A Fast Helmholtz Solver for Scattering by a Sound-soft Target in Sediment

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1 Introduction

We consider an efficient numerical method for computing time-harmonic acoustic scattering in a vertically layered media. One application for such problems is the detection of targets buried in a sediment. For this purpose it is useful to have a numerical approximation which can predict reasonably accurately backscatter by such targets. In this paper, we study scattering by sound-soft targets when the interface between the water and sediment is wavy. Such problems are typically modeled using a Helmholtz equation with varying coefficients.

With higher frequencies a finite element discretization leads to very large systems of linear equations. Often two-dimensional problems have millions of unknowns. It might be possible to solve these problems using a *LU* factorization with a nested dissection reordering of unknowns, but this approach cannot be used for three-dimensional problems which can have billions of unknows. For this reason, we consider the iterative solution of these problems. We employ an algebraic fictitious domain method [4, 6, 7, 8] which uses a right preconditioned GMRES method.

In a related work [12], it was noted that an iterative method with a separable preconditioner converges fast as long as the media is mainly layered in one direction or frequencies are reasonably low. We will use a separable preconditioner based on the perfectly layered media in our solution procedure. We embed the sound-soft target in a rectangular computation domain with a second order-absorbing boundary condition. Since the media is vertically layered with the wavy interface, our preconditioner coincides with the system matrix except the rows corresponding to unknowns near-by the interface and the target. Thus, we can reduce iterations on a small sparse subspace as has been shown in [7, 8]. This reduction makes our preconditioner extremely efficient as our numerical example demonstrates.

2 Model Problem

We are interested to compute the scattering of a time-harmonic acoustic pressure wave by a target which is buried in sediment. We model this situation with a Helmholtz equation with varying coefficients. Generally, it might be necessary to use elastic equations to model the wave inside the target, but in this investigation, we assume the target to be sound-soft. This means that a Dirichlet boundary condition can be posed on the surface of the target. The sediment is assumed be homogeneous and the surface between the water and the sediment is defined by $x_2 = f(x_1)$, where f is a given function.

We have a radiational wave from a point source in the water which is impinging the sediment and the target Ω. Furthermore, we could have a sensor in the water measuring the scattered wave. For computations, we truncate a rectangular domain Π enclosing the target and the source/sensor from the infinite domain. Figure 1 shows the set up of our model problem.

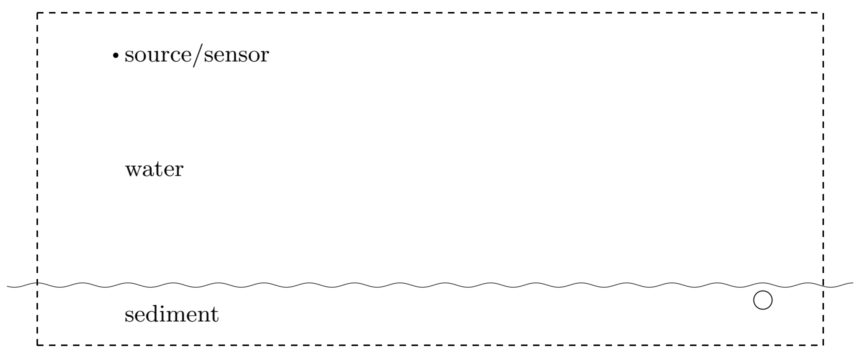

Fig. 1. The geometry of the model problem with a circular target Ω and a rectangular truncated domain Π given by the dashed line.

A time-harmonic acoustic pressure wave p satisfies the Helmholtz equation with varying coefficients

$$\begin{aligned} \nabla \cdot \frac{1}{\rho_1} \nabla p + \frac{k_1^2}{\rho_1} p = g & \qquad \text{in } x_2 > f(x_1), \\ \frac{1}{\rho_1} \frac{\partial p}{\partial n}\bigg|_+ = \frac{1}{\rho_2} \frac{\partial p}{\partial n}\bigg|_- & \qquad \text{on } x_2 = f(x_1), \\ \nabla \cdot \frac{1}{\rho_2} \nabla p + \frac{k_2^2}{\rho_2} p = 0 & \qquad \text{in } x_2 < f(x_1) \quad \text{and} \quad x \notin \Omega, \end{aligned} \tag{1}$$

where $k_1 = \frac{\omega}{c_1}$ and $k_2 = \frac{\omega}{c_2}$ are the wave numbers for the water and the sediment, respectively. A normal of the surface $x_2 = f(x_1)$ is denoted by n. The notation $\Big|_+$ refers to the value of a function or derivative when approaching

