The approximate integration in the Mortar method constraint

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Summary. The paper analyzes the approximation of the weak continuity constraint in the Mortar method, and provides error estimates in the L^2 and H^1 norms for generic discretization spaces, treating the presence of cross-points in the geometrical decomposition.

1 Introduction

The Mortar element method is a non overlapping non conforming domain decomposition technique for solving PDEs that weakens the continuity constraint of the solution by allowing jumps across the interfaces of the subdomains. Recently it has become of great interest especially for its flexibility in allowing the coupling of different physical models, the use of different discretization schemes and non matching grids at the interfaces of the decomposition. An important aspect of such a technique is related to the implementation of the weak constraint across the interfaces. It is in fact well known the the exact computation of the integrals appearing in the jump condition can give rise to non trivial problems when discrete functions defined on non matching grids are involved or when totally heterogeneous discretization spaces are used (as in the case of the wavelet/finite element coupling [2]).
A possible remedy is the use of quadrature formulas to evaluate such integrals. However it has been shown in [4] that if quadrature formulas based on the master or on the slave side of the interface are used, the results are not optimal for the best approximation error and the consistency error respectively. In [6] the authors propose to overcome the above problem by adopting a Petrov-Galerkin approach, namely by choosing a test space in which the quadrature formula is different from the one considered in trial space, and show numerical optimal results. On the other hand, the idea introduced in [2] consists of replacing the classical jump constraint by an approximated one where the trace on the *master edge* is replaced by its projection on a suitable defined auxiliary space. Even if this last approach can be more expensive (the

computation of the auxiliary projection requires the solution of a linear system), it allows to derive a rigorous analysis of the error and turns out to be applicable in a more general framework than the finite element method. Moreover, in [1] the authors show that the new technique provides an approach to the coding of non-conforming domain decompositions which allows to create a flexible, easily extendible and usable code. In particular, it is important to point out that by following the new method the introduction of a new type of discretization in an existing program does not require any modification to the pieces of the code that provide the tools of the discretization spaces already implemented: the programmer should implement methods to integrate trace functions with functions belonging to the auxiliary space, rather than with the functions belonging to all of the types of the discretizations already in the code unlike in the classical Mortar approach, where the realization of the jump condition, whatever the exact computation of the integrals or the use of quadrature formulas one decides to use, requires the programmer to be somehow familiar with all the libraries implementing the discretizations already in the code, to enter and modify the existing methods with the risk of breaking the existing portions of the code.

We present here the analysis of the Mortar method with the introduction of the approximate constraint in a general context, when generic approximation spaces are involved in each subdomain of the decomposition of $\Omega \subset \mathbb{R}^2$, providing H^1 and L^2-norm error estimates and we show some numerical results comparing the new technique with the classical mortar approach. The extension of such results to the three dimensional case is a work in progress.

2 The Mortar method with approximate constraint

We introduce the Mortar method through a very simple model problem, namely the Poisson equation, referring to [5] for more details and for proofs of the main results that we will recall throughout the section.

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain, and consider the following elliptic problem: given $f \in L^2(\Omega)$, find $u : \Omega \longrightarrow \mathbb{R}$ such that

$$-\Delta u = f, \text{ in } \Omega \qquad u = 0, \text{ on } \partial\Omega. \tag{1}$$

Let $\overline{\Omega} = \bigcup_{\ell=1}^{L} \overline{\Omega}_\ell$, be a fixed decomposition of Ω as the disjoint union of L polygonal subdomains Ω_ℓ and set $\Gamma_{\ell,\ell'} = \partial\Omega_\ell \cap \partial\Omega_{\ell'}$, and $S = \cup\Gamma_{\ell,\ell'}$. We denote by $\gamma_\ell^{(i)}$ the i-th side of the ℓ-th domain, so that we can write $\partial\Omega_\ell = \bigcup_i \gamma_\ell^{(i)}$. We do not fix a priori any restriction on the number of the sides of each polygon, and we assume that the decomposition is *geometrically conforming*, that is each edge $\gamma_\ell^{(i)}$ coincides with $\Gamma_{\ell,n} (= \partial\Omega_\ell \cap \partial\Omega_n)$ for some n, $1 \leq n \leq L$.

