A FETI-DP Formulation for Compressible Elasticity with Mortar Constraints

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Summary. A FETI-DP formulation for three dimensional elasticity problems on non-matching grids is considered. To resolve the nonconformity of the finite elements, a mortar matching condition is imposed on subdomain interfaces. The mortar matching condition are considered as weak continuity constraints in the FETI-DP formulation. A relatively large set of primal constraints, which include average and moment constraints over interfaces (faces) as well as vertex constraints, is further introduced to achieve a scalable FETI-DP method. A condition number bound, $C(1+\log(H/h))^2$, for the FETI-DP formulation with a Neumann-Dirichlet preconditioner is then proved for elasticity problems with discontinuous material parameters when the primal constraints are enforced on only some of the faces instead of all of them. These faces are called primal faces. An algorithm for selecting a quite small number of primal faces is described in [5].

1 A model problem

Let Ω be a polyhedral domain in $\mathbf{R}^3$. The space $H^1(\Omega)$ is the set of functions in $L^2(\Omega)$ which are square integrable up to first weak derivatives and equipped with the standard Sobolev norm: $\|v\|_{1,\Omega}^2 := |v|_{1,\Omega}^2 + \|v\|_{0,\Omega}^2$, where $|v|_{1,\Omega}^2 = \int_\Omega \nabla v \cdot \nabla v \, dx$ and $\|v\|_{0,\Omega} = \int_\Omega v^2 \, dx$. We assume that $\partial\Omega$ is divided into two parts $\partial\Omega_D$ and $\partial\Omega_N$ on which a Dirichlet boundary condition and a natural boundary condition are specified, respectively. The subspace $H_D^1(\Omega) \subset H^1(\Omega)$ is a set of functions having zero trace on $\partial\Omega_D$. For the elasticity problem, we introduce the vector valued Sobolev spaces

$$\mathbf{H}_D^1(\Omega) = \prod_{i=1}^{3} H_D^1(\Omega), \quad \mathbf{H}^1(\Omega) = \prod_{i=1}^{3} H^1(\Omega)$$

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equipped with the product norm.

We consider the following variational form of the compressible elasticity problem: find $\mathbf{u} \in \mathbf{H}^1_D(\Omega)$ such that

$$\int_\Omega G(\mathbf{x})\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v})\, d\mathbf{x} + \int_\Omega G(\mathbf{x})\beta(\mathbf{x})\nabla\cdot\mathbf{u}\,\nabla\cdot\mathbf{v}\, d\mathbf{x} = \langle \mathbf{F}, \mathbf{v}\rangle \quad \forall \mathbf{v} \in \mathbf{H}^1_D(\Omega), \tag{1}$$

where $G = E/(1+\nu)$ and $\beta = \nu/(1-2\nu)$ are material parameters depending on the Young's modulus $E > 0$ and the Poisson ratio $\nu \in (0, 1/2]$ bounded away from $1/2$. The linearized strain tensor is defined by

$$\varepsilon(\mathbf{u})_{ij} := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad i, j = 1, 2, 3,$$

and the tensor product and the force term are given by

$$\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) = \sum_{i,j=1}^{3} \varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}), \quad \langle \mathbf{F}, \mathbf{v}\rangle = \int_\Omega \mathbf{f}\cdot\mathbf{v}\, d\mathbf{x} + \int_{\partial\Omega_N} \mathbf{g}\cdot\mathbf{v} d\sigma.$$

Here $\mathbf{f}$ is the body force and $\mathbf{g}$ is the surface force on the natural boundary part $\partial\Omega_N$.

The space $\mathbf{ker}(\varepsilon)$ has the following six rigid body motions as its basis, which are three translations

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ \mathbf{r}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ \mathbf{r}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{2}$$

and three rotations

$$\mathbf{r}_4 = \frac{1}{H}\begin{pmatrix} x_2 - \widehat{x}_2 \\ -x_1 + \widehat{x}_1 \\ 0 \end{pmatrix},\ \mathbf{r}_5 = \frac{1}{H}\begin{pmatrix} -x_3 + \widehat{x}_3 \\ 0 \\ x_1 - \widehat{x}_1 \end{pmatrix},\ \mathbf{r}_6 = \frac{1}{H}\begin{pmatrix} 0 \\ x_3 - \widehat{x}_3 \\ -x_2 + \widehat{x}_2 \end{pmatrix}. \tag{3}$$

Here $\widehat{\mathbf{x}} = (\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) \in \Omega$ and H is the diameter of Ω. This shift and the scaling make the L_2-norm of the six vectors scale in the same way with H.

