Preconditioners for High Order Mortar Methods based on Substructuring

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Summary. A class of preconditioners for the Mortar Method based on substructuring is studied. We generalize the results of Achdou, Maday and Widlund (AMW99), obtained for the case of order one finite elements, to a wide class of discretization spaces including finite elements of any orders. More precisely, we show that the condition number of the preconditioned matrix grows at most polylogarithmically with the number of degrees of freedom per subdomain.

1 Introduction

We deal with the Mortar method, a nonconforming version of domain decomposition methods, that allows different discretizations and/or methods in different subdomains. Consequently, in an adaptive strategy, refinement can be carried out in each subdomain independently and it is possible to use in each subdomain the best suited method.

Here we face the problem of the efficient solution of the linear system arising from such discretization in order to make these techniques more competitive for real life applications. After elimination of the degrees of freedom internal to the subdomains, we need to find the traces of the solution on the subdomain boundaries, i.e. to solve the Schur complement system. The approach that we follow is the substructuring one, proposed by Bramble, Pasciak and Schatz (BPS86) in the framework of conforming domain decomposition. Such an approach was already applied to the Mortar method by Achdou, Maday and Widlund in (AMW99) for the case of order one finite elements. This consists in considering a suitable splitting of the nonconforming discretization space in terms of "edge" and "vertex" degrees of freedom and then using the related block-Jacobi type preconditioners. In this work we generalize the results of (AMW99) to a wide class of discretization spaces (including finite elements of any order) showing that the condition number of the preconditioned matrix grows at most polylogarithmically with the number of degrees of freedom per subdomain, analogously to what happens for the order one case.

Finally, we present numerical tests showing the scalability of the method for Q_1 and Q_2 finite elements.

2 The Mortar Method

At first we briefly introduce the Mortar method with its main properties (see (BMP94; W01)) and we focus, for simplicity, on the following simple model problem, even if the results of this paper can be easily extended to a more general situation. Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and $f \in L^2(\Omega)$, then we look for u satisfying

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial \mathbf{x}_j} \left(a_{ij}(\mathbf{x}) \frac{\partial u}{\partial \mathbf{x}_i} \right) = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{1}$$

The matrix $a(\mathbf{x}) = (a_{ij}(\mathbf{x}))_{i,j=1,2}$ is assumed to be, for almost all $\mathbf{x} \in \Omega$, symmetric positive definite with smallest eigenvalue $\geq \alpha > 0$ and largest eigenvalue $\leq \alpha'$, α, α' independent of $\mathbf{x}$.

In order to solve (1) we decompose the computational domain Ω as the union of L subdomains Ω_ℓ, $\Omega = \bigcup_{\ell=1,\ldots,L} \Omega_\ell$, which, for the sake of simplicity we assume to be quadrilateral (in general the constants in the inequalities will depend on the number of edges of the subdomains as well as on their aspect ratio). We follow the notation of (BP01): we set

$$\Gamma_{\ell n} = \partial\Omega_n \cap \partial\Omega_\ell, \qquad \mathcal{S} = \cup \Gamma_{\ell n} \tag{2}$$

and we denote by $\gamma_\ell^{(i)}$ $(i = 1, \ldots, 4)$ the i-th side of the ℓ-th domain so that $\partial\Omega_\ell = \bigcup_{i=1}^{4} \gamma_\ell^{(i)}$.

Definition 1. *We say that a decomposition is* geometrically conforming *if each edge $\gamma_\ell^{(i)}$ coincides with $\Gamma_{\ell n}$ for some n. If the decomposition is not geometrically conforming, then each interior edge $\gamma_\ell^{(i)}$ will be in general split as the union of several segments $\Gamma_{\ell n}$:*

$$\gamma_\ell^{(i)} = \bigcup_{n \in I_\ell^{(i)}} \Gamma_{\ell n}, \tag{3}$$

where $I_\ell^{(i)} = \{n : |\partial\Omega_n \cap \gamma_\ell^{(i)}| \neq 0\}$.

