time we studied here the zero order and therefore very simple transmission conditions. The use of higher order conditions is a possible way that can be further studied to obtain even better convergence results.

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