Optimal and optimized domain decomposition methods on the sphere

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1 Introduction

At the heart of numerical weather prediction algorithms lie a Laplace and positive definite Helmholtz problems on the sphere [12]. Recently, there has been interest in using finite elements [2] and domain decomposition methods [1, 10]. The Schwarz iteration [7, 8, 9] and its variants [9, 4, 5, 6, 3, 11] are popular domain decomposition methods.

In this paper, we introduce improved transmission operators for the Laplace problem on the sphere. In section 2, we review the Laplace operator on the sphere and recall the Schwarz iteration and its convergence estimates, previously published in [1]; we also give a new semidiscrete estimate which is substantially similar to the continuous one. In section 3, we introduce the framework of the optimized Schwarz iteration and give optimized operators. In section 4, we present numerical results that agree with the theoretical predictions.

2 The Laplace operator on the sphere

We take the Laplace operator in $\mathbb{R}^3$, given by

$$\mathcal{L}u = u_{xx} + u_{yy} + u_{zz},$$

rephrase it in spherical coordinates and set $\frac{\partial u}{\partial r} = 0$ to obtain

$$\mathcal{L}u = \frac{1}{\sin^2\varphi}\frac{\partial^2 u}{\partial\theta^2} + \frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial u}{\partial\varphi}\right),$$

where $\varphi \in [0, \pi]$ is the colatitude and $\theta \in [-\pi, \pi]$ the longitude.

2.1 The solution of the Laplace problem

We take a Fourier transform in θ but not in φ; this lets us analyze domain decompositions with latitudinal boundaries. The Laplacian becomes

$$\mathcal{L}\hat{u}(\varphi,m) = \frac{-m^2}{\sin^2\varphi}\hat{u}(\varphi,m) + \frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial\hat{u}(\varphi,m)}{\partial\varphi}\right),\ \varphi\in[0,\pi],\ m\in\mathbb{Z}.$$

For boundary conditions, the periodicity in θ is taken care of by the Fourier decomposition. The poles impose that $u(0,\theta)$ and $u(\pi,\theta)$ do not vary in θ. For $m \neq 0$ this is equivalent to

$$\hat{u}(0,m) = \hat{u}(\pi,m) = 0,\ m\in\mathbb{Z},\ m\neq 0.$$

For $m = 0$, the relation $u_\varphi(0,\theta) = -u_\varphi(0,\theta+\pi)$ leads to $\int_0^{2\pi} u_\varphi(0,\theta)\,d\theta = -\int_0^{2\pi} u_\varphi(0,\theta)\,d\theta$, i.e.,

$$\hat{u}_\varphi(0,0) = \hat{u}_\varphi(\pi,0) = 0.$$

If u is a solution of $\mathcal{L}u = f$ then so is $u+c$ ($c\in\mathbb{C}$), hence the ODE for $m=0$ is determined up to an additive constant.

With $m\neq 0$ fixed, the two independent solutions of $\mathcal{L}u = 0$ are

$$g_\pm(\varphi,m) = \left(\frac{\sin(\varphi)}{\cos(\varphi)+1}\right)^{\pm|m|},\ m\in\mathbb{Z}\setminus\{0\}.$$

For $m = 0$ the two independent solutions are

$$\hat{u}(\varphi,0) = C_1 + C_2\log\frac{1-\cos\varphi}{\sin\varphi}.$$

The solutions are defined on the domain $(0,\pi)$.

All the eigenvalues of $\mathcal{L}$ are of the form of $-n(n+1)$ for $n = 0,1,...$; in particular, they are non-positive (and $\mathcal{L}$ is negative semi-definite.)

2.2 The Schwarz iteration for $\mathcal{L}$ with two latitudinal subdomains

Let $b < a$. Begin with random "candidate solutions" u_0 and v_0. Define u_{k+1} and v_{k+1} iteratively by:

$$\begin{cases} \mathcal{L}u_{k+1} = f & \text{in } \Omega_1 = \{(\varphi,\theta)|0\le\varphi<a\}, \\ u_{k+1}(a,\theta) = v_k(a,\theta) & \theta\in[0,2\pi), \\ \mathcal{L}v_{k+1} = f & \text{in } \Omega_2 = \{(\varphi,\theta)|b<\varphi\le\pi\}, \\ v_{k+1}(b,\theta) = u_k(b,\theta) & \theta\in[0,2\pi); \end{cases} \tag{1}$$

(see figure 1.)

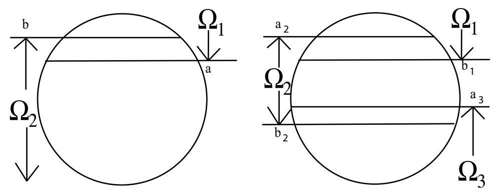

Fig. 1. Latitudinal domain decomposition. Left: two domains; right: multiple domains.

