Dual-primal iterative substructuring for almost incompressible elasticity

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1 Introduction

There exists a large number of publications devoted to the construction and analysis of finite element approximations for problems in solid mechanics, in which it is necessary to circumvent volumetric locking. Of special interest are nearly incompressible materials where standard low order finite element discretizations do not ensure uniform convergence in the incompressible limit. Methods associated with the enrichment or enhancement of the strain or stress field by the addition of carefully chosen basis functions have proved to be highly effective and popular. The key work dealing with enhanced assumed strain formulations is that of Simo and Rifai [1990]. Of exclusive interest in our paper are situations corresponding to a pure displacement based formulation which is obtained by a local static condensation of a mixed problem satisfying a uniform inf-sup condition. We work with conforming bilinear approximations for the displacement and a pressure space of piecewise constants. Unfortunately, the standard $Q1-P0$ pairing does not satisfy a uniform inf-sup condition. To obtain a stable scheme, we have to extract from the pressure space the so-called checkerboard modes. Although a lot of work has been done on stable discretization techniques, the construction of uniformly bounded iterative solvers in the incompressible limit is still an open problem. Some results can be found for multigrid solvers applied to a stable saddle point formulation, see, e.g., Wieners [2000] and Schöberl [1999]. Let us note that there are also recent results on FETI-DP and BDDC domain decomposition methods for mixed finite element discretizations of Stokes' equations, see Li and Widlund [2005], and almost incompressible elasticity, see Dohrmann [2004]. In this work, we propose a dual-primal iterative substructuring method for almost incompressible elasticity. Numerical results illustrate the performance and the scalability of our method in the incompressible limit.

2 Almost incompressible elasticity and finite elements

The equations of linear elasticity model the displacement of a homogeneous linear elastic material under the action of external and internal forces. The elastic body occupies a domain $\Omega \subset \mathbb{R}^2$, which is assumed to be polyhedral and of diameter one. We denote its boundary by $\partial\Omega$ and assume that one part of it, $\partial\Omega_D$, is clamped, i.e., with homogeneous Dirichlet boundary conditions, and that the rest, $\partial\Omega_N := \partial\Omega \setminus \partial\Omega_D$, is subject to a surface force $\mathbf{g}$, i.e., a natural boundary condition. We can also introduce a body force $\mathbf{f}$, e.g., gravity. With $\mathbf{H^1}(\Omega) := (H^1(\Omega))^2$, the appropriate space for a variational formulation is the Sobolev space $\mathbf{H^1_0}(\Omega, \partial\Omega_D) := \{\mathbf{v} \in \mathbf{H^1}(\Omega) : \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega_D\}$. The linear elasticity problem consists in finding the displacement $\mathbf{u} \in \mathbf{H^1_0}(\Omega, \partial\Omega_D)$ of the elastic body Ω, such that

$$\int_\Omega 2\mu\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) d\mathbf{x} + \int_\Omega \lambda \operatorname{div}\mathbf{u} \operatorname{div}\mathbf{v}\, d\mathbf{x} = \langle \mathbf{F}, \mathbf{v} \rangle \ \ \forall \mathbf{v} \in \mathbf{H^1_0}(\Omega, \partial\Omega_D). \quad (1)$$

Here μ and λ are the Lamé parameters, which are constant in view of the assumption of a homogeneous body, and which are assumed positive. Of particular interest is the incompressible limit, which corresponds to $\lambda \to \infty$. The Lamé parameters are related to the pair (E, ν), where E is Young's modulus and ν is Poisson's ratio by

$$E = \frac{\mu(2\mu + 3\lambda)}{\mu + \lambda}, \qquad \nu = \frac{\lambda}{2(\mu + \lambda)} \ .$$

Furthermore, $\varepsilon_{ij}(\mathbf{u}) := \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ is the linearized strain tensor, and

$$\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) = \sum_{i,j=1}^{2} \varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}), \quad \langle \mathbf{F}, \mathbf{v} \rangle := \int_\Omega \mathbf{f}^T \mathbf{v}\, d\mathbf{x} \ + \ \int_{\partial\Omega_N} \mathbf{g}^T \mathbf{v}\, d\sigma.$$

