Overlapping Schwarz preconditioners for Fekete spectral elements

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Summary. We construct and study overlapping Schwarz preconditioners for the iterative solution of elliptic problems discretized with spectral elements based on Fekete nodes (TSEM). These are a generalization to non-tensorial elements of the classical Gauss-Lobatto-Legendre hexahedral spectral elements (QSEM). Even if the resulting discrete problem is more ill-conditioned than in the classical QSEM case, the resulting preconditioned algorithm using generous overlap is optimal and scalable, since its convergence rate is bounded by a constant independent of the number of elements, subdomains and polynomial degree employed.

1 The model problem and SEM formulation

The recent trend toward highly parallel and high-order numerical solvers has led to increasing interest in domain decomposition preconditioners for spectral element methods; see [11, 17, 4, 5, 8, 6, 7]. While very successful algorithms have been constructed and analyzed for classical Gauss-Lobatto-Legendre hexahedral spectral elements (QSEM), many open problems remain for non-tensorial spectral elements. In this paper, we consider Fekete nodal spectral elements (TSEM) and propose an Overlapping Schwarz preconditioner that using generous overlap turns out to be optimal and scalable.

Let $\Omega \in \mathbb{R}^d, d = 2, 3$, be a bounded Lipschitz domain with piecewise smooth boundary. For simplicity, we consider a model elliptic problem in the plane ($d = 2$) and with homogeneous Dirichlet boundary data, but the techniques presented in this papers apply equally well to more general elliptic problems in three dimensions: Find $u \in V := H_0^1(\Omega)$ such that

$$a(u, v) := \int_\Omega (\alpha\, \mathbf{grad}\, u \cdot \mathbf{grad}\, v + \beta\, u\, v)\, \mathrm{d}\mathbf{x} = \int_\Omega f\, v\, \mathrm{d}\mathbf{x} \qquad \forall v \in V, \quad (1)$$

where α, $\beta > 0$ are piecewise constant in Ω and $f \in L^2(\Omega)$.

The variational problem (1) is discretized by the conforming spectral element method, either quadrangle based (QSEM) or triangle based (TSEM), which is a Galerkin method that employs a discrete space consisting of continuous piecewise polynomials of degree N; see [1, 5, 7] for a general introduction. Let $T_{\rm ref} = \{(r,s) : \ -1 \le r,s \le +1, \ r+s \le 0\}$ be the reference triangle and $\mathcal{P}_N(T_{\rm ref})$ the set of polynomials on $T_{\rm ref}$ of total degree $\le N$. Let $Q_{\rm ref}$ be the reference square $[-1,1]^2$ and $\mathbb{P}_N(Q_{\rm ref})$ the set of polynomials on $Q_{\rm ref}$ of degree $\le N$ in each variable. We assume that Ω is decomposed into K nonoverlapping triangular or quadrilateral finite elements Ω_k, $\overline{\Omega} = \bigcup_{k=1}^{K} \overline{\Omega}_k$, each of which is the image of $T_{\rm ref}$ or $Q_{\rm ref}$ by means of a suitable mapping, i.e., $\Omega_k = g_k(T_{\rm ref})$ or $\Omega_k = g_k(Q_{\rm ref})$. The intersection between two distinct Ω_k is either the empty set or a common vertex or a common side. We denote by H the maximum diameter of the subdomains $\Omega_k's$. The space V is discretized by continuous, piecewise polynomials of total degree $\le N$,

$$V_{K,N}^T = \{v \in V : v|_{\Omega_k} \circ g_k \in \mathcal{P}_N(T_{\rm ref}), \ 1 \le k \le K\},$$

or of degree $\le N$ in each variable,

$$V_{K,N}^Q = \{v \in V : v|_{\Omega_k} \circ g_k \in \mathbb{P}_N(Q_{\rm ref}), \ 1 \le k \le K\}.$$

QSEM and Gauss-Lobatto-Legendre points. We recall here the conforming quadrilateral spectral elements QSEM based on Gauss - Lobatto - Legendre (GLL) quadrature points, which also allows the construction of a very convenient tensor-product basis for $V_{K,N}^Q$. We denote by $\{\xi_i\}_{i=0}^N$ the set of GLL points of $[-1,1]$, and by σ_i the associated quadrature weights. Let $l_i(x)$ be the Lagrange interpolating polynomial of degree $\le N$ which vanishes at all the GLL nodes except ξ_i, where it equals one. The basis functions on the reference square $Q_{\rm ref}$ are defined by a tensor product as $l_i(x)l_j(y), \ 0 \le i,j \le N$. Each function of $\mathbb{P}_N(Q_{\rm ref})$ is expanded in this nodal GLL basis through its values at GLL nodes $u(\xi_i,\xi_j), \ 0 \le i,j \le N$. We replace each integral of (1) by GLL quadrature:

$$(u,v)_{K,N}^Q = \sum_{k=1}^{K} \sum_{i,j=0}^{N} (u \circ g_k)(\xi_i,\xi_j)(v \circ g_k)(\xi_i,\xi_j)|J_k^Q|\sigma_i\sigma_j, \tag{2}$$

where $|J_k^Q|$ is the Jacobian of g_k. This inner product is uniformly equivalent to the standard one on $\mathbb{P}_N(\Omega)$. We then obtain the discrete problem: Find $u \in V_{K,N}^Q$ such that

$$a_{K,N}^Q(u,v) = (f,v)_{K,N}^Q, \quad \forall v \in V_{K,N}^Q, \tag{3}$$

where $a_{K,N}^Q(\cdot,\cdot)$ is obtained from $a(\cdot,\cdot)$ by substituting each integral with the GLL quadrature rule described in (2). The matrix form of (3) is a linear system $A_Q\mathbf{u} = \mathbf{b}$, where A_Q is here the assembled QSEM matrix (positive

definite and symmetric for homogeneous material), $\mathbf{b}$ is the load vector and $\mathbf{u}$ is the vector of nodal values of the unknown function u.

TSEM and Fekete points. On triangular elements it is no longer possible to define spectral elements by tensor product as in QSEM. Let $\{\psi_j\}_{j=1}^n$ be an orthonormal basis of $\mathcal{P}_N(T_{ref})$ for the usual $L^2(T_{ref})$ inner product (for example, the Koornwinder-Dubiner polynomials may be used to constitute such a basis, see [8]). The Fekete points on T_{ref} are defined as the points $\{\hat{\mathbf{x}}_i\}_{i=1}^n$ that maximize the determinant of the Vandermonde matrix V with elements $V_{ij} = \psi_j(\hat{\mathbf{x}}_i)$, $1 \le i, j \le n$, where $n = (N+1)(N+2)/2$. For the TSEM introduced in [15], the Fekete points are used as approximation points and the Lagrange polynomials $\{\phi_i\}_{i=1}^n$ built on these points are used as basis functions. Among the main properties of Fekete points proved in [14], we recall that Fekete points are Gauss-Lobatto points for the cube, thus providing a strong link with the usual QSEM. Unlikely GLL points, a quadrature formula based on Fekete points is only exact for integrands in $\mathcal{P}_N(T_{ref})$. This fact has suggested for the TSEM to separate the sets of approximation and quadrature points, using the Fekete points $\{\hat{\mathbf{x}}_i\}_{i=1}^n$ for the first set and other points $\{\hat{\mathbf{y}}_i\}_{i=1}^m$ for the second set, imposing an exact integration of polynomials, e.g., in $\mathcal{P}_{2N}(T_{ref})$; see [9]. Given the values at the approximation points of a polynomial $u_N \in \mathcal{P}_N(T_{ref})$, one can set up interpolation and differentiation matrices to compute, at the quadrature points, the values of u_N and of its derivatives, respectively. For instance, denoting by $\underline{u}$ the vector of the $u_i = u_N(\hat{\mathbf{x}}_i)$, $1 \le i \le n$, and by $\underline{u}'$ that of the $u_N(\hat{\mathbf{y}}_i)$, $1 \le i \le m$, we have $\underline{u}' = V'V^{-1}\underline{u}$, where $V'_{ij} = \psi_j(\hat{\mathbf{y}}_i)$. On a generic triangle $\Omega_k = g_k(T_{ref})$, the same relation between $\underline{u}'$ and $\underline{u}$ holds true, provided that $u_i = (u_N \circ g_k)(\hat{\mathbf{x}}_i)$ and $u'_j = (u_N \circ g_k)(\hat{\mathbf{y}}_j)$. The TSEM requires of course the use of highly accurate integration rules based on Gauss points. Unfortunately, in practice such integration rules are difficult to define for large values of N (recent publications show that this is still an open subject of research). In the present case, we can use integration rules based on Gauss points for the quadrangle and then map them to T_{ref}; see [7]. On a generic triangle $\Omega_k = g_k(T_{ref})$:

$$(u, v)_{\Omega_k, N} = \sum_{j=1}^{m} u'_j \, v'_j \, |J_k^T(\hat{\mathbf{y}}_j)| \, \omega_j,$$

where $\omega_j > 0$, $1 \le j \le m$, are the quadrature weights and $|J_k^T|$ the Jacobian of the mapping g_k between T_{ref} and Ω_k. As for (3), we obtain a discrete problem

$$a_{K,N}^T(u, v) = (f, v)_{K,N}^T, \quad \forall v \in V_{K,N}^T, \tag{4}$$

that can be written in matrix form as a linear system $A_T \mathbf{u} = \mathbf{b}$. The TSEM matrix A_T is less sparse than the QSEM matrix A_Q and more ill-conditioned, since its condition number grows as $O(N^{2(d-1)})$ (see Sec. 3).