We are interested in studying the error terms $u_0 - u$ and $v_0 - u$ where $\mathcal{L}u = f$ since they solve equations (1) with $f = 0$. Hence for the remainder of this discussion, we will take $f = 0$.

Using the Fourier transform in θ, we can write $\hat{u}_{k+2}(b, m)$ explicitly in terms of $\hat{u}_k(b, m)$. This allows us to obtain a convergence rate estimate, which we recall from [1].

Theorem 1. *The Schwarz iteration on the sphere partitioned along two latitudes $b < a$ converges (except for the constant term.) The rate of convergence $|\hat{u}_{k+2}(b, m)/\hat{u}_k(b, m)|$ is*

$$C(m) = \left(\frac{\sin(b)}{\cos(b)+1}\right)^{2|m|} \left(\frac{\sin(a)}{\cos(a)+1}\right)^{-2|m|} < 1. \tag{2}$$

This convergence rate depends on the frequency m of u_k on the latitude b.

An analysis that is closer to the numerical algorithm would be to replace the continuous Fourier transform in θ by a discrete one.

Theorem 2. *(Semidiscrete analysis.) The Laplacian discretized in θ with n sample points:*

$$\mathcal{L}_n u = \frac{n^2}{4\pi^2 \sin^2 \varphi}\left(u\left(\varphi, \frac{j+1}{2\pi n}\right) - 2u(\varphi, j) + u\left(\varphi, \frac{j-1}{2\pi n}\right)\right) + \cot\varphi u_\varphi + u_{\varphi\varphi} \tag{3}$$

leads to a Schwarz iteration that converges with speed

$$\left(\frac{\sin(b)}{\cos(b)+1}\right)^{2|\tilde{m}|} \left(\frac{\sin(a)}{\cos(a)+1}\right)^{-2|\tilde{m}|} < 1$$

every two iterations, where

$$\tilde{m}^2 = \frac{n^2}{4\pi^2}(1 - \cos(2\pi k/n))$$

for the kth frequency.

The two contraction constants are very similar, and it is only possible to tell them apart on a logarithmic chart for the high frequencies (which converge quickly regardless.) For small values of m (ignoring $m = 0$ because that mode need not converge at all), the speed of convergence is very poor. The overall L^2 convergence rate (along the boundary) is given by $\sup_{m\geq 1} C(m) = C(1)$, and so the convergence rate of the Schwarz iteration deteriorates rapidly as $a - b$ vanishes.

While it is possible to prove that the Schwarz iteration converges regardless of subdomain shapes (so long as they are sufficiently "nice") and even regardless of the discretization (as long as it is sufficiently accurate) in the context of Sobolev spaces [7], it is difficult in general to obtain contraction constants as we have done here.

3 An optimized Schwarz iteration for $\mathcal{L}$ with latitudinal boundaries

We modify the transmission condition to obtain the following iteration:

$$\begin{cases} \mathcal{L}u_{k+1} = f & \text{in } \Omega_1 \\ \psi(\theta) * u_{k+1}(a,\theta) + \frac{\partial}{\partial\varphi}u_{k+1}(a,\theta) = \psi(\theta) * v_k(a,\theta) + \frac{\partial}{\partial\varphi}v_k(a,\theta) & \theta \in [0,2\pi), \\ \mathcal{L}v_{k+1} = f & \text{in } \Omega_2 \\ \xi(\theta) * v_{k+1}(b,\theta) + \frac{\partial}{\partial\varphi}v_{k+1}(b,\theta) = \xi(\theta) * u_k(b,\theta) + \frac{\partial}{\partial\varphi}u_k(b,\theta) & \theta \in [0,2\pi); \end{cases} \tag{4}$$

where ψ and ξ are distributions and Ω_1, Ω_2 are as previously defined. Choices include:

1. $(\psi * w)(\theta) = cw(\theta)$; that is, ψ is c times the point mass at $\theta = 0$. This results in a Robin transmission condition.
2. $(\psi * w)(\theta) = cw(\theta) + dw''(\theta)$. This results in a second order tangential transmission condition.
3. A nonlocal choice of ψ leading to an iteration that converges in two steps.

We have analyzed each case and obtained the following results.