Our finite element discretization is based on the conforming space $\mathbf{V}_h$ of continuous piecewise bilinear approximations on quadrilaterals. The quasi-uniform mesh is denoted by $\mathcal{T}_h$, and we assume that it has a macro-element structure, i.e., $\mathcal{T}_h$ is obtained by uniform refinement from a coarser mesh $\mathcal{T}_h^m$. To start with, we consider the abstract pair $(\mathbf{V}_h, M_h)$

$$\begin{array}{lll} 2\mu(\varepsilon(\mathbf{u}_h), \varepsilon(\mathbf{v}_h))_0 + (\operatorname{div}\mathbf{v}_h, p_h)_0 = \langle \mathbf{F}, \mathbf{v}_h \rangle & \forall \mathbf{v}_h \in \mathbf{V}_h \ , \\ (\operatorname{div}\mathbf{u}_h, q_h)_0 \qquad - \frac{1}{\lambda}(p_h, q_h)_0 \quad = 0 & \forall q_h \in M_h \ . \end{array}$$

In terms of static condensation, we can eliminate the pressure and obtain a displacement based formulation

$$\int_\Omega 2\mu\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) d\mathbf{x} + \int_\Omega \lambda\, \Pi_{M_h} \operatorname{div}\mathbf{u}\, \Pi_{M_h} \operatorname{div}\mathbf{v}\, d\mathbf{x} = \langle \mathbf{F}, \mathbf{v} \rangle \ \ \forall \mathbf{v} \in \mathbf{V}_h, \quad (2)$$

where Π_{M_h} denotes the L^2-projection onto M_h. It is well known that the choice $M_h = M_h^u$

$$M_h^u = \{q \in L_0^2(\Omega) \mid \ q|_K \in P_0(K),\ K \in \mathcal{T}_h\},$$

does not yield a uniform inf-sup condition and checkerboard modes in the pressure might be observed, see, e.g., Girault and Raviart [1986]. Thus it is necessary that M_h is a proper subset of M_h^u. There exist different possibilities to overcome this difficulty. One option is to work with macro-elements and to extract from M_h^u the checkerboard mode on each macro-element, as in Girault and Raviart [1986]. The restrictions of functions in M_h^u to a macro-element are spanned by the four functions depicted in Figure 1.

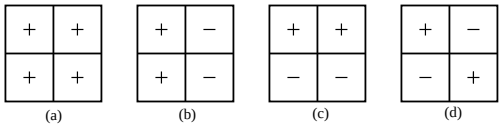

Fig. 1. Restrictions of the basis functions of M_h^u to a macro-element with $\pm$ indicating the sign inside the elements

The functions having the signs indicated in Figure 1 (d) are the local checkerboard modes p^c. To obtain a stable pairing, we have to work with $M_h = M_h^s$

$$M_h^s = \{q \in M_h^u \mid \ (q, p^c)_{0;K} = 0,\ K \in \mathcal{T}_h^m\}.$$

From now on, we call the choice $M_h = M_h^u$ unstable or not stabilized $Q1-P0$ formulation and the choice $M_h = M_h^s$ stabilized $Q1-P0$ formulation. The analysis and the implementation will be based on (2). We note that in both case the L^2-projection Π_{M_h} can be carried out locally.

3 The FETI-DP algorithm

Let the domain Ω be decomposed into nonoverlapping subdomains $\Omega_i, i = 1, \ldots, N$, each of which is the union of finite elements with matching finite element nodes across the interface Γ. The interface Γ is the union of subdomain edges and vertices. For each subdomain Ω_i, we assemble local stiffness matrices $K^{(i)}$ and local load vectors $\mathbf{f}^{(i)}$. By $\mathbf{u}^{(i)}$ we denote the local solution vectors of nodal values.

In the dual-primal FETI methods, we distinguish between dual and primal displacement variables the way the continuity of the solution in those variables is established. Dual displacement variables are those, for which the continuity is enforced by a continuity constraint and Lagrange multipliers $\boldsymbol{\lambda}$ and thus, continuity is not established until convergence of the iterative method is reached, as in the classical one-level FETI methods; see, e.g., Klawonn and Widlund [2001]. On the other hand, continuity of the primal displacement variables is enforced explicitly in each iteration step by subassembly of the

local stiffness matrices $K^{(i)}$ at the primal displacement variables. This subassembly yields a symmetric, positive definite stiffness matrix $\widetilde{K}$ which is not block diagonal anymore but coupled at the primal displacement variables. Let us note that this coupling yields a global problem which is necessary to obtain a numerically scalable algorithm.