2 Overlapping Schwarz Preconditioners

We now consider the iterative solution of the discrete systems $A\mathbf{u} = \mathbf{f}$ by the conjugate gradient method with an Overlapping Schwarz preconditioner; see e.g. [16, 13, 12] for a general introduction.

Let τ_0 be the coarse finite element triangulation of the domain Ω determined by the elements Ω_k, $k = 1, ..., K$, of characteristic diameter H. Let τ_N be the fine triangulation determined by either the Fekete or the GLL nodes introduced in each element Ω_k in Sections 2.1 and 2.2. Thus we can define two different coarse and fine triangulations and related overlapping partitions of Ω, according to the spectral element method at issue.

QSEM. The coarse triangulation τ_0 is given by quadrangles Ω_k providing a coarse problem with bilinear finite element ($N = 1$ in each direction). Then the local fine discretization τ_N is determined by GLL nodes in each quadrangle Ω_k. We define the overlapping partition of Ω by extending each subdomain Ω_k to a larger subdomain Ω'_k, consisting of all elements of τ_N within a certain distance from Ω_k; we measure this distance by the number δ of GLL points extending Ω_k in each direction. See Figure 1 (left) for a two-dimensional example.

TSEM. The coarse triangulation τ_0 is given by triangles Ω_k providing a coarse problem with linear finite element ($N = 1$). Then the local fine discretization τ_N is determined by Fekete nodes within each Ω_k. The overlapping partition of Ω is generated by extending each triangle Ω_k to a large subdomain Ω'_k consisting of all triangles sharing with Ω_k either a vertex or an edge. See Figure 1 (right) for a two-dimensional example. Overlapping techniques involving a smaller number of subdomains (e.g., sharing edges of Ω_k only) proved unsuccessful, whereas less generous overlapping partitions considering few nodes around Ω_k can not be designed straightforwardly since the internal Fekete nodes are not distributed regularly as in tensorial elements.

The overlapping Schwarz preconditioner B^{-1} for A is based on solving a) a coarse problem with linear or bilinear elements on the coarse mesh τ_0; b) local problems on the overlapping subdomains Ω'_k.

For the coarse solve, we need to define:

a1) a restriction matrix R_0; its transpose R_0^T interpolates coarse linear (resp. bilinear) functions on τ_0 to spectral elements functions on the fine Fekete (resp. GLL) mesh τ_N;

a2) a coarse stiffness matrix $A_0 = R_0 A R_0^T$ needed for the solution of the coarse problem with $N = 1$ on τ_0.

For the local solves, we need to define:

b1) restriction matrices R_k (with 0,1 entries) returning only the degrees of freedom inside each subdomain Ω'_k;

b2) local stiffness matrices $A_k = R_k A R_k^T$ needed for the solution of the kth local problem on Ω'_k with zero Dirichlet boundary conditions on $\partial\Omega'_k$.

These are the building blocks of the proposed preconditioners. The additive form of the overlapping Schwarz preconditioner is

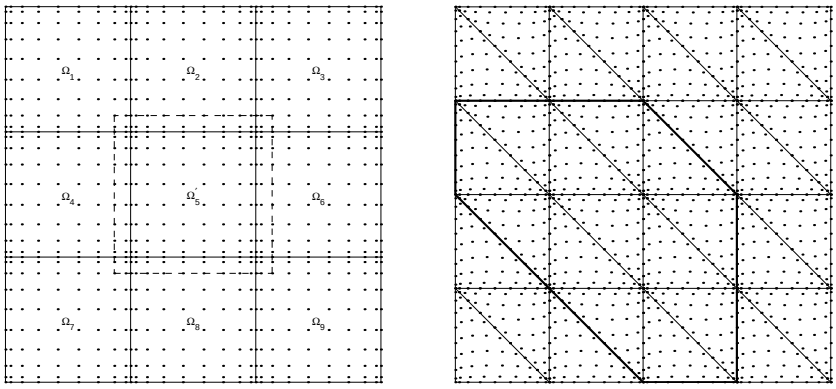

Fig. 1. Example of Ω'_k subdomains for QSEM with small overlap ($\delta = 2$, left) and TSEM with generous overlap (right).

$$B_{add}^{-1} = R_0^T A_0^{-1} R_0 + \sum_{k=1}^{K} R_k^T A_k^{-1} R_k, \tag{5}$$

Multiplicative and hybrid variants can be considered too, see [13, 16].