Theorem 3. *(Nonlocal operator.) If, for each m, $\hat{\psi}(m) = |m|/\sin a$ and $\hat{\xi}(m) = -|m|/\sin b$, $\mathcal{L}u_0 = 0$ in Ω_1 and $\mathcal{L}v_0 = 0$ in Ω_2, then $u_1 = 0$ and $v_1 = 0$.*

Corollary 1. *The iteration (4) is convergent (modulo the constant mode) if ψ and $-\xi$ are convolution operators that are positive definite, regardless of overlap.*

The corollary follows from the calculations in the proof of the preceding theorem. We do not assume that $a \neq b$.

Theorem 4. *(Robin conditions.) Let $\psi * w = cw$ and $2N$ be the number of discretization points along the latitude $\varphi = \pi/2$. As long as $c > 0$, we have a convergent algorithm. The contraction constant is*

$$C_0(N) = \min_c \max_{m\in[1,N]} \kappa_1(m,c) = \min_c \max_{m\in[1,N]} \frac{(c-|m|)^2}{(c+|m|)^2}.$$

The minimum is obtained at $c = \sqrt{N}$*, at which point the maximum contraction constant is*

$$C_0(N) = \frac{(\sqrt{N}-1)^2}{(\sqrt{N}+1)^2}.$$

For the second order tangential operator, a continuous analysis leads to:

Theorem 5. *(Second order tangential transmission condition.) Let*

$$\psi * w = cw + d\frac{\partial^2}{\partial\varphi^2}w, \tag{5}$$

with $c \geq 0$ *and* $d \leq 0$*,* $cd \neq 0$*. The best contraction constant is given by*

$$C_2(N) = \min_{c,d} \max_{m\in[1,N]} \kappa_2(m,c,d) = \min_{c,d} \max_{m\in[1,N]} \frac{(c-dm^2-m)^2}{(c-dm^2+m)^2}.$$

Choosing c, d *to obtain the smallest contraction gives*

$$C_2(N) = \left(\frac{\sqrt{2}(N+1)^2 \left(\frac{N}{(N+1)^2}\right)^{\frac{3}{4}} - 2N}{\sqrt{2}(N+1)^2 \left(\frac{N}{(N+1)^2}\right)^{\frac{3}{4}} + 2N} \right)^2$$

for the parameters

$$c = -Nd = 2\left(\frac{N}{4N^2+8N+4}\right)^{\frac{3}{4}}(N+1). \tag{6}$$

We can use a semidiscrete analysis to obtain a similar result.

Theorem 6. *(Second order tangential transmission operator, semidiscrete.) A semidiscrete analysis leads to slightly different parameters c and d given by*

$$\alpha = \frac{N\pi^4 + 8N^3\pi^2 - N^2(8\pi^2+\pi^4) + N\pi^4}{4\pi^4 - 64\pi^2N^2 + 256N^4},$$

$$c' = \frac{N(8n-\pi^2)}{2\alpha^{\frac{1}{4}}(8N^2-\pi^2)},$$

$$d' = \frac{2\alpha^{\frac{3}{4}}(8N^2-\pi^2)}{N(8N-\pi^2)}.$$

In the presence of overlap, an extra trigonometric term appears that prevents exact analytic solutions. If we neglect such trigonometric terms, the optimization problem becomes to minimize the moduli of

$$\frac{\hat{\psi}(m)\sin a - |m|}{\hat{\psi}(m)\sin a + |m|} \quad \text{and} \quad \frac{\hat{\xi}(m)\sin b + |m|}{\hat{\xi}(m)\sin b - |m|}.$$

If $a = b$, this is a nonoverlapping problem but possibly $a \neq \pi/2$. We adapt the preceding theorems.

Theorem 7. *To minimize the modulus of*

$$\frac{\hat{\psi}(m)\sin a - |m|}{\hat{\psi}(m)\sin a + |m|},$$

we can use $\psi = \sqrt{N}\csc a$ *(Robin case) and* $\psi = c\csc(a) + d\csc(a)\frac{\partial}{\partial\varphi}$ *(with* c, d *given by either of the second order tangential choices.)*

3.1 Multiple latitudinal subdomains

Let $l > 1$ and $u_{l+1}^{(k)}$, for $1 \leq k \leq n$, be the solutions of

$$\begin{cases} \mathcal{L}u_{l+1}^{(k)}(\varphi,\theta) = f & \text{in } \Omega_k \\ u_{l+1}^{(k)}(a_k,\theta) + \psi_k * \frac{\partial}{\partial\varphi}u_{l+1}^{(k)}(a_k,\theta) = u_l^{(k-1)}(a_k,\theta) + \psi_k * u_l^{(k-1)}(a_k,\theta) & \theta \in [0,2\pi) \text{ if } k > 1, \\ u_{l+1}^{(k)}(b_k,\theta) + \xi_k * \frac{\partial}{\partial\varphi}u_{l+1}^{(k)}(b_k,\theta) = u_l^{(k+1)}(b_k,\theta) + \xi_k * u_l^{(k+1)}(b_k,\theta) & \theta \in [0,2\pi) \text{ if } k < n; \end{cases}$$