We will use subscripts I, Δ, and Π, to denote the interior, dual, and primal displacement variables, respectively, and obtain for the local stiffness matrices, load vectors, and solution vectors of nodal values

$$K^{(i)} = \begin{bmatrix} K_{II}^{(i)} & K_{\Delta I}^{(i)T} & K_{\Pi I}^{(i)T} \\ K_{\Delta I}^{(i)} & K_{\Delta\Delta}^{(i)} & K_{\Pi\Delta}^{(i)T} \\ K_{\Pi I}^{(i)} & K_{\Pi\Delta}^{(i)} & K_{\Pi\Pi}^{(i)} \end{bmatrix}, \mathbf{u}^{(i)} = \begin{bmatrix} \mathbf{u}_I^{(i)} \\ \mathbf{u}_\Delta^{(i)} \\ \mathbf{u}_\Pi^{(i)} \end{bmatrix}, \mathbf{f}^{(i)} = \begin{bmatrix} \mathbf{f}_I^{(i)} \\ \mathbf{f}_\Delta^{(i)} \\ \mathbf{f}_\Pi^{(i)} \end{bmatrix}.$$

We also introduce the notation

$$\mathbf{u}_B = [\mathbf{u}_I\ \mathbf{u}_\Delta]^T, \mathbf{f}_B = [\mathbf{f}_I\ \mathbf{f}_\Delta]^T, \mathbf{u}_B^{(i)} = [\mathbf{u}_I^{(i)}\ \mathbf{u}_\Delta^{(i)}]^T, \text{ and } \mathbf{f}_B^{(i)} = [\mathbf{f}_I^{(i)}\ \mathbf{f}_\Delta^{(i)}]^T.$$

Accordingly, we define

$$K_{BB} = \operatorname{diag}_{i=1}^N(K_{BB}^{(i)}),\ \ K_{BB}^{(i)} = \begin{bmatrix} K_{II}^{(i)} & K_{\Delta I}^{(i)T} \\ K_{\Delta I}^{(i)} & K_{\Delta\Delta}^{(i)} \end{bmatrix},\ \ K_{\Pi B} = [K_{\Pi B}^{(1)} \dots K_{\Pi B}^{(N)}].$$

We note that K_{BB} is a block diagonal matrix. By subassembly in the primal displacement variables, we obtain

$$\widetilde{K} = \begin{bmatrix} K_{BB} & \widetilde{K}_{\Pi B}^T \\ \widetilde{K}_{\Pi B} & \widetilde{K}_{\Pi\Pi} \end{bmatrix},$$

where a tilde indicates the subassembled matrices and where

$$\widetilde{K}_{\Pi B} = [\widetilde{K}_{\Pi B}^{(1)} \cdots \widetilde{K}_{\Pi B}^{(N)}].$$

Introducing local assembly operators $R_\Pi^{(i)}$ which map from the local primal displacement variables $\mathbf{u}_\Pi^{(i)}$ to the global, assembled $\widetilde{\mathbf{u}}_\Pi$, we have

$$\widetilde{K}_{\Pi B}^{(i)} = R_\Pi^{(i)} K_{\Pi B}^{(i)}, \qquad \widetilde{\mathbf{u}}_\Pi = \sum_{i=1}^N R_\Pi^{(i)} \mathbf{u}_\Pi^{(i)}, \qquad \widetilde{K}_{\Pi\Pi} = \sum_{i=1}^N R_\Pi^{(i)} K_{\Pi\Pi}^{(i)} R_\Pi^{(i)T},$$

for $i = 1, \dots, N$. Due to the subassembly of the primal displacement variables, Lagrange multipliers have to be used only for the dual displacement variables $\mathbf{u}_\Delta$ to enforce continuity. We introduce a discrete jump operator B such that the solution $\mathbf{u}_\Delta$, associated with more than one subdomain, coincides when $B\mathbf{u}_B = 0$; the interior variables $\mathbf{u}_I$ remain unchanged and thus the corresponding entries in B remain zero. Since we assume pointwise matching

grids across the interface Γ, the entries of the matrix B are $0, 1$, and -1. However, we will otherwise use all possible constraints and thus work with a fully redundant set of Lagrange multipliers as in [Klawonn and Widlund, 2001, Section 5]. Thus, for an edge node common to four subdomains, we will use six constraints rather than choosing as few as three.

