Construction of a New Domain Decomposition Method for the Stokes Equations

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Summary. We propose a new domain decomposition method for the Stokes equations in two and three dimensions. The algorithm, we propose, is very similar to an algorithm which is obtained by a Richardson iteration of the Schur complement equation using a Neumann-Neumann preconditioner. A comparison of both methods with the help of a Fourier analysis shows clearly the advantage of the new approach. This could also be validated by numerical experiments.

1 Introduction

In this paper we study a Neumann-Neumann type algorithm for the Stokes equations. The last decade has shown, that these kind of domain decomposition methods are very efficient. Most of the theoretical and numerical work has been carried out for symmetric second order problems, see Roeck and Tallec [1991]. Then the method was extended to other problems, like the advection-diffusion equations (Achdou et al. [2000]) or recently the Stokes equations, c.f. Pavarino and Widlund [2002], Tallec and Patra [1997].

In the case of two domains consisting of the two half planes it is well known, that the Neumann-Neumann preconditioner is an exact preconditioner for the Schur complement equation for scalar equations like the Laplace problem (cf. Roeck and Tallec [1991]). As we will show, this property could not be transfered to the vector valued Stokes problem due to the incompressibility constraint.

We will construct a method, which preserves this property. The first preliminary numerical results clearly indicates a better convergence behavior.

2 The preconditioned Schur Complement equation

In order to make the presentation as simple as possible we restrict ourselves to the two dimensional case. But the extension to the three dimensional case is straightforward.

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. The Stokes problem is a simple model for incompressible flows and is defined as follows: We search for a velocity $\mathbf{u}$ and a pressure p, such that

$$\begin{aligned} -\nu\triangle\mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0 \qquad & \text{in } \Omega \\ \mathbf{u} = 0 \quad & \text{on } \partial\Omega. \end{aligned} \tag{1}$$

$\mathbf{f} \in [L^2(\Omega)]^d$ is a source term and ν is the viscosity. In the proceeding we denote the Stokes operator by $\mathcal{A}_{Stokes}(\mathbf{v}, q) := (-\nu\triangle\mathbf{v} + \nabla p, \nabla\cdot\mathbf{v})$.

2.1 Schur complement equation

Most of the domain decomposition methods for the Stokes equations use the classical sub-structuring or static condensation procedure. This means, that they end up with a Schur complement equation. Since the corresponding Steklov-Poincaré operator is badly conditioned, the application of suitable preconditioners is mandatory. One of the best-known preconditioner is the Neumann-Neumann preconditioner (cf. Tallec and Patra [1997], Ainsworth and Sherwin [1999], Pavarino and Widlund [2002]).

Assume a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ divided into two nonoverlapping subdomains Ω_1 and Ω_2. The interface is denoted by $\Gamma := \partial\Omega_1 \cap \partial\Omega_2$.

In case of the Stokes equations an additional problem occurs. If we assume, that $\mathbf{u}_i \in [H^1(\Omega_i)]^2$ satisfies the incompressibility constraint, i.e. $\nabla\cdot\mathbf{u}_i = 0$, then the Green's formula yields $\int_{\partial\Omega_i} \mathbf{u}_i\cdot\mathbf{n}_i ds = 0$ for the trace of $\mathbf{u}_i$ where $\mathbf{n}_i$ is the outward normal of Ω_i. Therefore we have to consider the subspace

$$H_*^{\frac{1}{2}}(\Gamma) := \{\boldsymbol{\varphi} \in [H_{00}^{\frac{1}{2}}(\Gamma)]^2 \mid \int_\Gamma \boldsymbol{\varphi}\cdot\mathbf{n}_i ds = 0\}$$

of the trace space taking into account the homogeneous boundary conditions on $\partial\Omega_i \cap \partial\Omega$. We consider the operator

$$\begin{aligned} \Sigma : H_*^{\frac{1}{2}}(\Gamma) \times [L^2(\Omega)]^2 \quad &\to \quad [H^{-\frac{1}{2}}(\Gamma)]^2 \\ (\mathbf{u}_\Gamma, \mathbf{f}) \quad &\mapsto \quad \frac{1}{2}\left(\nu\frac{\partial\mathbf{u}_1}{\partial\mathbf{n}_1} - p_1\mathbf{n}_1\right)\Big|_\Gamma + \frac{1}{2}\left(\nu\frac{\partial\mathbf{u}_2}{\partial\mathbf{n}_2} - p_2\mathbf{n}_2\right)\Big|_\Gamma \end{aligned}$$