For each ℓ, let $\mathcal{V}_h^\ell$ be a family of finite dimensional subspaces of $H^1(\Omega_\ell) \cap C^0(\bar{\Omega}_\ell)$, depending on a parameter $h = h_\ell > 0$ and satisfying homogeneous boundary conditions on $\partial\Omega \cap \partial\Omega_\ell$, and denote by

$$X_h = \prod_{\ell=1}^{L} \mathcal{V}_h^\ell.$$

According to the Mortar approach, in order to impose weak continuity to the solution across the interfaces of the decomposition, we start by choosing the so called *non mortars* (or *slave*) *sides* $\gamma_n^{(k)}$. More precisely, since each edge of the conforming decomposition coincides with the intersection of two adjacent subdomains, it is possible to write that $\gamma_n^{(k)} \equiv \gamma_\ell^{(i)} \equiv \Gamma_{\ell n}$ for some indices ℓ and i. Then we choose one side (say $\gamma_\ell^{(i)}$) as master side and the other as slave side of the common edge $\Gamma_{\ell n}$, intersection of the two adjacent master subdomain $\overline{\Omega}_\ell$ and slave subdomain $\overline{\Omega}_n$ respectively. Moreover, in order to simplify the notations, we use the compact index $m = (n, k)$ to signify that the integer m is related to the slave side of the interface. Therefore we can rewrite the decomposition of the skeleton as follows:

$$S = \bigcup_m \overline{\gamma}_m \qquad \text{with} \qquad \gamma_m \cap \gamma_{m'} = \emptyset.$$

For $v \in \prod_\ell H^1(\Omega_\ell)$, let denote by v^+ and v^- the two $L^2(S)$ functions whose restriction to each edge of the skeleton coincides with the trace on that edge corresponding to the master and to the slave subdomain respectively:

$$v^+_{|\gamma_m} = v^\ell_{|\gamma_m} \quad \text{and} \quad v^-_{|\gamma_m} = v^n_{|\gamma_m}.$$

On each slave side γ_m we define a multiplier space $M_h^m \subset L^2(\gamma_m)$ and we introduce the following weak continuity constraint which appears in the classical Mortar approach:

$$\int_S (v^+ - v^-)\lambda \, ds = 0, \qquad \forall \lambda \in M_h \sim \prod_{m \in I} M_h^m. \tag{2}$$

As already pointed out in the introduction, an important aspect of the Mortar technique is related to the implementation of the weak constraint (2) across the interfaces. The problem arises when, within the jump condition, one has to compute the integrals $\int_{\gamma_m} v^+_{|\gamma_m} \lambda_m$ for each interface when $v^+_{|\gamma_m}$ and λ_m belong to different types of discretization. It is in fact well known that the exact computation becomes extremely technical when the intersections of the supports of unrelated triangular meshes have to be computed and when totally heterogeneous functions are involved (it can happen that the integral of the product of unrelated functions can not be exactly computed, as in the coupling of wavelets and finite elements). The idea proposed in [2] consists in replacing

the classical jump constraint by an approximate one where the trace on the master edge v^+ is substituted by its projection on a suitable chosen auxiliary space. Precisely, on each slave side γ_m let us introduce an auxiliary space $U_{\delta,m} \subset L^2(\gamma_m)$ depending on a parameter $\delta = \delta_m$. For all $\zeta \in L^2(\gamma_m)$, let $P_m(\zeta) \in U_{\delta,m}$ be the unique element of $U_{\delta,m}$ such that

$$\int_{\gamma_m} P_m(\zeta)\eta \, ds = \int_{\gamma_m} \zeta\eta \, ds, \qquad \forall \eta \in U_{\delta,m}, \tag{3}$$

and let us define the projection operator $P : L^2(S) \longrightarrow U_\delta = \prod_{m\in I} U_{\delta,m}$ as follows: for $\zeta \in \mathrm{L}^2(S)$

$$P(\zeta) = \Big(P_m(\zeta_m)\Big)_m, \qquad \text{with} \qquad \zeta_m = \zeta_{|\gamma_m}.$$

