OBDD: Overlapping Balancing Domain Decomposition Methods and Generalizations to the Helmholtz Equation

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1 Introduction

Balancing Domain Decomposition (BDD) methods belong to the family of preconditioners based on nonoverlapping decomposition of subregions and they have been tested successfully on several challenging large scale applications. Here we extend the BDD algorithms to the case of overlapping subregions and we name them as Overlapping Balancing Domain Decomposition (OBDD) algorithms. Like on the BDD methods, coarse space and weighting matrices play crucial roles in making both the proposed algorithms scalable with respect to the number of subdomains as well as to make the local Neumann subproblems on the overlapping subregions consistent on each iteration of the preconditioned system. These new algorithms also differ from the standard overlapping additive Schwarz method (ASM) of hybrid form since those are based on Dirichlet local problems on the overlapping subregions. This difference motivated us to generalize the OBDD algorithms to the Helmholtz equation where we use the Sommerfeld boundary condition for the local problems and a combination of partition of unity and plane waves for the coarse problem.

1.1 Balancing Domain Decomposition Methods

To have a clear picture of the OBDD algorithms, we first provide a short review of two-level Balancing Domain Decomposition (BDD) methods introduced in [Man93, TW04]. BDD methods are iterative substructuring algorithms, i.e. methods where the interior degrees of freedom of each of the nonoverlapping substructures are eliminated. Hence the discrete problem

$$Ax = f \tag{1}$$

obtained from a finite element discretization method applied to the domain Ω is reduced and posed on the interface $\Gamma = \cup_{i=1}^{N} \Gamma_i$. Here $\Gamma_i = \partial\Omega_i \backslash \partial\Omega_D$ are the local interfaces and $\partial\Omega_D$ the Dirichlet part of $\partial\Omega$. The linear system is then reduced to the form

$$Su = g,$$

where

$$S = \sum_{i=1}^{N} R_i^T S_i R_i,$$

where the matrices S_i are the local Schur complements and R_i are the regular restriction operators from nodal values on Γ to Γ_i. To simplify the exposition, we consider that the matrix A comes from a finite element discretization of the Poisson problem and therefore, the Schur complement matrices S_i are symmetric positive semi-definite (the kernel consists of constant functions) when $\partial\Omega_i \cap \partial\Omega_D = \emptyset$, or positive definite otherwise. To build the BDD preconditioner, weighting matrices D_i on the interface are constructed so that

$$\sum R_i^T R_i D_i = I_\Gamma \tag{2}$$

forms a partition of unity on the interface Γ. The weighting matrices D_i, for the Poisson problem with constant coefficient, can be chosen as the diagonal matrix defined as zero at the nodes on $\Gamma \backslash \Gamma_i$ and the reciprocal of the number of subdomains a node $x \in \Gamma_i$ is associated with. The preconditioner is of the hybrid type given by

$$T_{BDD} = P_0 + (I - P_0)(\sum_{i=1}^{N} T_i)(I - P_0), \tag{3}$$

where the coarse problem P_0 is simply the orthogonal projection (on the S-norm) onto the coarse space V_0. The coarse space V_0 is defined as the span of the basis functions $D_i R_i^T n_i$ where each column vector n_i, up to considerations of what to do when $\partial\Omega_i \cap \partial\Omega_D \neq \emptyset$, is a vector that generates the null space of S_i, i.e. the column vector $[1, 1, 1, 1, \ldots, 1]^T$ on nodes of Γ_i. Hence,

$$P_0 = R_0^T (R_0 S R_0^T)^{-1} R_0 S, \tag{4}$$

where the columns of the matrix R_0^T are formed by all columns of $D_i R_i^T n_i$.