where $(\mathbf{u}_i, p_i) \in [H^1(\Omega_i)]^2 \times L_0^2(\Omega_i)$ are the unique solutions of the local Stokes problems

$$\begin{aligned} \mathcal{A}_{Stokes}(\mathbf{u}_i, p_i) &= (\mathbf{f}, 0) \quad \text{in } \Omega_i \\ \mathbf{u}_i = 0 \quad \text{on } \partial\Omega_i \cap \partial\Omega, \qquad & \mathbf{u}_i = \mathbf{u}_\Gamma \quad \text{on } \Gamma. \end{aligned}$$

It is clear, that the equation

$$\text{Find } \boldsymbol{\phi} \in H_*^{\frac{1}{2}}(\Gamma) \mid \quad \langle\Sigma(\boldsymbol{\phi}, 0), \psi\rangle = \langle -\Sigma(0, \mathbf{f}), \psi\rangle\,, \quad \forall\psi \in H_*^{\frac{1}{2}}(\Gamma) \tag{2}$$

is satisfied by the restriction of the continuous solution (1) on the interface Γ.

2.2 Neumann-Neumann preconditioner

The Neumann-Neumann preconditioner of the Steklov-Poincaré operator $\mathbb{S} := \Sigma(\cdot, 0)$ is defined by

$$\mathcal{T} : (H^{-\frac{1}{2}}(\Gamma))^2 \ \rightarrow \ H_*^{\frac{1}{2}}(\Gamma), \qquad \boldsymbol{\phi} \ \mapsto \ \left(\frac{1}{2}(\mathbf{v}_{1,j} + \mathbf{v}_{2,j})|_\Gamma\right)_{j=1}^2 .$$

where $\mathbf{v}_i = (v_{i,1}, v_{i,2}) \in [H^1(\Omega_i)]^2$ satisfies

$$\begin{aligned} \mathcal{A}_{Stokes}(\mathbf{v}_i, q_i) &= 0 \quad \text{in } \Omega_i \\ \mathbf{v}_i = 0 \quad \text{on } \partial\Omega_i \cap \partial\Omega, \qquad & \frac{\partial \mathbf{v}_i}{\partial \mathbf{n}_i} - q_i \mathbf{n}_i = \boldsymbol{\phi} \quad \text{on } \Gamma. \end{aligned}$$

In order to keep the presentation simple we consider the following Richardson iteration of equation (2): Starting with an initial guess $\boldsymbol{\varphi}^0 \in H_*^{\frac{1}{2}}(\Gamma)$ we obtain

$$\boldsymbol{\varphi}_{k+1} = \boldsymbol{\varphi}_k - \mathcal{T}(\mathbb{S}\boldsymbol{\varphi}_k + \Sigma(0, f)), \qquad k = 0, 1, 2, \ldots . \tag{3}$$

Please notice that all $\boldsymbol{\varphi}_{k+1}$, $k \in \mathbb{N}$, satisfy $\int_{\partial\Omega_i} \boldsymbol{\varphi}_{k+1} \cdot \mathbf{n}_i ds = 0$. Thus after a proper initialization all iterations $\boldsymbol{\varphi}_k$ are elements of $H_*^{\frac{1}{2}}(\Gamma)$. Of course, in a practical implementation the Richardson iteration (3) would be replaced by a suitable Krylov method.

3 Smith Factorization

We first recall the definition of the Smith factorization of a matrix with polynomial entries and apply it to the Stokes system.

Theorem 1. *Let n be an integer and A an invertible $n \times n$ matrix with polynomial entries with respect to the variable λ: $A = (a_{ij}(\lambda))_{1 \le i,j \le n}$.*
Then, there exist matrices E, F and a diagonal matrix D with polynomial entries satisfying $A = EDF$.