$x_2 = f(x_1)$ from the side $x_2 > f(x_1)$. Similarly $\Big|_-$ refers to the value of a function or derivative when approaching $x_2 = f(x_1)$ from the side $x_2 < f(x_1)$. The angular frequency is denoted by ω. The sound speed in the water is c_1 and in the sediment it is c_2. The wave attenuation in the sediment is modeled by the imaginary part of the complex-valued speed c_2. The densities for the water and the sediment are ρ_1 and ρ_2, respectively. The right-hand side g is non zero due to the point source.

On the boundary of the sound-soft target Ω we pose a Dirichlet boundary condition

$$p = 0 \qquad \text{on } \partial\Omega. \tag{2}$$

We denote the truncated rectangular domain by Π. On the artificial boundary $\partial\Pi$ we pose a second-order absorbing boundary condition

$$\frac{1}{\rho}\frac{\partial p}{\partial n} - i\frac{k}{\rho}p - i\frac{1}{2k}\frac{\partial}{\partial s}\frac{1}{\rho}\frac{\partial p}{\partial s} = 0 \tag{3}$$

on the faces of $\partial\Pi$ together with the condition $\partial p/\partial n = ik\frac{3}{2}p$ at the corners of $\partial\Pi$. In the previous n denotes the unit outward normal vector of $\partial\Pi$ and s denotes the unit tangent vector of $\partial\Pi$. Furthermore, the wave number function k and the density function ρ are defined by

$$k = \begin{cases} k_1,\, x_2 \geq f(x_1) \\ k_2,\, x_2 < f(x_1) \end{cases} \qquad \text{and} \qquad \rho = \begin{cases} \rho_1,\, x_2 \geq f(x_1) \\ \rho_2,\, x_2 < f(x_1). \end{cases}$$

A similar absorbing boundary condition for homogeneous media has been considered in [1].

3 Finite Element Discretization

We discretize the equations (1) together with the Dirichlet boundary condition (2) and the absorbing boundary condition (3) with linear finite elements. We use meshes which are orthogonal and uniform except near the target Ω and the interface. There it is locally perturbed so that the boundary $\partial\Omega$ is approximated well. An algorithm generating such meshes is presented in [3]. An example of a locally perturbed mesh is shown in Figure 2. The meshes have to be sufficiently fine, say, at least 10 grid points per the wave length, so that they can approximate properly the oscillatory solution. The discretization leads to a systems of linear equations

$$Ap = g, \tag{4}$$

where the matrix A is symmetric and complex-value, but not Hermitean.

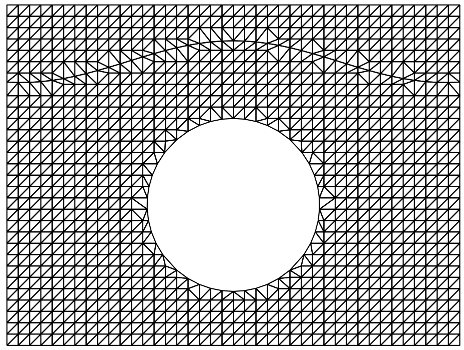

Fig. 2. A part of a locally perturbed mesh for a circular target and a sinusoidal surface of sediment.

4 Separable Preconditioner

We describe first the construction of our separable preconditioner and after that we consider in Section 5 the algebraic extension of the original system of linear equations (4) to have the same dimension as the preconditioner.