We now consider a non conforming domain decomposition method, based on the above splitting of the domain Ω, for the solution of problem (1). First we introduce the corresponding functional setting, hence let

$$X = \prod_\ell \{u_\ell \in H^1(\Omega_\ell)| \ u_\ell = 0 \text{ on } \partial\Omega \cap \partial\Omega_\ell\}, \qquad T = \prod_\ell H_*^{1/2}(\partial\Omega_\ell),$$

with $H_*^{1/2}(\Omega_\ell) = H^{1/2}(\partial\Omega_\ell)$ if $\partial\Omega_\ell \cap \partial\Omega = \emptyset$ and $H_*^{1/2}(\partial\Omega_\ell) = \{\eta \in H^{1/2}(\partial\Omega_\ell), \ \eta|_{\partial\Omega_\ell \cap \partial\Omega} \equiv 0\} \sim H_{00}^{1/2}(\partial\Omega_\ell \setminus \partial\Omega)$ otherwise. We denote by $\|\cdot\|_{1/2,\partial\Omega_\ell}$ the related norm, and by $\|\cdot\|_{-1/2,\ell}$ the norm of the corresponding dual space.

On X and T we introduce the following broken norm and semi-norm:

$$\|u\|_X = \left(\sum_\ell \|u\|_{1,\Omega_\ell}^2\right)^{1/2}, |u|_X = \left(\sum_\ell |u|_{1,\Omega_\ell}^2\right)^{1/2}, \|\eta\|_T = \left(\sum_\ell \|\eta_\ell\|_{1/2,\partial\Omega_\ell}^2\right)^{1/2}.$$

For each ℓ, let $\mathcal{V}_h^\ell$ be a family of finite dimensional subspaces of $H^1(\Omega_\ell) \cap C^0(\bar\Omega_\ell)$, depending on a parameter $h = h_\ell > 0$ and satisfying an homogeneous boundary condition on $\partial\Omega \cap \partial\Omega_\ell$.

Let $T_h^\ell = \mathcal{V}_h^\ell|_{\partial\Omega_\ell}$, and, for each edge $\gamma_\ell^{(i)}$ of the subdomain Ω_ℓ,

$$T_{\ell,i} = \{\eta \ : \ \eta \text{ is the trace on } \gamma_\ell^{(i)} \text{ of some } u_\ell \in \mathcal{V}_h^\ell\}$$
$$T_{\ell,i}^0 = \{\eta \in T_{\ell,i} \ : \ \eta = 0 \text{ at the vertices of } \gamma_\ell^{(i)}\}.$$

We set

$$X_h = \prod_{\ell=1}^L \mathcal{V}_h^\ell \subset X, \qquad T_h = \prod_{\ell=1}^L T_h^\ell \subset T \qquad (4)$$

and we define a composite bilinear form $a_X : X \times X \longrightarrow \mathbb{R}$ as follows:

$$a_X(u,v) = \sum_\ell \int_{\Omega_\ell} \sum_{i,j} a_{ij}(\mathbf{x})\frac{\partial u_\ell}{\partial\mathbf{x}_i}\frac{\partial v_\ell}{\partial\mathbf{x}_j} \, d\mathbf{x}. \qquad (5)$$

The bilinear form a_X is clearly not coercive on X. In order to obtain a well posed problem, we will then consider proper subspaces of X consisting of functions that satisfy a suitable *weak continuity* constraint defined, according to the Mortar method, by choosing a splitting of the skeleton $\mathcal{S}$ as the disjoint union of a certain number of subdomain sides $\gamma_\ell^{(i)}$ called "multiplier sides". We denote by $I \subset \{1,\ldots,L\}\times\{1,\ldots,4\}$ the set of indices (l,i) such that $\gamma_l^{(i)}$ is a multiplier side, while $I^* \subset \{1,\cdots,L\}\times\{1,\cdots,4\}$ will denote the index-set corresponding to "trace sides" ("mortars" or "master sides" in the usual terminology).