The local operators T_i are defined as

$$T_i = D_i R_i^T S_i^+ R_i D_i S \tag{5}$$

where S_i^+ is the pseudo inverse of the local Schur complement S_i. We remark that each local Neumann problem S_i^+ is solved up to a constant when the $\partial\Omega_i \cap \partial\Omega_D = \emptyset$. The compatibility condition is guaranteed because a coarse problem is solved right before; if y belongs to the range of $(I - P_0)$, and using the definition of P_0 (an orthogonal projection in S-norm), we have

$(D_i R_i^T n_i, Sy)_\Gamma = 0$ (inner product on Γ), and so $(n_i, R_i D_i Sy)_{\Gamma_i} = 0$ (inner product on Γ_i). Hence, $R_i D_i Sy$ is perpendicular to the null space of S_i and so the local problem $S_i x_i = D_i R_i Sy$ satisfies the compatibility condition, and we say that the local problems are balanced.

1.2 Overlapping Balancing Domain Decomposition Methods

We generalize the nonoverlapping BDD method to the overlapping domain case. This is done by maintaining the BDD structure described above. We replace the Schur complement matrix S to the whole matrix A. We replace the restriction operator R_i to Γ_i to a restriction operator R_i^δ to all nodes of the extended subdomain $\overline{\Omega}_i^\delta \backslash \partial\Omega_D$ (including also the boundary nodes on $\partial\Omega_i^\delta \backslash \partial\Omega_D$). We replace the Neumann problem S_i^+ by a Neumann problem $(A_i^\delta)^+$ on Ω_i^δ with Neumann boundary condition on $\partial\Omega_i^\delta \backslash \partial\Omega_D$ and zero Dirichlet boundary condition on $\partial\Omega_i^\delta \cap \partial\Omega_D$. We replace the partition of unity (2) to a partition of unity on $\overline{\Omega} \backslash \partial\Omega_D$

$$\sum_{i=1}^{N} (R_i^\delta)^T R_i^\delta D_i^\delta = I_{\overline{\Omega} \backslash \partial\Omega_D}, \tag{6}$$

where the weighting matrix D_i^δ is a diagonal matrix with diagonal elements given by the regular partition of unity seeing on the theory of Schwarz methods. Similarly, the coarse space V_0^δ also is based on this partition of unity (with some modification near $\partial\Omega_D$ to satisfy Dirichlet boundary conditions). The coarse problem P_0^δ is the orthogonal projection (on the A-norm) onto the space V_0^δ and the OBDD preconditioner is define as

$$T_{OBDD} = P_0^\delta + (I - P_0^\delta)(\sum_{i=1}^{N} T_i^\delta)(I - P_0^\delta), \tag{7}$$

where the local problems are given by

$$T_i^\delta = D_i^\delta (R_i^\delta)^T (A_i^\delta)^+ R_i^\delta D_i^\delta A. \tag{8}$$

The same arguments about BDD compatibilities hold here: if y belongs to the range of $(I - P_0^\delta)$ we have $(D_i^\delta (R_i^\delta)^T n_i^\delta, Ay)_{\Omega \backslash \partial\Omega_D} = 0$, and so $(n_i^\delta, R_i^\delta D_i^\delta Ay)_{\Omega_i^\delta \backslash \partial\Omega_D} = 0$. Hence, $R_i^\delta D_i^\delta Ay$ is perpendicular to the vector n_i^δ (a column vector of ones on nodes of Ω_i^δ when $\Omega_i^\delta \cap \partial\Omega_D =$). The vector n_i^δ spans a space that contains the kernel of A_i^δ, and so the local Neumann problems $A_i^\delta x_i = D_i^\delta R_i^\delta Ay$ satisfy the local compatibility condition.

1.3 Advantages and Disadvantages between BDD and OBDD

We note that differently from BDD methods, the OBDD methods work on the whole finite element function space without eliminating any variables. Hence

we solve $Ax = b$ instead of $Su = g$. As a first consequence, we avoid completely the local Dirichlet solvers required on the BDD methods to compute residuals as well as to build the coarse matrix. This is a considerable advantage for the OBDD methods since these BDD local Dirichlet solvers require exact solvers in each iteration of the preconditioned system, and more dramatically, specially in three dimensional problems, a large number of preprocessing exact local Dirichlet solvers are required to build the coarse matrix. We note also that the coarse matrix of the proposed OBDD methods are the same size as those of BDD methods, i.e. one degree of freedom per subdomain. However, the OBDD coarse matrices are more sparse than those of BDD ones since they do require connectivity among neighbors of the neighbors subdomains. Another advantage for using OBDD methods is that they are less sensitive to the roughness of the boundary of the subdomains (in general boundaries of extended subdomains are smoother than nonoverlapping subdomains).