Remark that the auxiliary space will have to be chosen in such a way that the integrals of the form $\int_{\gamma_m} \zeta\eta \, ds$ are (easily) computable provided ζ is any trace function on the slave side and $\eta \in U_{\delta,m}$. Therefore, introducing the following approximate integration

$$\int_S (P(v^+) - v^-)\lambda \, ds = 0, \qquad \forall \lambda \in M_h, \tag{4}$$

and the associated discrete constrained space $\mathcal{X}_h^* = \{v_h \in X_h, \quad \text{s.t. (4) holds}\}$, we consider the following problem:

Problem 1. Find $u_h \in \mathcal{X}_h^*$, such that for all $v_h \in \mathcal{X}_h^*$,

$$\sum_{\ell=1}^{L} \int_{\Omega_\ell} \nabla u_\delta \nabla v_\delta = \int_\Omega f v_h.$$

Remark again that Problem 1 is derived from the classical method by simply replacing the jump condition (2) with (4). Moreover, even if this last approach requires the solution of a linear system for the computation of the auxiliary projection, thus resulting more expensive with respect to other possible solutions, it allows to derive a rigorous analysis of the error and turns out to be applicable in a more general framework than the finite element method. In particular, in [5] the author show error estimates for Problem 1 for generic choices of discretization spaces. We recall here for completeness the main result, referring to the paper for more details. Introducing the notations

- $\|\cdot\|_{1,*} = \Big(\sum_\ell \|\cdot\|^2_{H^1(\Omega_\ell)}\Big)^{1/2}$ is the broken H^1 norm,
- h_m is the discretization parameter acting as "mesh sizes" on γ_m,
- $\hat{h} = \max_m\{h_m\}, \quad \check{h} = \min_m\{h_m\}, \quad \hat{\delta} = \max_m\{\delta_m\}, \quad \check{\delta} = \min_m\{\delta_m\},$

and denoting by $H = \max\{\hat{h}, \hat{\delta}\}$ and $h = \min\{\check{h}, \check{\delta}\}$, let u_h be the solution of problem (1), and u the true solution of problem (1) verifying $u \in H^s(\Omega)$

for some $s > 1$. Under suitable and fairly standard assumptions on the multiplier space and on the approximation and auxiliary spaces, the following error estimates holds:

$$\|u-u_h\|_{1,*} \lesssim (1+|log_2 h|)\, H^{s-1}\|u\|_{s,*}, \quad \|u-u_h\|_{0,\Omega} \lesssim (1+|log_2 h|)\, H^{s}\|u\|_{s,*}.$$

Remark that in the analysis of the error "natural" norms (namely $H^{1/2}$ on the interfaces) are involved. This give rise to the loss of a logarithmic factor in the estimates (when *c*ross-points/wire basket are present in the decomposition), but allows to apply the analysis in a general framework (even when non mesh-dependent spaces as in [3] are involved).

2.1 Numerical results

We conclude the presentation of the method by showing some numerical applications. In Section 2.2 we consider the coupling of finite element and wavelet schemes and in Section 2.3 we compare the Mortar method with approximate constraint with the classical approach in the finite element framework.

2.2 Wavelet/finite element coupling

We recall that such an approach allows to overcome one of the drawbacks of wavelet type methods, which perform in a very promising way on academic examples, but whose application to real life problems is seriously limited by the issue of geometry (tensor product-like domains). Moreover, in the wavelet/FEM coupling it is not possible to compute exactly the integral of a wavelet type function times a piecewise polynomial defined on an unstructured grid since wavelets are (in general) not known in closed form. Therefore we apply the technique proposed in the paper and we show some examples which refer to the computation of the numerical solution of the Poisson problem (1) when the domain Ω is the reference square $[0,1]^2$ containing holes in different numbers, shapes and positions. Triangular meshes are used to describe the holes profile, so that finite element type discretizations are used in the corresponding subdomains, while wavelet analysis is performed in the presence of tensorial-type meshes (subdomains without holes) (see Figures 1 and 2).