For each $m = (\ell,i) \in I$ let a finite dimensional multiplier space M_h^m (also depending on the parameter h) on γ_m,

$$M_h^m \subset L^2(\gamma_m), \qquad\qquad dim(M_h^m) = dim(T_h^{m,0}),$$

be given. We set:

$$M_h = \{\eta \in H^{-1/2}(\mathcal{S}), \ \forall m \in I \ \ \eta|_{\gamma_m} \in M_h^m\} \sim \prod_{m \in i} M_m.$$

The *constrained* approximation and trace spaces $\mathcal{X}_h$ and $\mathcal{T}_h$ are then defined as follows:

$$
\begin{aligned}
\mathcal{X}_h &= \{v_h \in X_h, \int_S [v_h]\lambda\, ds = 0, \, \forall \lambda \in M_h\} \\
\mathcal{T}_h &= \{\eta \in T_h, \int_S [\eta]\lambda\, ds = 0, \, \forall \lambda \in M_h\}.
\end{aligned}
\tag{6}
$$

The elements of $\mathcal{X}_h$ can be obtained by applying to any element of X_h a correction operator $\mathcal{P}_h : X_h \to \mathcal{X}_h$, whose action consists in suitably modify its argument to impose the constraint; remark that $\mathcal{P}_h$ is a projector.

Thus we can introduce the following discrete problem:

Problem 1. Find $u_h \in \mathcal{X}_h$ such that for all $v_h \in \mathcal{X}_h$

$$
a_X(u_h, v_h) = \int_\Omega f v_h.
$$

It is not difficult to choose the class M_h of multipliers in such a way to guarantee ellipticity uniformly with respect to the mesh-size parameter h and to the number L of subdomains.

Then it can be proved that for all $h > 0$, Problem (1) admits a unique solution u_h which satisfies the following error estimate (BP01):

$$
\|u - u_h\|_X \lesssim \left(\inf_{v_h \in \mathcal{X}_h} \|u - v_h\|_X + \inf_{\lambda \in M_h} \left\|\frac{\partial u}{\partial \nu} - \lambda\right\|_{-1/2, \mathcal{S}} \right)
\tag{7}
$$

with $\|\cdot\|_{-1/2, \mathcal{S}}$ denoting the norm of T', dual of T.

3 Substructuring Preconditioners for the Mortar Element Method

In this section we focus on a class of preconditioners for the linear system arising from the discretization by the Mortar method. We will follow the "substructuring" approach first introduced in (BPS86) and already studied in the case of the Mortar Finite Element method in (AMW99). The main idea of these preconditioners consists in distinguishing three types of degrees of freedom: *interior* degrees of freedom (corresponding to basis functions vanishing on the skeleton and supported on one sub-domain), *edge* degrees of freedom, and *vertex* degrees of freedom. Consequently we can split the functions $u \in \mathcal{X}_h$ as the sum of three suitably defined components: $u = u^0 + u^E + u^V$. Moreover, when expressed in a basis related to such a splitting, substructuring preconditioners can be written in a block diagonal form.

Let us now examine in details how the splitting is constructed. Given any discrete function $w = (w_\ell)_{\ell=1,\cdots,L} \in X_h$ we can split it in a unique way as the sum of an *interior* function $w^0 \in \mathcal{X}_h^0$ and a discrete lifting, performed subdomainwise of its trace $\eta(w) = (w^\ell|_{\Omega_\ell})_{\ell=1,\cdots,L}$ which by abuse of notation we will denote by $R_h(w)$ (rather than using the heavier notation $R_h(\eta(w))$):

$$w = w^0 + R_h(w), \qquad w^0 \in \mathcal{X}_h^0, \tag{8}$$

with $R_h(w) = (R_h^\ell(w_\ell))_{\ell=1,\dots,K}$, $R_h^\ell(w_\ell)$ being the unique element in $\mathcal{V}_h^\ell$ satisfying

$$R_h^\ell(w_\ell) = w_\ell \qquad \text{on } \Gamma_\ell,$$

$$\int_{\Omega_\ell} \sum_{i,j} a(\mathbf{x}) \frac{\partial}{\partial \mathbf{x}_i} \frac{\partial}{\partial \mathbf{x}_j} R_h^\ell(w_\ell) v_h^\ell \, d\mathbf{x} = 0, \quad \forall v_h \in \mathcal{V}_h^\ell \cap H_0^1(\Omega_\ell).$$