The proposed OBDD algorithms have also their disadvantages. The first one is the extra cost when working with extended subdomains. Hence for effective performance in terms of CPU time and memory allocation, small overlap is a common practice. The second disadvantage is that the condition number obtained by the OBDD methods are $O(1 + H/(\delta h))$ while the BDD methods are $O(1 + \log(H/h)^2)$. Numerically we show that for the minimum overlap case, the preconditioned systems associated to OBDD present small condition numbers, so the linear bound is comparable to the two log factors for the BDD. For three dimensional problems, the ratio H/h would be smaller and therefore, the linear bound of the OBDD would get closer to the two logs bound of the BDD. The third disadvantage is that the inner products and the vector sums inside the PCG/BDD (GMRES/BDD) are done only on the interfaces nodes while on the PCG/OBDD (GMRES/OBDD) they are done on all the nodes. We note however that in the proposed algorithms, after the first iteration of the OBDD, only on the extended boundary interfaces will have nonzero residuals and will maintain if during PCG iterations RASHO coarse problems [CDS03, Sar02b] are considered (since the RASHO coarse basis functions are built to have zero residual on non interface nodes). Hence a large saving on performancing $A*v$ to compute residuals is possible. The BDD methods nowadays are well developed for several applications such as discontinuous coefficients, two and three dimensional elasticity, plates and shells, and recently also extended to saddle point problems. For two and three dimensional elasticity and for discontinuous coefficients problems, we can apply some of the ideas in [Sar03, Sar02b] to design and analyze OBDD algorithms. Extensions of OBDD algorithms to the saddle point problems are not trivial and are very interesting subjects for future research.

2 The Finite Element Formulation

Consider the Helmholtz problem

$$\begin{aligned} -\Delta u^* - (k(x))^2 u^* &= f \quad \text{in} \quad \Omega \\ u^* &= g_D \quad \text{on} \quad \partial\Omega_D \\ \frac{\partial u^*}{\partial n} &= g_N \quad \text{on} \quad \partial\Omega_N \\ \frac{\partial u^*}{\partial n} + iku^* &= g_S \quad \text{on} \quad \partial\Omega_S \end{aligned} \tag{9}$$

where Ω is a bounded polygonal region in $\Re^2$ with a diameter of size $O(1)$. The $\partial\Omega_D$, $\partial\Omega_N$, and $\partial\Omega_S$ are disjoint parts of $\partial\Omega$ where the Dirichlet, Neumann, and Sommerfeld boundary conditions are imposed. We note that the methods developed here also holds for polyhedral regions in $\Re^3$. From a Green's formula and conjugation of the test functions, we can reduce into the following variational form: find $u^* - u_D^* \in H_D^1(\Omega)$ such that,

$$\begin{aligned} a(u^*, v) &= \int_\Omega (\nabla u^* \cdot \nabla \bar{v} - k^2 u^* \bar{v})\, dx - ik \int_{\partial\Omega_S} u^* \bar{v}\, ds \\ &= \int_\Omega f\bar{v}\, dx + \int_{\partial\Omega_N} g\bar{v}\, ds = F(v),\ \forall v \in H_D^1(\Omega), \end{aligned} \tag{10}$$

where u_D^* is an extension of g_D to $H^1(\Omega)$, and $H_D^1(\Omega)$ is the space of $H^1(\Omega)$ functions which vanishes on $\partial\Omega_D$. To treat the Poisson's problem, we let $k = 0$ and $\partial\Omega_S = \emptyset$.

Let $\mathcal{T}^h(\Omega)$ be a shape regular quasi-uniform triangulation of Ω and let $V \subset H_D^1(\Omega)$ be the finite element space consisting of continuous piecewise linear functions associated with the triangulation which vanish on $\partial\Omega_D$. Eliminating u_D we obtain the following discrete problem: Find $u \in V$ such that

$$a(u, v) = f(v), \quad \forall\ v \in V. \tag{11}$$

Using the standard basis functions, (11) can be rewritten as a linear system of equations of ther form (1).