where $0 = a_1 < a_2 < \ldots < a_n$, $b_1 < b_2 < \ldots < b_n = \pi$, $\Omega_k = \{(\varphi,\theta)|\varphi \in (a_k,b_k)\}$, $a_k < b_k$, $k = 1,\ldots,n$ and $\cup_k[a_k,b_k] = [0,\pi]$ (see figure 1.) Once more using a Fourier transform in θ, one can show that the same optimal operators lead to convergence in n steps. The iteration leads to a matrix whose entries look like $\frac{\hat{\psi}(m)\sin a - |m|}{\hat{\psi}(m)\sin a + |m|}$ and one may heuristically use the same operators as in the two-subdomain case.

4 Numerical results

We have written a semispectral solver for the various transmission operators we have described and the numerical results are summarized in figure 2:

(a) We have computed 18 iterates of the Schwarz iteration and plotted the error at each even iteration to match with the analysis in the text. The transmission operators are Robin, second order tangential with coefficients (c,d) (dash-dot), second order tangential with coefficients (c',d') (dashed) and a discretized optimal operator (solid.) The slopes are the contraction constants. The bump at step 2 is because $\mathcal{L}u_0 \neq 0$.

(b) The decay of the contraction constant as the number of subdomains increases. The x axis is the number of subdomains and the y axis is the contraction constant. The x marks and diamonds are for the Robin and optimal operators, respectively, and the circles and squares are for the choices (c,d) and (c',d') of second order tangential operators. The truncation frequency is $N = 50$ in all cases; there are 101 points along the equator.

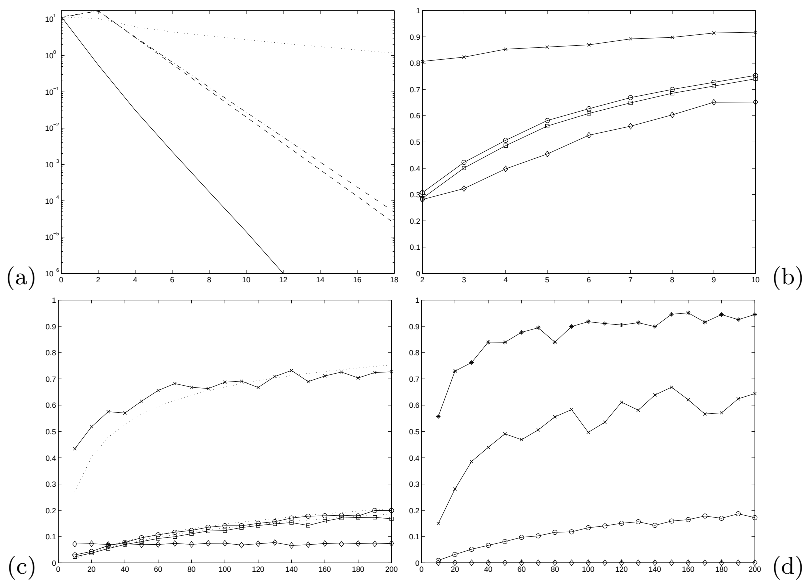

Fig. 2. (a): iterates of the various Schwarz algorithms (two subdomains, no overlap, semispectral code.) (b): contraction constants as a function of the number of subdomains (no overlap.) (c), (d): contraction constants as a function of the truncation frequency (two subdomains.) (c) is without overlap, (d) is one grid interval of overlap.

(c) Depiction of the behavior of the contraction-every-two-steps constant as we increase the discretization parameter N, two subdomains, no overlap. The number of points along the equator is $2N+1$. The line with x marks is a Robin algorithm, the line with circles is with the second order operator and the diamonds is the optimal operator. The two circled lines are for the two choices (c, d) and (c', d') (slightly better) of the second order transmission parameters. Dotted lines are predictions from our analysis. The optimal operator does not lead to convergence in two steps due to the discretization.

(d) Same as (c), but with a single grid length of overlap. Since we have overlap, we include the Dirichlet operator as the * line. The optimal transmission operator behaved vastly better in the overlap case (exhibiting apparently superlinear convergence.)

5 Conclusions

We have given optimal and optimized transmission operators for the Laplace problem on the sphere and have shown that they perform much better than the classical iteration with a Dirichlet condition. We have computed convergence rates for the Robin condition and two choices of second-order tangential operators, and compared them against the optimal nonlocal operator. A similar analysis for the positive definite Helmholtz problem will be detailed in a later paper.

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