Thus the spaces X_h of unconstrained functions and $\mathcal{X}_h$ of constrained functions can be split as direct sums of an interior and of a (respectively unconstrained or constrained) trace component:

$$X_h = X_h^0 \oplus R_h(T_h), \qquad \mathcal{X}_h = \mathcal{X}_h^0 \oplus R_h(\mathcal{T}_h). \tag{9}$$

We can easily verify that $a_X : X_h \times X_h \to \mathbb{R}$ satisfies

$$a_X(w,v) = a_X(w^0, v^0) + a_X(R_h(w), R_h(v)) := a_X(w^0, v^0) + s(\eta(w), \eta(v)),$$

where the *discrete Steklov-Poincaré* operator $s : T_h \times T_h \to \mathbb{R}$ is defined by

$$s(\xi, \eta) := \sum_\ell \int_{\Omega_\ell} (a(\mathbf{x}) \nabla R_h^\ell(\xi)) \cdot \nabla R_h^\ell(\eta).$$

We note that the problem of preconditioning the matrix A associated to the discretization of a_X, reduces to finding good preconditioners for the matrices A_0 and S corresponding respectively to the bilinear forms $a_X : \mathcal{X}_h^0 \times \mathcal{X}_h^0 \longrightarrow \mathbb{R}$ and $s : T_h \times T_h \longrightarrow \mathbb{R}$. The matrix A_0 is block diagonal since the coupling between subdomains is taken into account only by the Steklov-Poincaré operator. The blocks of A_0 (which are in fact stiffness matrices corresponding to standard Dirichlet solvers) are widely studied in the literature; here we concentrate only on the preconditioning of the discrete Steklov-Poincaré operator S.

3.1 The splitting of the trace space

The space of constrained skeleton functions $\mathcal{T}_h$ defined in (6) can be further split as the sum of *vertex* and *edge* functions. More specifically, if we denote by $\mathfrak{L} \subset \prod_{\ell=1}^L H_*^{1/2}(\partial \Omega_\ell)$ the space

$$\mathfrak{L} = \{(\eta_\ell)_{\ell=1,\cdots,L}, \ \eta_\ell \text{ is linear on each edge of } \Omega_\ell\}, \tag{10}$$

then we can define the space of constrained *vertex* functions as

$$\mathcal{T}_h^V = \mathcal{P}_h \mathfrak{L} \tag{11}$$

with $\mathcal{P}_h$ the correction operator imposing the constraint. In the following we will make the (not restrictive) assumption $\mathfrak{L} \subset \mathcal{T}_h$, which yields $\mathcal{T}_h^V \subset \mathcal{T}_h$.

We then introduce the space of constrained *edge* functions $\mathcal{T}_h^E \subset \mathcal{T}_h$ defined by

$$\mathcal{T}_h^E = \{\eta = (\eta_\ell)_{\ell=1,\cdots,L} \in \mathcal{T}_h, \ \eta_\ell(A) = 0, \ \forall \text{ vertex } A \text{ of } \Omega_\ell\} \tag{12}$$

and it is quite simple to check that a function in $\mathcal{T}_h^E$ is uniquely defined by its value on trace edges, the value on multiplier edges being forced by the constraint.

Thus, it can be easily verified that

$$\mathcal{T}_h = \mathcal{T}_h^V \oplus \mathcal{T}_h^E \tag{13}$$

and that each $\eta \in \mathcal{T}_h$ can be decomposed in a unique way as

$$\eta = \eta^V + \eta^E, \quad \text{with} \quad \eta^V \in \mathcal{T}_h^V \text{ and } \eta^E \in \mathcal{T}_h^E.$$

The preconditioner

The preconditioner that we consider for S is of block-Jacobi type with blocks related to edges and vertexes. More specifically we can assemble the preconditioner $\hat{s}$ as

$$\hat{s} : \mathcal{T}_h \times \mathcal{T}_h \longrightarrow \mathbb{R}$$
$$\hat{s}(\eta, \xi) = b^V(\eta^V, \xi^V) + b^E(\eta^E, \xi^E) \tag{14}$$

with

$$b^V : \mathcal{T}_h^V \times \mathcal{T}_h^V \longrightarrow \mathbb{R} \qquad \text{such that} \qquad b^V(\eta^V, \eta^V) \simeq s(\eta^V, \eta^V)$$