More details can be found in Wloka et al. [1995]. We first take formally the Fourier transform of system (1) with respect to y (dual variable is k). We keep the partial derivatives in x since in the sequel we shall consider a model problem where the interface between the subdomains is orthogonal to the x direction. We note

$$\hat{A}_{Stokes} = \begin{pmatrix} -\nu(\partial_{xx} - k^2) & 0 & \partial_x \\ 0 & -\nu(\partial_{xx} - k^2) & ik \\ \partial_x & ik & 0 \end{pmatrix}. \tag{4}$$

We perform the Smith factorization of $\hat{A}_{Stokes}$ by considering it as a matrix with polynomials in ∂_x entries. Applying the inverse Fourier transform yields

$$A_{Stokes} = EDF \tag{5}$$

where $D_{11} = D_{22} = 1$ and $D_{33} = -\nu\triangle^2$ and

$$E := T_2^{-1} \begin{pmatrix} -\nu\triangle\partial_y & \nu\partial_{xxx} & -\nu\partial_x \\ 0 & T_2 & 0 \\ \partial_{xy} & -\partial_{xx} & 1 \end{pmatrix}, \qquad F := \begin{pmatrix} \nu\partial_{yy} & \nu\partial_{yx} & \partial_x \\ 0 & -\nu\triangle & \partial_y \\ 0 & 1 & 0 \end{pmatrix}$$

where T_2 is a differential operator in y-direction whose symbol is $i\nu k^3$.

This suggests that the derivation of a DDM for the bi-Laplacian is a key ingredient for a DDM for the Stokes system. One should note that a stream function formulation gives the same differential equation for the stream function.

4 The new algorithm

The algorithm is derived by an Neumann-Neumann algorithm for the Bi-Laplacian using the Smith factorization (5). For details we refer to Dolean et al. [2005], Nataf and Rapin [2005].

The new algorithm is very similar to the algorithm given by (3). Again, each iteration step requires the solution of two local boundary value problems with Dirichlet and Neumann boundary conditions. But this time we distinguish between tangential parts and normal parts of the velocity and impose different boundary conditions for each part.

In order to write the resulting algorithm in an intrinsic form, we introduce the stress $\boldsymbol{\sigma}(\mathbf{u}, p) = \nu\frac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n}$ on the interface for a velocity $\mathbf{u}$ and a pressure p. For any vector $\mathbf{u}$ its normal (resp. tangential) component on the interface is $\mathbf{u}_n$ (resp. $\mathbf{u}_\tau$). We denote by $\boldsymbol{\sigma}_n$ and $\boldsymbol{\sigma}_\tau$ the normal and tangential parts of $\boldsymbol{\sigma}$, respectively. We consider a decomposition of the domain into non overlapping subdomains: $\bar{\Omega} = \cup_{i=1}^N \bar{\Omega}_i$ and denote by Γ_{ij} the interface between subdomains Ω_i and Ω_j, $i \neq j$. The new algorithm for the Stokes system reads:

ALGORITHM 1 *Starting with an initial guess satisfying* $\mathbf{u}^0_{i,\tau_i} = \mathbf{u}^0_{j,\tau_j}$ *and* $\boldsymbol{\sigma}^0_{i,n_i} = -\boldsymbol{\sigma}^0_{j,n_j}$ *on* Γ_{ij}*, the correction step is defined as follows for* $1 \leq i \leq N$*:*

$$\begin{aligned} A_{Stokes}(\tilde{\mathbf{u}}_i^{n+1}, \tilde{p}_i^{n+1})^T &= 0 \text{ in } \Omega_i, \qquad \tilde{\mathbf{u}}_i^{n+1} = 0 \text{ on } \partial\Omega_i \cap \partial\Omega \\ \tilde{\mathbf{u}}_{i,n_i}^{n+1} &= -(\mathbf{u}_{i,n_i}^n - \mathbf{u}_{j,n_j}^n)/2 \text{ on } \Gamma_{ij} \\ \sigma_{\tau_i}(\tilde{\mathbf{u}}_i^{n+1}, \tilde{p}_i^{n+1}) &= -(\sigma_{\tau_i}(\tilde{\mathbf{u}}_i^n, \tilde{p}_i^n) + \sigma_{\tau_j}(\tilde{\mathbf{u}}_j^n, \tilde{p}_j^n))/2 \text{ on } \Gamma_{ij} \end{aligned}$$

followed by an update step:

$$\begin{aligned} A_{Stokes}(\mathbf{u}_i^{n+1}, p_i^{n+1})^T &= f \text{ in } \Omega_i \qquad \mathbf{u}_i^{n+1} = 0 \text{ on } \partial\Omega_i \cap \partial\Omega \\ \mathbf{u}_{i,\tau_i}^{n+1} &= \mathbf{u}_{i,\tau_i}^n + (\tilde{\mathbf{u}}_{i,\tau_i}^{n+1} + \tilde{\mathbf{u}}_{j,\tau_j}^{n+1})/2 \text{ on } \Gamma_{ij} \\ \boldsymbol{\sigma}_{n_i}(\mathbf{u}_i^{n+1}, p_i^{n+1}) &= \boldsymbol{\sigma}_{n_i}(\mathbf{u}_i^n, p_i^n) \\ &\quad + (\boldsymbol{\sigma}_{n_i}(\tilde{\mathbf{u}}_i^{n+1}, \tilde{p}_i^{n+1}) - \boldsymbol{\sigma}_{n_j}(\tilde{\mathbf{u}}_j^{n+1}, \tilde{p}_j^{n+1}))/2 \text{ on } \Gamma_{ij}. \end{aligned}$$

The boundary conditions in the correction step involve the normal velocity and the tangential stress whereas in the update step they involve the tangential velocity and the normal stress. In 3D, the algorithm has the same definition. By construction, it converges in two steps.

Theorem 2. *For a domain $\Omega = \mathbb{R}^2$ divided into two non overlapping half planes, the algorithm 1 converges in two iterations.*

5 Analysis of the Neumann-Neumann Algorithm

Here we focus on the Neumann-Neumann algorithm and we will use the Smith factorization in order prove that the Neumann-Neumann algorithm (3) does not converge in at most two steps in the case of the plane $\Omega = \mathbb{R}^2$ divided into the two half planes $\Omega_1 := (-\infty, 0) \times \mathbb{R}$ and $\Omega_1 := (0, \infty) \times \mathbb{R}$. Therefore the Neumann-Neumann preconditioner is not an exact preconditioner.

5.1 Reformulation of the algorithm

For the above decomposition the Smith factorization enables us to formulate the Neumann-Neumann algorithm (3) of the Stokes equations solely in terms of the second velocity components. The third equation of (5) gives $-\triangle^2 z = g$ with $z = (F(\mathbf{u}, p))_3 = u_2$ and $g = (E^{-1}(\mathbf{f}, 0))_3$. Then the first velocity and the pressure component can be eliminated in the interface conditions using the Stokes equations. Let us define $\mathcal{L}u := -\nu\triangle u$.

We end up with the following algorithm: Starting with an initial guess

$$u_1^n = u_2^n, \quad \frac{\partial}{\partial \mathbf{n}_1}(\mathcal{L} - \nu\partial_{yy})u_1^n = -\frac{\partial}{\partial \mathbf{n}_2}(\mathcal{L} - \nu\partial_{yy})u_2^n \quad \text{on } \Gamma$$

the correction step for $n = 1, 2, \ldots$ is given by

$$-\nu\triangle^2 v_i^n = 0 \quad \text{in } \Omega_i \tag{6}$$

$$\frac{\partial v_i^n}{\partial \mathbf{n}_i} = -\frac{1}{2}\left(\frac{\partial u_1^{n-1}}{\partial \mathbf{n}_1} + \frac{\partial u_2^{n-1}}{\partial \mathbf{n}_2}\right) \quad \text{on } \Gamma \tag{7}$$

$$(\mathcal{L} - \nu\partial_{yy})v_i^n = -\frac{1}{2}\left(\mathcal{L}u_i^{n-1} - \mathcal{L}u_{3-i}^{n-1}\right) \quad \text{on } \Gamma \tag{8}$$

for $i = 1, 2$. The update step is defined by

$$-\nu\triangle^2 u_i^n = g \quad \text{in } \Omega_i, \tag{9}$$

$$u_i^n = u_i^{n-1} + \frac{1}{2}\left(v_1^n + v_2^n\right) \quad \text{on } \Gamma \tag{10}$$

$$\frac{\partial}{\partial \mathbf{n}_i}(\mathcal{L} - \nu\partial_{yy})u_i^n = \frac{\partial}{\partial \mathbf{n}_i}(\mathcal{L} - \nu\partial_{yy})u_i^{n-1} + \frac{1}{2}\frac{\partial}{\partial \mathbf{n}_i}(\mathcal{L} - \nu\partial_{yy})\left(v_1^n + v_2^n\right) \quad \text{on } \Gamma \tag{11}$$

with $g = (E^{-1}(\mathbf{f}, 0))_3$ and $i = 1, 2$.