We can now reformulate the finite element discretization of (2) as

$$\begin{bmatrix} K_{BB} & \widetilde{K}_{\Pi B}^T & B^T \\ \widetilde{K}_{\Pi B} & \widetilde{K}_{\Pi\Pi} & O \\ B & O & O \end{bmatrix} \begin{bmatrix} \mathbf{u}_B \\ \widetilde{\mathbf{u}}_\Pi \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_B \\ \widetilde{\mathbf{f}}_\Pi \\ \mathbf{0} \end{bmatrix}. \tag{3}$$

Elimination of the primal variables $\widetilde{\mathbf{u}}_\Pi$ and of the interior and dual displacement variables $\mathbf{u}_B$ leads to a a reduced linear system of the form

$$F\boldsymbol{\lambda} = \mathbf{d},$$

where the matrix F and the right hand side $\mathbf{d}$ are formally obtained by block Gauss elimination. Let us note that the matrix F is never built explicitly but that in every iteration appropriate linear systems are solved; see Farhat et al. [2000], Klawonn and Widlund [2004] or Klawonn and Rheinbach [2005] for further details.

To define the FETI-DP Dirichlet preconditioner M^{-1}, we introduce a scaled jump operator B_D; this is done by scaling the contributions of B associated with the dual displacement variables from individual subdomains. We define $B_D = [B_D^{(1)}, \ldots, B_D^{(N)}]$, where the $B_D^{(i)}$ are defined as follows: each row of $B^{(i)}$ with a nonzero entry corresponds to a Lagrange multiplier connecting the subdomain Ω_i with a neighboring subdomain Ω_j at a point $x \in \partial\Omega_{i,h} \cap \partial\Omega_{j,h}$. We obtain $B_D^{(i)}$ by multiplying each such row of $B^{(i)}$ with $1/|\mathcal{N}_x|$, where $|\mathcal{N}_x|$ denotes the multiplicity of the interface point $x \in \Gamma$. This scaling is called multiplicity scaling and is suitable for homogeneous problems; see Klawonn and Widlund [2004]. Our preconditioner is then given in matrix form by

$$M^{-1} = B_D R_\Gamma^T S R_\Gamma B_D^T = \sum_{i=1}^{N} B_D^{(i)} R_\Gamma^{(i)T} S^{(i)} R_\Gamma^{(i)} B_D^{(i)T}. \tag{4}$$

Here, $R_\Gamma^{(i)}$ are restriction matrices that restrict the degrees of freedom of a subdomain to its interface and $R_\Gamma = \mathrm{diag}_i(R_\Gamma^{(i)})$.

We have to decide how to choose the primal displacement variables. The simplest choice is to choose them as certain selected primal vertices of the subdomains, see Farhat et al. [2001], where this approach was first considered. Following the notation introduced in Klawonn et al. [2002], we will denote the FETI-DP algorithm which uses exclusively selected vertices as primal displacement constraints as Algorithm A. Unfortunately, Algorithm A does not yield uniform bounds in the incompressible limit. To obtain better convergence properties, we have to introduce additional constraints. These constraints are

averages over the edges, which are enforced to have the same values across the interface. This variant has been introduced in Klawonn et al. [2002] for scalar problems and is denoted by Algorithm B.

For our FETI-DP algorithm B, we have the following condition number estimate, cf. Klawonn and Wohlmuth [2005],

Theorem 1. *The condition number for the choice $M_h = M_h^s$ satisfies*

$$\kappa(M^{-1}F) \leq C\,(1 + \log(H/h))^2.$$

Here, $C > 0$ is independent of h, H, and the values of the Poisson ratio ν.

4 Numerical results

We apply Algorithms A and B to (2), where $\Omega = (0,1)^2$ and the Young modulus is defined as $E = 210$. We will present results for different Poisson ratios ν. Algorithm A uses all subdomain vertices as primal constraints and Algorithm B, additionally, uses all edge averages as primal constraints. For the experiments in Table 1, we use a structured grid with 240×240 macro elements ($= 480 \times 480$ elements). In small portions of the boundary in all four corners of the unit square homogeneous Dirichlet boundary conditions were applied (see Figure 2) and the domain was subjected to a volume force directed towards $(1,1)^T$. The domain was decomposed into 64 square subdomains with 7 442 d.o.f. each; this results in an overall problem of 462 722 d.o.f. The stopping criterion is a relative residual reduction of 10^{-10}. The experiments were carried out on two Opteron 248 (2.2 Ghz) 64-bit processors. The differences in computing time between the unstable and the stabilized $Q1 - P0$ element, e.g., for $\nu = 0.4$, are due to the different sparsity patterns of the stiffness matrices. The stabilized $Q1 - P0$ element leads to approximately 50% more nonzero entries in the corresponding stiffness matrix.