2.3 Coupling finite elements with non-matching grids

In this section we test the influence that the parameter δ (acting as mesh size of the auxiliary space $U_{\delta,m}$ on the interfaces) has on the behavior of the numerical solution. In doing that, we compare the classical Mortar method and the new technique with the approximate constraint. Recall that the two approaches differ in the computation of the integrals $\int_{\gamma_m} v^+_{|\gamma_m} \lambda_m$ that appear in the constraint: precisely such quantities are computed exactly in the first approach while they are replaced by $\int_{\gamma_m} P_m(v^+_{|\gamma_m})\lambda_m$ in the second one.

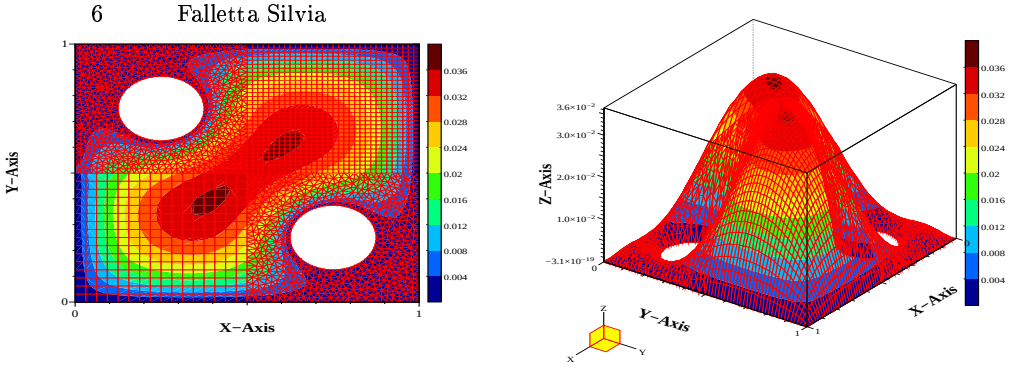

Fig. 1. A 2×2 D.D.: the unit square contains two circular holes in the second and fourth subdomains. Wavelets of level j = 4 in the first subdomain and j = 5 in the third one while finite elements defined on unstructured meshes are used in the other subdomains.

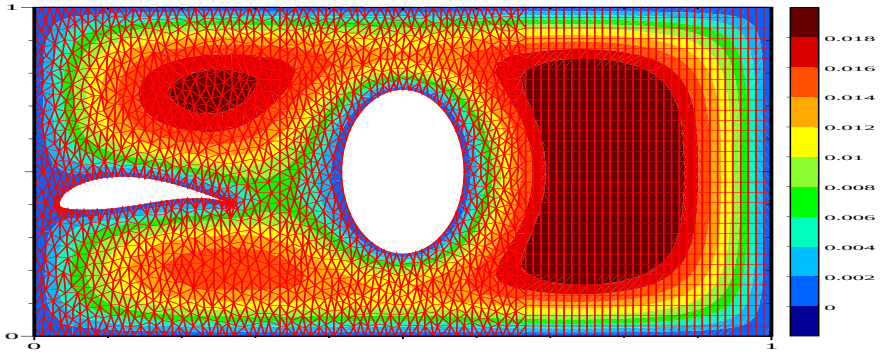

Fig. 2. (a): A 3×1 D.D. The domain contains two holes, the first having a wing profile shape and the second a circular shape. Wavelet discretization space is used in the last subdomain.

To fix the ideas we consider a decomposition of $\Omega = [0,1]^2$ into two rectangles $\Omega_1 = [0,.5] \times [0,1]$ and $\Omega_2 = [.5,1] \times [0,1]$ and finite element approximations in each and we always refer to the model problem (1), where the right hand side f is chosen in such a way that the exact solution (plotted in Figure 3) is given by

$$u(x,y) = x(1-x)y(1-y)cos(50(x-.5)y).$$

In Table 1 we show L^2 and H^1 semi-norm of the error between the approximate and the exact solution when exact integrals are used in the cases of 256/256 and 1024/1024 number of nodes. In Table 2 we show the behavior of the errors for both cases with respect to different values of the parameter δ.