All the domains and subdomains are assumed to be open; i.e., boundaries are not included in their definitions. The superscript T means the adjoint of an operator.

3 Notations

Given the domain Ω and triangulation $\mathcal{T}^h(\Omega)$, we assume that a domain partition has been applied and resulted in N non-overlapping connected subdomains $\Omega_i, i = 1, \ldots N$ of size $O(H)$, such that

$$\overline{\Omega} = \cup_{i=1}^N \overline{\Omega}_i \quad \text{and} \quad \Omega_i \cap \Omega_j = \emptyset, \quad \text{for} \quad j \neq i.$$

We define the overlapping subdomains Ω_i^δ as follows. Let Ω_i^1 be the one-overlap element extension of Ω_i, where $\Omega_i^1 \supset \Omega_i$ is obtained by including all

the immediate neighboring elements $\tau_h \in \mathcal{T}^h(\Omega)$ of Ω_i such that $\overline{\tau}_h \cap \overline{\Omega}_i \neq \emptyset$. Using the idea recursively, define a δ-extension overlapping subdomains Ω_i^δ

$$\Omega_i = \Omega_i^0 \subset \Omega_i^1 \subset \cdots \subset \Omega_i^\delta.$$

Here the integer $\delta \geq 1$ indicates the level of element extension and δh is the approximate length of the extension. We note that this extension can be coded easily through the knowledge of the adjacent matrix associated to the mesh.

4 Local Problems: Definitons of D_i^δ, R_i^δ and T_i^δ

Consider a partition of unity on $\overline{\Omega}$ with the following usual properties: $\sum_{i=1}^N \theta_i^\delta(x) = 1, \quad 0 \leq \theta_i^\delta(x) \leq 1$, and $|\nabla \theta_i^\delta(x)| \leq C/(\delta h)$, when $x \in \overline{\Omega}$, and $\theta_i^\delta(x)$ vanishes on $\overline{\Omega} \backslash \overline{\Omega}_i^\delta$; for details see [Sar02a, TW04]. The diagonal weighting matrices D_i^δ are defined to have diagonal elements values equal to $\theta_i^\delta(x)$ at nodes $x \in \overline{\Omega}$.

Let us denote by V_i^δ, $i = 1, \cdots, N$, the local space of functions in $H^1(\Omega_i^\delta)$ which are continuous and piecewise linear on the elements of $\mathcal{T}^h(\Omega_i^\delta)$ and which vanish on $\partial\Omega_D \cap \partial\Omega_i^\delta$. We remark that we do not assume that functions in V_i^δ have to vanish on the whole $\partial\Omega_i^\delta$. We then define the corresponding restriction operator R_i^δ

$$R_i^\delta : V \to V_i^\delta, \quad i = 1, \cdots, N.$$

and obtain (6) and the following subspace decomposition

$$D_i^\delta (R_i^\delta)^T V_i^\delta \subset V \quad \text{and} \quad V = \sum_{i=1}^N D_i^\delta (R_i^\delta)^T V_i^\delta.$$

To define the local solvers, we introduce the local bilinear forms on V_i^δ by

$$a_{\Omega_i^\delta}(u_i, v_i) = \int_{\Omega_i^\delta} (\nabla u_i \cdot \nabla \bar{v}_i - k^2 u_i \bar{v}_i)\, dx - ik \int_{\partial\Omega_i^\delta \backslash (\partial\Omega_D \cup \partial\Omega_N)} u_i \bar{v}_i \, ds. \quad (12)$$