2 FETI-DP formulation

2.1 Finite elements and mortar matching condition

We divide the domain Ω into a geometrically conforming partition $\{\Omega_i\}_{i=1}^N$ and we assume that the coefficients $G(\mathbf{x})$ and $\beta(\mathbf{x})$ are positive constants in each subdomain

$$G(\mathbf{x})|_{\Omega_i} = G_i, \quad \beta(\mathbf{x})|_{\Omega_i} = \beta_i.$$

Since we confine our study to the compressible elasticity problem, we can associate the conforming P_1-finite element space $\mathbf{X}_i$ to a quasi-uniform triangulation τ_i of each subdomain Ω_i. In addition, functions in the space $\mathbf{X}_i$ satisfy the Dirichlet boundary condition on $\partial\Omega_i \cap \partial\Omega_D$. The triangulations $\{\tau_i\}_{i=1}^N$ may not match across subdomain interfaces. We associate the finite element space $\mathbf{W}_i$ to the boundary of subdomain Ω_i; it is the trace space of $\mathbf{X}_i$ on $\partial\Omega_i$. Throughout this paper, we will use H_i and h_i to denote the diameter of Ω_i and the typical mesh size of τ_i, respectively.

For each interface (face) $F^{ij} = \partial\Omega_i \cap \partial\Omega_j$, we will choose the one with larger $G(\mathbf{x})$ as the mortar side and the other as the nonmortar side. We then introduce the finite element space on the interface F^{ij}

$$\mathbf{W}_{ij} = \left\{ \mathbf{w} \in \mathbf{H}_0^1(F^{ij}) \;:\; \mathbf{w} = \mathbf{v}|_{F^{ij}} \text{ for } \mathbf{v} \in \mathbf{X}_n(ij) \right\},$$

where $n(ij)$ denotes the nonmortar side. A Lagrange multiplier space $\mathbf{M}_{ij}$, which depends on the space $\mathbf{W}_{ij}$ is given. We refer to [4] for the detailed construction of the dual Lagrange multiplier space and to [1] for the standard Lagrange multiplier space. The mortar matching condition is written as

$$\int_{F_{ij}} (\mathbf{v}_i - \mathbf{v}_j) \cdot \boldsymbol{\lambda} \, ds = 0 \quad \forall \boldsymbol{\lambda} \in \mathbf{M}_{ij}, \; \forall F_{ij}. \tag{4}$$

For each subdomain Ω_i, we define the set m_i containing the subdomain indices j which are mortar sides of interfaces $F \subset \partial\Omega_i$:

$$m_i := \{ j \;:\; \Omega_i \text{ is the nonmortar side of } F (:= \partial\Omega_i \cap \partial\Omega_j) \; \forall F \subset \partial\Omega_i \}.$$

We then introduce the finite element spaces on the interfaces

$$\mathbf{W} = \prod_{i=1}^N \mathbf{W}_i, \quad \mathbf{W}_n = \prod_{i=1}^N \prod_{j \in m_i} \mathbf{W}_{ij}, \quad \mathbf{M} = \prod_{i=1}^N \prod_{j \in m_i} \mathbf{M}_{ij}.$$

2.2 Primal constraints

Selection of primal constraints is important in achieving scalability of FETI-DP algorithms as well as making each subdomain problem invertible. FETI-DP algorithms have been developed for elasticity problems with conforming discretization [3] and numerical results in [2] further show that primal constraints with faces average and vertex constraints provide a scalable algorithm for three dimensional problems. Klawonn and Widlund [7] considered various types of primal constraints for elasticity problems with discontinuous coefficients. Their primal constraints are edge average and edge moment constraints, and vertex constraints. Furthermore they introduced the concepts of an acceptable face path and an acceptable vertex path in an attempt to reduce the number of primal constraints. For the case of mortar constraints, we are able to construct primal constraints based on faces. Thus, in [6], we introduce face average constraints for three dimensional elliptic problems with mortar discretizations and show that the condition number is bounded by a polylogarithmic function of the subdomain problem size independently of the mesh parameters and the coefficients.