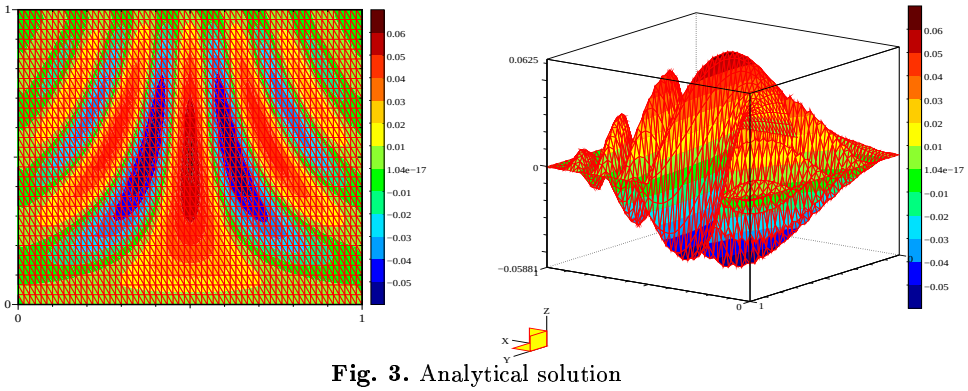

Fig. 3. Analytical solution

Nodes	L^2 norm	H^1 semi-norm
256/256	0.00201031	0.00204565
1024/1024	0.000503404	0.000511633

Table 1. Global error in the L^2 and H^1 norms with respect to the number of nodes with exact computation of integrals.

Nodes: 256 / 256			Nodes: 1024 / 1024		
	L^2 norm	H^1 semi-norm		L^2 norm	H^1 semi-norm
δ	Approx. integral	Approx. integral	δ	Approx. integral	Approx. integral
8	0.00245223	0.0170961	24	0.000503542	0.000550747
10	0.00203049	0.00363632	26	0.000503434	0.00051738
12	0.00201145	0.00212979	28	0.000503409	0.000512209
14	0.00201035	0.00204711	30	0.000503404	0.000511652
16	0.00201031	0.00204565	32	0.000503404	0.000511633
18	0.00201032	0.00204616	34	0.000503404	0.000511644
20	0.00201043	0.00205476	36	0.000503406	0.000511837
30	0.00201031	0.00204575	40	0.000503419	0.000516155

Table 2. Behavior of L^2 and H^1 error with respect to the parameter δ for the approximate integration.

In Table 3 we now compare the error behavior of the two methods (the classical approach and the approximate constraint) for different choices of the meshes and for values of the mesh size of the auxiliary space $\delta = h^\tau$, where h denotes the maximum between the mesh size of the master and slave side of the interface and, roughly speaking, $\tau < 1$ is a suitable chosen parameter which allows to balance the approximation error associated to each subdomain and the contribution that corresponds to the introduction of the auxiliary projection.

	L^2 norm	L^2 norm	H^1 semi-norm	H^1 semi-norm
Nodes	Exact integral	Approx. integral	Exact integral	Approx. integral
100/64	0.00151034	0.00151044	0.123656	0.123657
100/225	0.001484	0.00148373	0.121936	0.121938
256/289	0.000989383	0.000989398	0.0509703	0.0509703
256/361	0.000986929	0.000986939	0.0508623	0.0508626
529/441	0.000580285	0.000580295	0.0254181	0.0254184
729/625	0.000441467	0.000441472	0.0185628	0.018563
729/841	0.000439254	0.000439258	0.0184722	0.0184726

Table 3. Comparison between exact and approximate integration

2.4 Conclusions

We conclude with some remarks for the coupling of finite elements with non-matching grids in the three dimensional case. The idea of replacing the exact computation of the integral appearing in the jump constraint by an approximate one avoids the difficult task of coding the intersections of the supports of discrete functions living on different meshes of the (bi-dimensional) interfaces. A possible remedy can be the choise of $\mathbb{Q}_1$ elements on quadrilateral meshes for the auxiliary space U_δ and the numerical tests performed for the 2D case suggest that the mesh size δ can be chosen coarser than the coarsest of the mesh sizes of the approximation spaces involved in the subdomains. Moreover, the new technique allows to handle the approximation spaces as independent as possible from the implementation point of view, and the introduction of a new discretization in an existing code turns out to be particularly easy and does not require any modification to the methods already implemented, which is an essential feature that a commercial library should satisfy.

References

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