5.2 Some Fourier Analysis

We start with the reformulated algorithm (6)-(8), (9)-(11). Again, using the linearity of the scheme, we obtain for the error $\tilde{e}_i^n$ in the n-th iteration step in subdomain Ω_i the update formula $\tilde{e}_i^n = \tilde{e}_i^{n-1} + \tilde{z}_i^n$ where $\tilde{z}_i^n$ satisfies

$$-\nu\triangle^2 \tilde{z}_i^n = 0 \quad \text{in } \Omega_i \tag{12}$$

$$\tilde{z}_i^n = \frac{1}{2}(v_1^n + v_2^n) \quad \text{on } \Gamma \tag{13}$$

$$\partial_x(-\nu\partial_{xx} - 2\nu\partial_{yy})\tilde{z}_i^n = \frac{1}{2}\partial_x(-\nu\partial_{xx} - 2\nu\partial_{yy})(v_1^n + v_2^n) \quad \text{on } \Gamma. \tag{14}$$

v_1^n, v_2^n are the solutions of the correction step (6)-(8) with right hand side

$$H_{NN}^n := -\frac{1}{2}\nu\left(\frac{\partial\triangle\tilde{e}_1^n}{\partial\mathbf{n}_1} + \frac{\partial\triangle\tilde{e}_2^n}{\partial\mathbf{n}_2}\right)\Big|_{x=0}, \quad K_{NN}^n := -\frac{1}{2}\left(\frac{\partial\tilde{e}_1^n}{\partial\mathbf{n}_1} + \frac{\partial\tilde{e}_2^n}{\partial\mathbf{n}_2}\right)\Big|_{x=0}.$$

Let us start with the correction step. After Fourier transform we obtain

$$\nu(-\partial_{xxxx} + 2k^2\partial_{xx} - k^4)\hat{v}_i^n(x,k) = 0.$$

For a fixed k these are ordinary differential equations in x with solutions

$$\hat{v}_1^n(x,k) = C_{11}^n \exp(|k|x) + C_{12}^n x\exp(|k|x) \tag{15}$$

$$\hat{v}_2^n(x,k) = C_{21}^n \exp(-|k|x) + C_{22}^n x\exp(-|k|x). \tag{16}$$

Using the interface conditions (7) we get

$$\hat{K}_{NN}^{n-1} = |k|C_{11}^n + C_{12}^n, \qquad -\hat{K}_{NN}^{n-1} = -|k|C_{21}^n + C_{22}^n.$$

The second interface condition (8) yields

$$\hat{H}_{NN}^{n-1} = -\nu|k|^2 C_{11}^n - 2\nu|k|C_{12}^n, \quad -\hat{H}_{NN}^{n-1} = \nu|k|^2 C_{21}^n + 2\nu|k|C_{22}.$$

Thus, we have four linear equations for the four unknowns C_{11}^n, C_{12}^n, C_{21}^n, and C_{22}^n. After simple computations we obtain

$$C_{11}^n = \tfrac{2}{3}\tfrac{1}{|k|}\hat{K}_{NN}^{n-1} + \tfrac{\hat{H}_{NN}^{n-1}}{3\nu|k|^2}, \quad C_{12}^n = \frac{1}{3}\hat{K}_{NN}^{n-1} - \frac{\hat{H}_{NN}^{n-1}}{3\nu|k|}$$

$$C_{21}^n = \tfrac{2}{3}\tfrac{1}{|k|}\hat{K}_{NN}^{n-1} - \tfrac{\hat{H}_{NN}^{n-1}}{3\nu|k|^2}, \quad C_{22}^n = -\frac{1}{3}\hat{K}_{NN}^{n-1} - \frac{\hat{H}_{NN}^{n-1}}{3\nu|k|}.$$

Next, we use the solutions of the correction step in order to compute the right hand side of the update step

$$\tilde{f}^n := \frac{1}{2}\left(\hat{v}_1^n + \hat{v}_2^n\right)\big|_{x=0} = \frac{1}{2}(C_{11}^n + C_{21}^n) = \frac{2}{3}\frac{\hat{K}_{NN}^{n-1}}{|k|}$$

$$\tilde{g}^n := \left(\frac{1}{2}\partial x(-\nu\partial_{xx} - 2\nu\partial_{yy})(\hat{v}_1^n + \hat{v}_2^n)\right)\Big|_{x=0} = \frac{2}{3}|k|\hat{H}_{NN}^{n-1}.$$