We will now select primal constraints on each face for the elasticity problems with mortar discretization. For an interface F^{ij}, we consider the rigid body motions $\{\mathbf{r}_i\}_{i=1}^{6}$ as in (2) and (3), where H is the diameter of the interface F^{ij} and $\widehat{\mathbf{x}}$ is a point in F^{ij}. We define a projection $\mathbf{Q} : \mathbf{H}^{1/2}(F^{ij}) \rightarrow \mathbf{M}_{ij}$ by

$$\int_{F^{ij}} (\mathbf{Q}(\mathbf{w}) - \mathbf{w}) \cdot \boldsymbol{\phi} \, ds = 0 \quad \forall \boldsymbol{\phi} \in \mathbf{W}_{ij}.$$

We then construct the projected rigid body motions $\{\mathbf{Q}(\mathbf{r}_i)\}_{i=1}^{6}$. Since the space $\mathbf{M}_{ij}$ contains the translational rigid body motions, $\mathbf{Q}(\mathbf{r}_i) = \mathbf{r}_i$ for $i = 1, 2, 3$. We now consider the following constraints on the face F^{ij}

$$\int_{F^{ij}} (\mathbf{v}_i - \mathbf{v}_j) \cdot \mathbf{Q}(\mathbf{r}_l) \, ds = 0 \quad \forall l = 1, \cdots, 6.$$

For $\{\mathbf{Q}(\mathbf{r}_l)\}_{l=1}^{3}$, these constraints are nothing but the average matching conditions across the interface (face). The remaining constraints with $\{\mathbf{Q}(\mathbf{r}_l)\}_{l=4}^{6}$ are similar to the moment matching constraints which were introduced for fully primal edges in [8] except that our constraints use the projected rotations and are imposed on faces. We call $\{\mathbf{Q}(\mathbf{r}_l)\}_{l=4}^{6}$ the moment constraints.

To reduce the size of the coarse problem, we select only some faces as primal among all the faces and we impose the primal constraints over only them. For the remaining (non-primal faces), we assume that they satisfy an acceptable face path condition. This assumption makes it possible for the FETI-DP method to have a condition number bound comparable to when all faces are chosen to be primal.

Definition 1. (Acceptable face path) *For a pair of subdomains (Ω_i, Ω_j) having the common face F^{ij} with $G_i \leq G_j$, an acceptable face path is a path $\{\Omega_i, \Omega_{k_1}, \cdots, \Omega_{k_n}, \Omega_j\}$ from Ω_i to Ω_j such that the coefficient G_{k_l} of Ω_{k_l} satisfy the conditions*

$$TOL * (1 + log(H_i/h_i))^{-1} (1 + log(H_{k_l}/h_{k_l}))^2 * G_{k_l} \geq G_i \tag{5}$$

and the path from one subdomain to another is always through a primal face.

Furthermore, we choose some of the vertices as primal vertices at which we impose a point-wise matching condition. We assume that enough primal vertices are taken so as to make each local problem invertible. Based on these primal constraints, we introduce the following subspaces

$$\begin{aligned} \widetilde{\mathbf{W}} : = \{\mathbf{w} \in \mathbf{W} \ : \ & \mathbf{w} \text{ satisfies vertex constraints at the primal vertices} \\ & \text{and the face constraints across the primal faces}\}, \\ \widetilde{\mathbf{W}}_n : = \{\mathbf{w}_n \in \mathbf{W}_n \ : \ & \mathbf{w}_n \text{ satisfies zero average and zero moment} \\ & \text{constraints for each primal faces}\}. \end{aligned}$$

For $\mathbf{w}_n \in \widetilde{\mathbf{W}}_n$, let $E(\mathbf{w}_n) \in \mathbf{W}$ be the zero extension of $\mathbf{w}_n$ to the whole interface, i.e., mortar and nonmortar interfaces. We can easily see that $E(\mathbf{w}_n)$ belongs to $\widetilde{\mathbf{W}}$.

2.3 The FETI-DP equation

Let $A^{(i)}$ denote the stiffness matrix of the bilinear form

$$a_i(\mathbf{u}_i, \mathbf{v}_i) := G_i \int_{\Omega_i} \varepsilon(\mathbf{u}_i) : \varepsilon(\mathbf{v}_i)\, dx + G_i \beta_i \int_{\Omega_i} \nabla \cdot \mathbf{u}_i \nabla \cdot \mathbf{v}_i \, dx,$$

and let $S^{(i)}$ be the Schur complement of the matrix $A^{(i)}$. The matrix $B^{(i)}$ denotes the mortar matching matrix for the unknowns of $\partial\Omega_i$ and the mortar matching condition for $\mathbf{w} = (\mathbf{w}_1, \cdots, \mathbf{w}_N) \in \mathbf{W}$ can then be written as

$$\sum_{i=1}^{N} B^{(i)} \mathbf{w}_i = 0.$$

Let V_c be the set of unknowns at the primal vertices, let $V_c^{(i)}$ be the restriction of V_c on the subdomain Ω_i, and let the mapping $R_c^{(i)} : V_c \rightarrow V_c^{(i)}$ denote a restriction. The matrix $B^{(i)}$ and the vector $\mathbf{w}_i \in \mathbf{W}_i$ are ordered as