For the case $k = 0$, i.e. the Poisson problem, $a_{\Omega_i^\delta}$ reduces to the regular H^1-seminorm inner product. For the case $k \neq 0$, i.e. the Helmholtz case, the bilinear form $a_{\Omega_i^\delta}$ considers Sommerfeld boundary condition on $\partial\Omega_i^\delta \backslash \partial\Omega_{N \cup D}$, Neumann on $\partial\Omega_i \cap \partial\Omega_N$ and Dirichlet on $\partial\Omega_i \cap \partial\Omega_D$; see also [CCEW98]. The associated local problems are defined as $\tilde{T}_i^\delta : V \to V_i^\delta$ by: for any $u \in V$

$$a_{\Omega_i^\delta}(\tilde{T}_i^\delta u, v) = a(u, D_i^\delta (R_i^\delta)^T v), \quad \forall v \in V_i^\delta, \ i = 1, \cdots, N, \quad (13)$$

and let $T_i^\delta = D_i^\delta (R_i^\delta)^T \tilde{T}_i^\delta$ to obtain (8). When $k = 0$ and Ω_i^δ is floating subdomain, the matrix A_i^δ is singular. To obtain the compatibility condition (Poisson problem) or to accelerate the algorithm (Helmholtz problem) we next introduce the coarse problems.

5 Coarse Problems: Definitions of R_0^δ and P_0^δ

We note that some of the functions $\vartheta_i^\delta = I_h\theta_i^\delta$ cannot be used as a coarse basis functions since some of them do not satisfy the zero Dirichlet boundary condition on $\partial\Omega_D$ and therefore, do not belong to V. Hence we modify those just on a neighborhood δh layer near $\partial\Omega_D$. This is done by defining a smooth cut-off function ϕ_δ on a δh layer near $\partial\Omega_D$ and define the coarse basis functions $\vartheta_i^\delta = I_h(\phi_\delta\theta_i^\delta)$. Here I_h is the regular pointwise interpolation operator to V.

For the Poisson's problem, we define the coarse space V_0^δ as the span of the coarse basis functions $\vartheta_i^\delta, i = 1, \cdots, N$.

For the Helmholtz's problem, we combine the ϑ_i^δ with N_p planar waves. The basis functions for the coarse space V_0^δ are given by $I_h(\vartheta_i^\delta Q_j), i = 1, \ldots, N$ and $j = 1, \cdots, N_p$, where $Q_j(x) = e^{ik\Theta_j^T x}$, where $\Theta_j^T = (\cos(\theta_j), \sin(\theta_j))$, with $\theta_j = (j-1) \times \frac{\pi}{N_p}, j = 1, \cdots, N_p$; see also [FML00] for the use of plane waves for FETI-H methods.

We define the restriction matrix $R_0^\delta : V \to V_0^\delta$ consisting of columns ϑ_i^δ (Poisson) or $I_h(\vartheta_i^\delta Q_j)$ (Helmholtz). We define $P_0^\delta : V \to V_0^\delta$ by: for any $u \in V$

$$a(P_0^\delta u, v) = a(u, v), \quad \forall v \in V_0^\delta,$$

and in matrix notation, $P_0^\delta = (R_0^\delta)^T (A_0^\delta)^{-1} R_0^\delta$, where $A_0^\delta = R_0^\delta A (R_0^\delta)^T$.

For the Poisson case, we have [KS]:

Theorem 1.

$$a(u,u) \preceq a(T_{OBDD}u, u) \preceq (1 + \frac{H}{\delta h}) a(u,u).$$

6 Numerical Experiments

Below we present numerical results for solving the Helmholtz's problem on the unit square with the following boundary condition: Dirichlet $g_D = 1$ on west side, homogeneous Neumann on north and south sides, and homogeneous Sommerfeld on east side; see [FML00]. For the Poisson's equation including discussions on the parallel implementations see Kimn and Bourdin [KB].

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Table 1. Number of iterations (PGMREZ) to solve Helmholtz equation with the Guided Wave Problem, Wave coarse space $N_p = 4$, Tol=10^{-6}, $k = 20$

ovlp =1, n =	33	65	129	257
sub = 4x4	18	22	43	82
sub = 8x8	9	11	14	21
sub = 16x16		8	10	13
sub = 32x32			8	10

Table 2. Number of Iterations (PGMREZ) to solve Helmholtz equation with the Guided Wave Problem, Wave coarse space $N_s = 8$, Tol=10^{-6}, $k = 20$

ovlp =1, n =	33	65	129	257
sub = 4x4	14	18	25	48
sub = 8x8	7	7	8	9
sub = 16x16		4	4	4
sub = 32x32			2	2

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