These preconditioners are associated with the space decomposition $V_{K,N} = V_0 + \sum_{k=1}^{K} V_k$, where either $V_{K,N} \equiv V_{K,N}^T$ or $V_{K,N} \equiv V_{K,N}^Q$. V_0 is the subspace of $V_{K,N}$ consisting of piecewise linear or bilinear functions on the coarse mesh τ_0 and

$$V_k = \{v \in V_{K,N}^T : v = 0 \text{ at all the Fekete nodes outside } \Omega'_k \text{ and on } \partial\Omega'_k\}$$

in the case of triangles, and

$$V_k = \{v \in V_{K,N}^Q : v = 0 \text{ at all the GLL nodes outside } \Omega'_k \text{ and on } \partial\Omega'_k\}$$

in the case of quadrangles. Defining the operators $T_k : V_{K,N} \longrightarrow V_k$ by $a_{K,N}(T_k u, v) = a_{K,N}(u, v)\ \forall v \in V_k,\ 0 \le k \le K$ where $a_{K,N} \equiv a_{K,N}^T$ for TSEM and $a_{K,N} \equiv a_{K,N}^Q$ for QSEM, then (5) is exactly the matrix form of the additive Schwarz operator $T_{add} = T_0 + T_1 + \cdots + T_K$. The theory developed by Casarin [3] for QSEM and scalar symmetric positive definite problems allows to transfer the main domain decomposition results from the finite elements to QSEM (see e.g. Toselli and Widlund [16]).

Theorem 1. *The condition number of the overlapping Schwarz QSEM operator is bounded by*

$$cond(T_a) \le C(1 + \frac{H}{\tilde{\delta}}),$$

with $\tilde{\delta} = \min_k\{\mathrm{dist}(\partial\Omega_k, \partial\Omega'_k)\}$ *and the constant* C *independent of* $N, H, \tilde{\delta}$.

In case of generous overlap $\tilde{\delta} = CH$, we have a constant upper bound for both $cond(T_a)$ and the number of iterations; this was already proved in Pavarino [10] for more general hp finite elements. For unstructured hp elements on nontensorial elements, both analyzes in [3] and [10] are no longer valid and preconditioners with small overlap are not known; the only theory available is for nonoverlapping methods in I. Bica Ph.D. Thesis [2]. Nevertheless, we can build preconditioners with generous overlap as shown before and the numerical results of the next section show that they are optimal and scalable, hence we conjecture that a bound as in Theorem 1 also holds for TSEM.

3 Numerical results

In this section, we report the results of numerical experiments for the overlapping Schwarz preconditioner applied to the model problem (1) discretized with triangular spectral elements using Fekete nodes. We consider an homogeneous material with $\alpha = \beta = 1$. The computational domain is $\Omega = [-1, 1]^2$ and the body force f is consistent with $u(x) = \sin(\pi x)\sin(\pi y)$ as exact solution of (1). The mesh is obtained by first dividing Ω into $K = k^2$ identical squares and then by dividing similarly each of them into two triangles. The grid-size parameter H is chosen equal to $2/k$. The resulting discrete problem is solved by the preconditioned conjugate gradient (PCG) method without or with Schwarz preconditioner (5), this last one without or with the coarse solver $R_0^T A_0^{-1} R_0$. The initial guess is zero and the stopping criterion is $|\mathbf{r}^{(\nu)}|/|\mathbf{b}^{(\nu)}| \leq 10^{-7}$, where $\mathbf{r}^{(\nu)}$ is the νth residual. In Table 1, we report the iteration counts (It.), spectral condition number ($\kappa_2(A)$) and extreme eigenvalues (λ_{max}, λ_{min}), fixing $H = 1/2$ (32 subdomains) and varying the degree N from 3 to 15. Columns 2-3 refers to CG, columns 4-7 refer to PCG without coarse solver, and columns 8-11 refer to PCG with coarse solver. The same quantities are reported in Tables 2 fixing now $N = 3$ and varying $1/H$ from 2 to 10. These results are also plotted in Fig. 2 and 3, that clearly show that while the very ill-conditioned original TSEM matrix has a condition number that grows as $O(N^4 H^{-2})$, the overlapping Schwarz preconditioned operator is optimal and scalable (i.e. independent of N and H).