$$B^{(i)} = \begin{pmatrix} B_r^{(i)} & B_c^{(i)} \end{pmatrix}, \quad \mathbf{w}_i = \begin{pmatrix} \mathbf{w}_r^{(i)} \\ \mathbf{w}_c^{(i)} \end{pmatrix},$$

where c stands for the unknowns at the primal vertices in $V_c^{(i)}$ and r stands for the remaining unknowns. We then assemble vectors and matrices of each subdomains

$$\mathbf{w}_r = \begin{pmatrix} \mathbf{w}_r^{(1)} \\ \vdots \\ \mathbf{w}_r^{(N)} \end{pmatrix}, \quad B_r = \begin{pmatrix} B_r^{(1)} & \ldots & B_r^{(N)} \end{pmatrix}, \quad B_c = \sum_{i=1}^{N} B_c^{(i)} R_c^{(i)}.$$

Since the primal face constraints are the mortar constraints, we express them by using an appropriate matrix R

$$R^t (B_r \mathbf{w}_r + B_c \mathbf{w}_c) = 0,$$

where $\mathbf{w}_c$ represents the unknowns at the global primal vertices.

By introducing Lagrange multipliers $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ for the primal face constraints and for the mortar matching constraints, respectively, we get the following mixed formulation of (1)

$$\begin{pmatrix} S_{rr} & S_{rc} & B_r^t R & B_r^t \\ S_{cr} & S_{cc} & B_c^t R & B_c^t \\ R^t B_r & R^t B_c & 0 & 0 \\ B_r & B_c & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{w}_r \\ \mathbf{w}_c \\ \boldsymbol{\mu} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{g}_r \\ \mathbf{g}_c \\ 0 \\ 0 \end{pmatrix}.$$

We now eliminate all the unknowns except $\boldsymbol{\lambda}$ and obtain

$$F_{DP}\boldsymbol{\lambda} = \mathbf{d}.$$

This matrix F_{DP} satisfies the well-known relation

$$\langle F_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle = \max_{\mathbf{w} \in \widetilde{\mathbf{W}}} \frac{\langle B\mathbf{w}, \boldsymbol{\lambda} \rangle^2}{\langle S\mathbf{w}, \mathbf{w} \rangle},$$

where

$$S = \operatorname{diag}(S^{(i)}), \quad B = \left(B^{(1)} \, \dots \, B^{(N)} \right).$$

We now introduce the Neumann-Dirichlet preconditioner M^{-1} given by

$$\langle M\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle = \max_{\mathbf{w}_n \in \widetilde{\mathbf{W}}_n} \frac{\langle BE(\mathbf{w}_n), \boldsymbol{\lambda} \rangle^2}{\langle SE(\mathbf{w}_n), E(\mathbf{w}_n) \rangle},$$

where $E(\mathbf{w}_n)$ is the zero extension of $\mathbf{w}_n$ into the space $\mathbf{W}$. From the fact that $E(\mathbf{w}_n)$ belongs to $\widetilde{\mathbf{W}}$ for $\mathbf{w}_n \in \widetilde{\mathbf{W}}_n$, we obtain

$$\langle M\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle = \max_{\mathbf{w}_n \in \widetilde{\mathbf{W}}_n} \frac{\langle BE(\mathbf{w}_n), \boldsymbol{\lambda} \rangle^2}{\langle SE(\mathbf{w}_n), E(\mathbf{w}_n) \rangle} \leq \max_{\mathbf{w} \in \widetilde{\mathbf{W}}} \frac{\langle B\mathbf{w}, \boldsymbol{\lambda} \rangle^2}{\langle S\mathbf{w}, \mathbf{w} \rangle} = \langle F_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle. \quad (6)$$

Therefore the lower bound of the FETI-DP operator is bounded from below by 1.

3 Condition number analysis

In the following, we will provide several lemmas which will be used to obtain the upper bound of the FETI-DP operator. For a face $F \subset \partial\Omega_i$, the space $H_{00}^{1/2}(F)$ consists of the functions whose zero extension onto the whole boundary $\partial\Omega_i$ belongs to the space $H^{1/2}(\partial\Omega_i)$ and it is equipped with the norm

$$\|v\|_{H_{00}^{1/2}(F)} := \left(|v|^2_{H^{1/2}(F)} + \int_F \frac{v(x)^2}{\operatorname{dist}(x, \partial F)} \, ds \right)^{1/2}.$$

We note that we can extend this norm to the product space $\mathbf{H}_{00}^{1/2}(F) := [H_{00}^{1/2}(F)]^3$ by using the usual product norm. We now provide several inequalities for the mortar projection of functions.