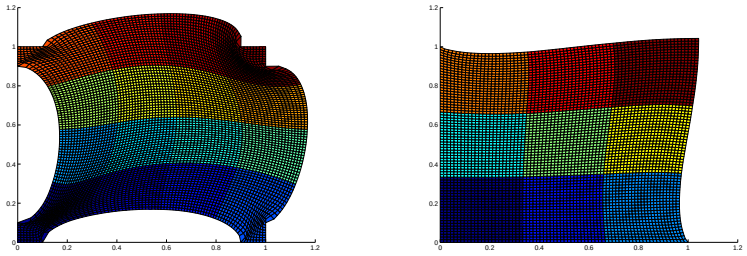

Fig. 2. Deformed configuration for the experiments in Table 1 (left) and for the experiments in Table 2 (right). In both cases a coarser grid than used in the calculations is depicted.

ν	It.	$\lambda_{\max}$	$\lambda_{\min}$	Time	It.	$\lambda_{\max}$	$\lambda_{\min}$	Time
Alg. B		(**stabilized**)				(**not stabilized**)		
0.4	23	**6.98**	1.0075	$55s$	23	**6.98**	1.0075	$47s$
0.49	23	**6.81**	1.0079	$55s$	23	**6.86**	1.0086	$47s$
0.499	24	**6.79**	1.0078	$56s$	23	**6.79**	1.0090	$47s$
0.4999	24	**6.79**	1.0078	$56s$	29	**6.48**	1.0087	$53s$
0.49999	24	**6.79**	1.0080	$56s$	55	**39.98**	1.0088	$80s$
0.499999	25	**6.79**	1.0076	$57s$	97	**366**	1.0086	$124s$
0.4999999	25	**6.79**	1.0078	$57s$	131	**3 632**	1.0096	$159s$
Alg. A		(**stabilized**)				(**not stabilized**)		
0.4	53	**42.52**	1.012	$82s$	53	**42.52**	1.012	$81s$
0.49	103	**316**	1.017	$139s$	67	**85.93**	1.015	$78s$
0.499	192	**3 037**	1.018	$241s$	137	**723**	1.017	$143s$
0.4999	270	$\mathbf{3.02} \times 10^4$	1.020	$332s$	220	**7 069**	1.020	$221s$
0.49999	368	$\mathbf{3.02} \times 10^5$	1.020	$445s$	315	$\mathbf{7.05} \times 10^4$	1.021	$310s$
0.499999	465	$\mathbf{3.02} \times 10^6$	1.022	$558s$	> 500	$\mathbf{7.05} \times 10^5$	1.037	$> 486s$
0.4999999	> 500	$\mathbf{3.02} \times 10^7$	1.032	$> 599s$	> 500	$\mathbf{7.05} \times 10^6$	1.159	$> 484s$

Table 1. Algorithms B and A, 462 722 d.o.f. and 64 subdomains.

Algorithm B			$\nu = \mathbf{0.4999999}$			$\nu = \mathbf{0.4}$		
N	Mesh	d.o.f.	It.	$\lambda_{\max}$	$\lambda_{\min}$	It.	$\lambda_{\max}$	$\lambda_{\min}$
4	48×48	4 802	17	2.51	1.0011	13	2.19	1.0015
9	72×72	10 658	21	3.38	1.0020	19	3.47	1.0024
16	96×96	18 818	24	4.03	1.0023	22	4.13	1.0025
36	144×144	42 050	26	4.53	1.0024	24	4.64	1.0025
64	192×192	74 498	27	4.69	1.0024	25	4.80	1.0026
100	240×240	116 162	29	4.75	1.0022	26	4.86	1.0025
144	288×288	167 042	29	4.78	1.0023	27	4.88	1.0026
256	384×384	296 450	30	4.79	1.0022	30	4.91	1.0024
576	576×576	665 858	32	4.80	1.0021	32	4.77	1.0024
1 024	768×768	1 182 722	32	4.80	1.0021	33	4.81	1.0024

Table 2. Numerical scalability of Algorithm B, $Q_1 - P_0$ (stabilized).

For the experiments in Table 2, the unit square is decomposed into 4 to 1 024 subdomains with 1 250 d.o.f. each. Homogeneous Dirichlet boundary conditions are applied on the bottom and the left side. Again, a volume force directed towards $(1, 1)^T$ is applied. The calculations were carried out on a single Opteron 144 (1.8 Ghz) 64-bit processor. We used as a stopping criterion the relative residual reduction of 10^{-14}.

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