Domain embedding and fictitious domain methods are based on very efficient preconditioners on simple shaped domains. In our particular case the simple shaped domain is the whole rectangle Π, that is, we neglect the target Ω when we construct a preconditioner. Our separable preconditioner is based on the observation that the density ρ and the wave number k depend only on the x_2 coordinate for the perfectly layered media. Due to this we can express our preconditioner in a tensor product form

$$B = A_1 \otimes M_2 + M_1 \otimes (A_2 - \widetilde{M_2}).$$

This preconditioner coincides with the matrix obtained by discretizing the problem (1) without the target Ω together with the boundary condition (3) except on a part of the left and right boundary of Π. The dimension of the matrices A_1 and M_1 is the same as the number of nodes in the x_1 direction and they are given by

$$A_1 = \frac{1}{h} \begin{pmatrix} 1 - ihk/2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 - ihk/2 \end{pmatrix}$$

and

$$M_1 = h \begin{pmatrix} 1/2 + i/(2hk) & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1/2 + i/(2hk) \end{pmatrix}.$$

The matrices A_2, M_2, and $\widetilde{M}_2$ can be considered to correspond one-dimensional problems in the x_2 direction and their dimension is the number of nodes in the x_2 direction. They can be assembled from the element matrices

$$A_2^e = \frac{1}{h\rho_e} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad M_2^e = \frac{h}{2\rho_e} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \widetilde{M}_2^e = \frac{k_e^2 h}{2\rho_e} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where ρ_e and k_e are the density and the wave number on the element e. Due to the absorbing boundary condition the following additions have to made to these matrices: add $-ik/(2\rho)$ into the first and last diagonal entry of A_2, add $i/(2k\rho)$ into the first and last diagonal entry of M_2, and add $ik/(2\rho)$ into the first and last diagonal entry of $\widetilde{M}_2$. Systems of linear equations with the matrix B can be solved efficiently using, for example, the fast direct solver considered in [5].

5 Extended Linear System

Now we extend the original system of linear equations (4) to have the same size as the preconditioner B. We will accomplish this by using the so-called absorbing extension [9]. The idea is to pose another problem in Ω which is in this case a Helmholtz problem in Ω with an absorbing boundary condition on $\partial\Omega$. Furthermore, we introduce one sided coupling between the problems in $\Pi \setminus \Omega$ and Ω.

After a suitable permutation of rows and columns the preconditioner has a block form

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where the first block row corresponds to the unknowns outside Ω. Thus, B_{11} has the same size as A in (4). We denote the extended system matrix by C. It has the block form

$$C = \begin{pmatrix} A & B_{12} \\ 0 & B_{22} + D \end{pmatrix},$$

where D is a such diagonal matrix that $B_{22} + D$ corresponds to a Helmholtz problem in Ω with a first-order absorbing boundary condition on $\partial\Omega$. Particularly, the discretization is based on the orthogonal mesh without local adaptation to the boundary $\partial\Omega$. The extended system of linear equations reads

$$Cu = C \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} g \\ 0 \end{pmatrix} = f.$$

The vector q has to be zero, since the matrix block $B_{22} + D$ is non singular, and, thus, p satisfies also the original problem (4). For more details on the extension procedure we refer to [4, 6, 9].

6 Reduction to Sparse Subspace

We solve the right preconditioned system of linear equations

$$CB^{-1}v = f, \quad u = B^{-1}v. \tag{5}$$

Our sparse subspace X is defined by $X = \text{range}(C - B)$. The jth component x_k of an arbitrary vector x in X can be nonzero only if the jth row of B and C do not coincide. Hence, the subspace X is called sparse. For the problems considered in this paper the dimension of X is very small compared to the size of the linear system (5).

Next we consider the reduction to the sparse subspace in the case of general right-hand vector f. We denote $\hat{v} = v - f$ and then we have

$$CB^{-1}\hat{v} = f - CB^{-1}f = -(C - B)B^{-1}f = \hat{f} \in X$$

where we have used the identity $CB^{-1} = I + (C - B)B^{-1}$. Furthermore, $\hat{v}$ satisfies

$$\left[I + (C - B)B^{-1}\right]\hat{v} = \hat{f} \tag{6}$$

and $\hat{v} \in X$. The reduced equation (6) is well suited for iterating on the subspace X.