and

$$b^E : \mathcal{T}_h^E \times \mathcal{T}_h^E \longrightarrow \mathbb{R} \qquad b^E(\eta, \xi) = \sum_{(\ell,i) \in I^*} b_{\ell,i}(\eta_\ell, \xi_\ell)$$

where for any trace side $\gamma_\ell^{(i)}$, $(\ell, i) \in I^*$, $b_{\ell,i} : T_{\ell,i}^0 \times T_{\ell,i}^0 \longrightarrow \mathbb{R}$ is a symmetric bilinear form satisfying for all $\eta \in T_{\ell,i}^0$

$$b_{\ell,i}(\eta, \eta) \simeq \|\eta\|_{H_{00}^{1/2}(\gamma_\ell^{(i)})}.$$

Denoting by H_l the diameter of Ω_l and writing conventionally $H/h = \min_l \{H_l/h_l\}$ then, under suitable regularity assumptions on the subdomains and on the spaces considered (see (BP04)), we can prove the following theorem providing bounds for the condition number of the preconditioned matrix.

Theorem 1. *Let S and $\hat{S}$ be the matrices obtained by discretizing respectively s and $\hat{s}$. Then it holds*

$$Cond(\hat{S}^{-1}S) \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right)^4.\tag{15}$$

Moreover, if the decomposition is geometrically conforming then

$$Cond(\hat{S}^{-1}S) \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right)^2.\tag{16}$$

The proof of Theorem 1 follows essentially the guidelines of the proofs of the analogous results in (BPS86; AMW99); due to space constraint, we do not present it but we refer to (BP04).

4 Numerical tests

Finally we performed numerical experiments to test the scalability of the method for Q_1 and Q_2 finite elements. The model problem is the Poisson equation on the unit square Ω with homogeneous Dirichlet boundary conditions. A uniform, geometrically conforming, decomposition of Ω in $K = N \times N$ equal square subdomains of size $H \times H$ with $H = 1/N$ is considered. In each subdomains Ω_k, a uniform mesh $\mathcal{T}^k$ is built and Q_1, Q_2 finite elements are used in each square.

In order to study the dependence on H (size of the subdomains) and on h (finest meshsize of the finite element spaces), we set $n_k = n$ for all k; hence $h_k = h = H/n$ and $H/h = n$. This corresponds to a non–conforming implementation of the standard domain decomposition method. Then, we tested the preconditioners for several combinations of N and n with n in the range $[5, 40]$ and N in the range $[4, 32]$.

The preconditioned conjugate gradient iteration was stopped when the residual norm decreased by a factor of 10^{-5} and the experiment were carried out in MATLAB.

Tables 1 show the number of conjugate gradient iterations for reducing the residual of a factor 10^{-5} for Q_1 (left) and Q_2 (right) finite elements respectively. For the edge block of the preconditioner we considered the square root of the stiffness matrix associated to the discretization of the operator $-d^2/dx^2$ by P_1 and P_2 finite elements on each edge with homogeneous Dirichlet boundary conditions at the extrema.

The results are in close agreement with the theory: the condition number of the preconditioned matrix grows at most polylogarithmically with the number of degrees of freedom per subdomain, as indicated by theorem (1).

A complete set of numerical tests showing the scalability of the method for Q_1 and Q_2 finite elements can be found in (BP04).

$K= N^2$	n=5 # iter.	n=10 # iter.	n=20 # iter.	n=40 # iter.	$K= N^2$	n=5 # iter.	n=10 # iter.	n=20 # iter.	n=40 # iter.
16	23	25	26	27	16	25	25	27	29
64	24	26	27	28	64	27	28	30	31
144	24	26	27	29	144	27	28	30	31
256	24	26	27	29	256	27	28	30	32
400	24	26	27	28	400	27	28	30	31
576	24	26	27	28	576	27	28	30	31
784	24	26	27	28	784	27	28	30	31
1024	23	26	27	28	1024	27	28	30	31

Table 1. Number of conjugate gradient iterations needed for reducing the residual of a factor 10^{-5}, for different combinations of the number $K = N^2$ of subdomains and n elements per edge (n^2 elements per subdomains) and for Q_1 finite elements (left) and Q_2 finite elements (right).

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