Table 1. CG and PCG preconditioners for the model problem (1) with $\alpha = \beta = 1$ and $\Omega = [-1, 1]^2$. Iteration counts, condition number and extreme eigenvalues fixed $1/H = 2$ and varying N.

	CG		PCG				PCG + coarse			
N	It.	$\kappa_2(A)$	It.	λ_{max}	λ_{min}	$\kappa_2(\tilde{A})$	It.	λ_{max}	λ_{min}	$\kappa_2(\tilde{A})$
3	28	84.34	12	12.99	2.66	4.87	13	13.00	3.34	3.88
6	85	729.37	13	12.99	2.67	4.85	13	12.99	3.35	3.87
9	206	4819.90	13	12.99	2.67	4.85	13	12.99	3.35	3.87
12	299	8899.07	13	12.99	2.67	4.85	13	12.99	3.35	3.87
15	456	21738.04	13	12.99	2.67	4.85	13	12.99	3.35	3.87

Table 2. CG and PCG preconditioners for the model problem (1) with $\alpha = \beta = 1$ and $\Omega = [-1, 1]^2$. Iteration counts, condition number and extreme eigenvalues fixed $N = 3$ and varying $1/H$.

	CG		PCG				PCG + coarse			
1/H	It.	$\kappa_2(A)$	It.	λ_{max}	λ_{min}	$\kappa_2(\tilde{A})$	It.	λ_{max}	λ_{min}	$\kappa_2(\tilde{A})$
2	28	84.34	12	12.99	2.66	4.87	13	12.99	3.34	3.88
3	39	190.08	14	12.99	1.41	9.16	13	12.99	2.34	5.53
4	48	337.99	16	12.99	0.84	15.37	14	13.00	1.81	7.17
5	56	527.00	18	12.99	0.55	23.40	15	13.00	1.53	8.46
6	63	753.65	20	12.99	0.39	33.23	17	13.00	1.37	9.43
7	69	1007.16	22	12.99	0.28	44.85	18	13.00	1.27	10.15
8	90	1352.64	25	13.00	0.22	58.27	19	13.00	1.21	10.69
9	98	1710.50	27	13.00	0.17	73.48	20	13.00	1.17	11.10
10	104	2104.60	30	13.00	0.14	90.48	20	13.00	1.13	11.41

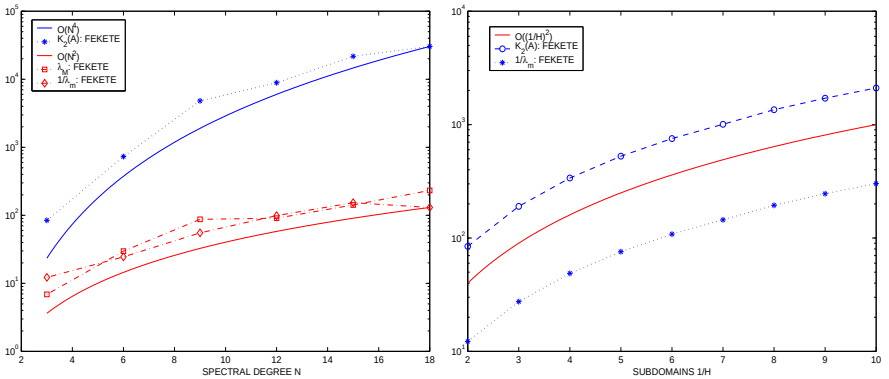

Fig. 2. Condition number and extreme eigenvalues of the unpreconditioned stiffness matrix as a function of N (left) and of $1/H$ (right)

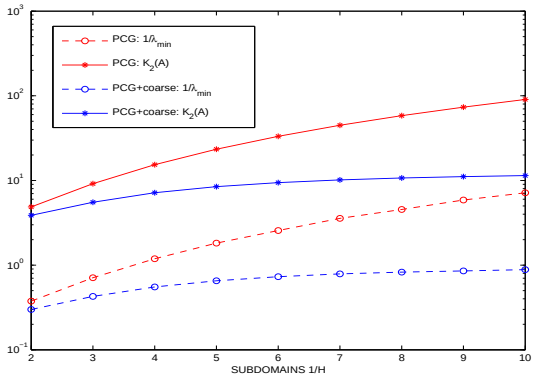

Fig. 3. Condition number and extreme eigenvalues of the overlapping Schwarz preconditioned matrix with (blue lines) and without (red lines) coarse solver as a function of $1/H$

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