Again, after Fourier transform the solutions of (12) are given by

$$\hat{z}_1^n(x,k) = D_{11}^n \exp(|k|x) + D_{12}^n x \exp(|k|x),$$
$$\hat{z}_2^n(x,k) = D_{21}^n \exp(-|k|x) + D_{22}^n x \exp(-|k|x)$$

using that the solutions vanish at infinity. Inserting the boundary condition (13) yields $D_{11}^n = D_{21}^n = \tilde{f}^n = \frac{2}{3}\frac{\hat{K}_{NN}^n}{|k|}$. Now, we consider the second transmission condition (14). Then we can derive

$$D_{12}^n = -\frac{2}{3}\frac{1}{\nu|k|}\hat{H}_{NN}^{n-1} + \frac{2}{3}\hat{K}_{NN}^{n-1}, \quad D_{22}^n = -\frac{2}{3}\frac{\hat{H}_{NN}^{n-1}}{\nu|k|} - \frac{2}{3}\hat{K}_{NN}^{n-1}.$$

This result can be used to compute $\hat{H}_{NN}^n$ and $\hat{K}_{NN}^n$. They are given by

$$\hat{K}_{NN}^n = \hat{K}_{NN}^{n-1} - \frac{1}{2}\left(\frac{\partial \hat{z}_1^n}{\partial x} - \frac{\partial \hat{z}_2^n}{\partial x}\right)\Big|_{x=0} = -\frac{1}{3}\hat{K}_{NN}^{n-1}$$

resp.

$$\hat{H}_{NN}^n = \hat{H}_{NN}^{n-1} - \frac{1}{2}\left(-\nu\partial_{xx}(\hat{z}_1^n - \hat{z}_2^n)\right)|_{x=0} = -\frac{1}{3}\hat{H}_{NN}^{n-1}.$$

Let us summarize the result

Theorem 3. *Consider the case $\Omega = \mathbb{R}^2$. If the domain Ω is divided into the two half planes, the preconditioned Richardson iteration (3) of the Schur complement equation converges. Moreover, the error is reduced by the factor 3 in each iteration step.*

6 Preliminary Numerical Results

The domain $\Omega = (-A, B) \times (0, 1)$ is decomposed into two subdomains $\Omega_1 = (-A, 0) \times (0, 1)$ and $\Omega_2 = (0, B) \times (0, 1)$. We compare the new algorithm to the iterative version of the Neumann-Neumann algorithm. The stopping criteria is that the jumps of the normal derivative of the tangential component of the velocity are reduced by the factor 10^{-4}. In table 1 (left) $A = B = 1$, we see that both algorithms are not sensitive with respect to the mesh size. Of course, due to the discrete approximation we cannot expect the optimal convergence in two steps. But we only need one more step to achieve the error bound. We have also varied the width of the subdomains, (middle table). As expected the convergence of the Neumann-Neumann method deteriorates. For large aspect ratios, the method diverges (– in the table), since there exists an eigenvalue of the operator corresponding to the Richardson iteration with a modulus larger than 1. But still in this case convergence can be enforced by its use as a preconditioner in Krylov method as it is usually the case. Our new algorithm seems to be surprisingly robust with respect to the subdomain widths. For moderate variations we always need 3 iterations steps. If we choose very thin

subdomains, for instance $A = 1$, $B = 20$, the stopping criterion is achieved in only 7 steps. In table 1 (right), we have added a reaction term $c > 0$ to the first two equations of the Stokes system. For instance c may be the inverse of the time step in a time-dependent computation. We see that the new algorithm is fairly stable.

h	new algo	N-N
0.02	3	10
0.025	3	12
0.05	3	11
0.5	3	11
0.1	3	11
0.2	3	10

B	new algo	N-N
1	3	11
2	3	12
3	3	11
5	3	15
10	3	–
20	7	–

c	new algo	N-N
0.001	3	11
0.01	3	16
0.1	3	19
1	3	19
10	3	16
100	3	10

Table 1. Number of iterations for different mesh sizes (left), aspect ratio (middle) and different reaction terms (right).

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