Definition 2. *(***Mortar projection***) The mortar projection $\pi_{ij} : \mathbf{L}^2(F^{ij}) \rightarrow \mathbf{W}_{ij}$ is given by*

$$\int_{F^{ij}} (\pi_{ij}(\mathbf{v}) - \mathbf{v}) \cdot \boldsymbol{\psi} \, ds = 0 \quad \forall \boldsymbol{\psi} \in \mathbf{M}_{ij}.$$

Lemma 1. *For $F^{ij}(=\partial\Omega_i \cap \partial\Omega_j)$, a primal face with $G_i \leq G_j$, and for $\mathbf{w} \in \widetilde{\mathbf{W}}$, we have*

$$G_i \|\pi_{ij}(\mathbf{w}_i - \mathbf{w}_j)\|^2_{H^{1/2}_{00}(F^{ij})} \leq C \left\{ \left(1 + log\frac{H_i}{h_i}\right)^2 |\mathbf{w}_i|^2_{S_i} + \frac{G_i}{G_j}\left(1 + log\frac{H_j}{h_j}\right)\left(1 + log\frac{H_j}{h_j} + \frac{h_j}{h_i}\right)|\mathbf{w}_j|^2_{S_j}\right\},$$

where $|\mathbf{w}_l|^2_{S_l} = \langle S_l \mathbf{w}_l, \mathbf{w}_l\rangle$ for $l = i, j$.

Lemma 2. *For a non-primal face $F = \partial\Omega_i \cap \partial\Omega_j$ with $G_i \leq G_j$, assume that there is an acceptable face path $\{\Omega_i, \Omega_{k_1}, \cdots, \Omega_{k_n}, \Omega_j\}$. Then, for $\mathbf{w} \in \widetilde{\mathbf{W}}$, we have*

$$G_i \|\pi_{ij}(\mathbf{w}_i - \mathbf{w}_j)\|^2_{H^{1/2}_{00}(F)} \leq C \left\{ \left(1 + log\frac{H_i}{h_i}\right)^2 |\mathbf{w}_i|^2_{S_i} + L * \sum_{l=1}^{n}\left(1 + log\frac{H_i}{h_i}\right)\frac{G_i}{G_{k_l}}|\mathbf{w}_{k_l}|^2_{S_{k_l}} + \frac{G_i}{G_j}\left(1 + log\frac{H_j}{h_j}\right)\left(1 + log\frac{H_j}{h_j} + \frac{h_j}{h_i}\right)|\mathbf{w}_j|^2_{S_j}\right\},$$

where $\mathbf{w}_i = \mathbf{w}|_{\partial\Omega_i}$, $\mathbf{w}_j = \mathbf{w}|_{\partial\Omega_j}$, and the constant L is the number of subdomains on the acceptable face path.

To bound the term $(G_i/G_j)(h_j/h_i)$ by a constant independently of mesh parameters, we need to impose an assumption on mesh sizes.

Assumption on mesh sizes For the subdomains Ω_i and Ω_j which have a common face F with $G_i \leq G_j$, the mesh sizes h_i and h_j satisfy

$$\frac{h_j}{h_i} \leq C\left(\frac{G_j}{G_i}\right)^{\gamma} \text{ for some } 0 \leq \gamma \leq 1. \tag{7}$$

By combining Lemmas 1 and 2 with the assumption on the mesh sizes and the acceptable face path condition (5), we have the following upper bound for the FETI-DP operator.

Lemma 3. *Assume that the mesh sizes satisfy the assumption (7) and that every non-primal face satisfies the acceptable face path condition with given TOL and L. We then have*

$$\langle F_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle^2 = \max_{\mathbf{w}\in\widetilde{\mathbf{W}}} \frac{\langle B\mathbf{w}, \boldsymbol{\lambda}\rangle^2}{\langle S\mathbf{w}, \mathbf{w}\rangle} \leq C(TOL, L) \max_{i=1,\cdots,N}\left\{\left(1 + log\frac{H_i}{h_i}\right)^2\right\}\langle M\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle,$$

where the constant C depends on the TOL and L but not on the mesh parameters and the coefficients G_i.

The lower bound in (6) and the upper bound from Lemma 3 lead to the following condition number bound.

Theorem 1. *Under the assumptions in Lemma 3, we obtain the condition number bound*

$$\kappa(M^{-1}F_{DP}) \leq C(TOL, L) \max_{i=1,\cdots,N} \left\{ \left(1 + log\frac{H_i}{h_i}\right)^2 \right\}.$$

Here the constant C is independent of the mesh parameters and the coefficients G_i, but depends on TOL and L, the maximum face path length.

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