If $r \in X$ then the Krylov subspace

$$\text{span}\{r, CB^{-1}r, \cdots, (CB^{-1})^{k-1}\}$$

is a subspace of X. Thus, any iterative method based on the Krylov subspace for the solution of $CB^{-1}v = f$ generate a sequence of approximate solutions v^k in the subspace X provided that the initial iterate is $v^0 = f$. Moreover, the basic operation

$$(C - B)B^{-1}r, \quad r \in X$$

which is repeated during the iterations requires the solutions $B^{-1}r$ on the range of $(C - B)^T$. The dimension of this range is usually the same order as the dimension of X. Hence we apply the partial solution technique [2, 10] for this evaluation. This can reduce the computational cost of these solutions to be order of N floating point operations, where N is the size of the linear system (5).

7 Numerical Example

The geometry of our example problem is a cross cut of the experiment set up in [11]. The interface between water and sediment is given by $x_2 = (0.0368\,\mathrm{m})\cos(360°x_1/(0.75\,\mathrm{m}))$. The target is circular and its diameter is one feet (0.3048 m) and its center is at $(0\,\mathrm{m}, -0.2524\,\mathrm{m})$. Thus, the target is 0.1 m below the median level of the interface. The speed of sound in the water is $c_1 = 1495\,\mathrm{m/s}$ and the speed of the sound in the sediment is $c_2 = (1668 - 16.8i)\,\mathrm{m/s}$. Here, the imaginary part of the speed accounts for wave attenuation. The density for the water and sediment are $\rho_1 = 1000\,\mathrm{kg/m}^3$ and $\rho_2 = 2000\,\mathrm{kg/m}^3$, respectively. The point source is located at $(-10.7\,\mathrm{m}, 3.8\,\mathrm{m})$. We have chosen the frequency to be 20 kHz which corresponds about the wavelength 0.075 m in water.

Our computational domain is $[-12\,\mathrm{m}, 1\,\mathrm{m}] \times [-1\,\mathrm{m}, 4.5\,\mathrm{m}]$ and the mesh is based on 2601×1101 grid. Thus, the mesh step size in the direction of coordinate axes is 0.005 m and we have about 15 nodes per wavelength. We have plotted the scattered field intensity level in Figure 3. The computations were performed on a PowerBook G4 with 1.3 GHz processor and 0.5 Gbytes of memory. The solution required about 5 minutes. The preconditioned GMRES method needed 36 iterations to reduce the norm of the residual by the factor 10^{-6}. The extended linear system (5) has about 2.86 million unknowns while the dimension of the sparse subspace X is 18418. Thus, memory and computational savings due to the use of subspace iterations are indeed extensive.

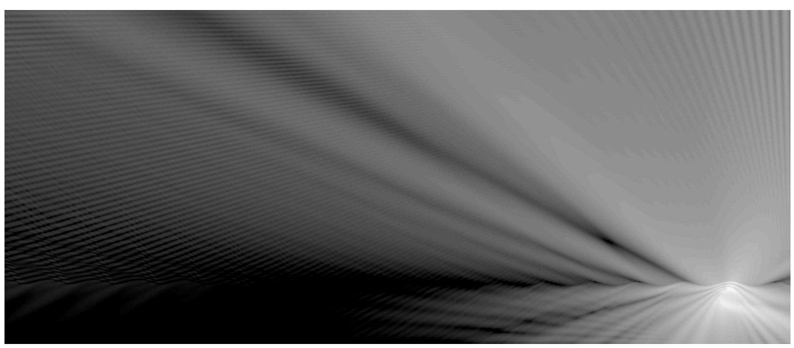

Fig. 3. The scattered field intensity level $\log_{10}|p_\mathrm{s}|^2$; the difference between white and black is 60 dB.

8 Conclusions and Future Research

We proposed a fast iterative method for computing the scattering in nearly vertically layered media. The main ingredients of our approach leading to computational efficiency are a fast direct solver for a separable preconditioner and a GMRES method iterating on a small sparse subspaces. The numerical example demonstrates that problems with millions of unknows can be solved on a contemporary PC in a few minutes.

For considering more practical problems several generalizations has to made. The proposed method can be extended in a straight forward manner for three-dimensional problems. Typical targets are elastic instead of sound-soft. In such a case one possible approach is to perform a domain decomposition to a small near-field domain and a large far-field domain. For far-field problems similar techniques to the one presented in this paper can used. Due to the small size of near-field problems more traditional approached are sufficiently fast for them.

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