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Talk: Time- and Space-Decomposition Methods for Parabolic Problems and Applications in Multiphysics Problems.

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Outline of the talk

1.) Introduction

2.) Decomposition-methods

3.) Time- and Space Decomposition methods

4.) Combined Time- and Space Decomposition Methods

5.) A priori Error-Estimate

6.) Numerical experiments

7.) Future Works

Motivation and Ideas

Decoupling of multi physics problems to simpler physics problems

Embedding the physical characteristics to the numerical methods (conservation of physics)

Parallelization and accelerating the solver-process

Higher order methods for time and space

Methods for non-smooth and degenerate problems

Fast computations for complicate and decoupable problems

Model-Equation

Systems of parabolic-differential equations with first order time-derivation and second order spatial-derivations

$$\frac{\partial c}{\partial t} = f(c) + Ac + Bc \ , \ \text{in } \Omega \times (0,T) \ , \quad (1)$$
$$c(x,t) = g(x,t) \ , \ \text{on } \partial\Omega \times (0,T) \ \text{(Boundary-Condition)} \ ,$$
$$c(x,0) = c_0(x) \ , \ \text{in } \Omega \ \text{(Initial-Condition)} \ ,$$

where $c = (c_1, \ldots, c_n)^t$ and $f(c) = (f_1(c), \ldots, f_n(c))^t$,

$$A = \begin{pmatrix} -v_{11} \cdot \nabla & \cdots & -v_{n1} \cdot \nabla \\ \cdots & \cdots & \cdots \\ -v_{1n} \cdot \nabla & \cdots & -v_{nn} \cdot \nabla \end{pmatrix}, B = \begin{pmatrix} \nabla D_{11} \cdot \nabla & \ldots & \nabla D_{n1} \cdot \nabla \\ \cdots & \cdots & \cdots \\ \nabla D_{1n} \cdot \nabla & \ldots & \nabla D_{nn} \cdot \nabla \end{pmatrix} \quad ,$$

Convection- and diffusion-operator with $A, B : X \rightarrow X$ and $X = \mathbb{R}^n$ a matrix-space.

sufficient smoothness $c_i \in C^{2,1}(\Omega, [0,T])$ for $i = 1, \ldots, n$

Decomposition Methods

Ideas :

Decoupling the time-scales, space-scales.

Decoupling the multi-physics.

Time-adaptivity, Space-adaptivity.

Parallelization in Time and Space.

Methods :

Operator-Splitting and Variational Splitting Methods (Time).

Iterative and extended Operator Splitting Methods (Time).

Waveform-Relaxation-Methods (Time).

Schwarz Wave form relaxation method (Space).

Additive and Multiplicative Schwarz method (Space).

Partition of Units combined with Splitting methods (Time and Space).

Spatial decomposition method : Overlapping Schwarz wave form relaxation method

Given the following model problem

$$u_t + Lu = f \text{ , in } \Omega \times (0, T) \text{ ,} \tag{2}$$

$$\overline{\Omega} \times (0, T) := \overline{\Omega}_1 \times (0, T) \cup \overline{\Omega}_2 \times (0, T) \text{ ,}$$

$$u(x, 0) = u_0 \text{ , (Initial-Condition) ,}$$

$$u = g \text{ , on } \partial\Omega \times (0, T) \text{ ,}$$

where L denotes for each time t a second-order partial differential operator $Lu = -\nabla D \nabla u + v \nabla u + cu$ for the given coeffients $D \in \mathbb{R}^+, v \in \mathbb{R}^n, c \in \mathbb{R}^+$, and n is the dimension of the space.

Schwarz-Waveform Relaxation method

We consider the method for two half steps, associated with the two subdomains and we solve 2 subproblems

$$\begin{aligned} & u_{1t} + L u_1^n = f \text{ , in } \Omega_1 \times (0,T) \ , \\ & u_1(x,0) = u_{10} \ , \ \text{(Initial-Condition)} \ , \\ & u_1^n = g \text{ , on } L_0 = \partial\Omega \times (0,T) \cap \partial\Omega_1 \times (0,T) \ , \\ & u_1^n = u_2^{n-1} \text{ , on } L_2 = \partial\Omega_1 \times (0,T) \backslash \partial\Omega \times (0,T) \ , \end{aligned} \tag{3}$$

$$\begin{aligned} & u_{2t} + L u_2^n = f \text{ , in } \Omega_2 \times (0,T) \ , \\ & u_2(x,0) = u_{20} \ , \ \text{(Initial-Condition)} \ , \\ & u_2^n = g \text{ , on } L_3 = \partial\Omega \times (0,T) \cap \partial\Omega_2 \times (0,T) \ , \\ & u_2^n = u_1^n \text{ , on } L_1 = \partial\Omega_2 \times (0,T) \backslash \partial\Omega \ \times (0,T); , \end{aligned} \tag{4}$$

Error of an Overlapping Schwarz wave form relaxation for the scalar convection reaction diffusion equation

We consider the convection diffusion reaction equation, given by

$$u_t = D u_{xx} - \nu u_x - \lambda u \ , \tag{5}$$

defined on the domain $\Omega = [0, L]$ for $T = [T_0, T_f]$, with the following initial and boundary conditions

$$u(0,t) = f_1(t), \quad u(L,t) = f_2(t), \quad u(x, T_0) = u_0 \ .$$

To solve the model problem using overlapping Schwarz wave form relaxation method, we subdivide the domain Ω in two overlapping sub-domains $\Omega_1 = [0, L_2]$ and $\Omega_2 = [L_1, L]$, where $L_1 < L_2$ and $\Omega_1 \bigcap \Omega_2 = [L_1, L_2]$ is the overlapping region for Ω_1 and Ω_2.

The convergence and error-estimates of $e^{k+1} = u - u_1^{k+1}$ and $d^{k+1} = u - u_2^{k+1}$ given by (3) and (4) respectively, are presented by the following theorem

Theorem 1. *Let e^{k+1} and d^{k+1} be the error from the solution of the subproblems (3) and (4) by Schwarz wave form relaxation over Ω_1 and Ω_2, respectively, then*

$$||e^{k+2}(L_1, t)||_\infty \leq \gamma ||e^k(L_1, t)||_\infty \, ,$$

and

$$||d^{k+2}(L_2, t)||_\infty \leq \gamma ||d^k(L_1, t)||_\infty \, ,$$

where

$$\gamma = \frac{\sinh(\beta L_1)}{\sinh(\beta L_2)} \frac{\sinh(\beta(L_2 - L)}{\sinh(\beta(L_1 - L))} < 1 \, ,$$

with $\beta = \frac{\sqrt{\nu^2 + 4D\lambda}}{2D}$.

Proof see [Geiser & Daoud, in review to NMPDE, 2006]

Time-Decomposition methods : Sequential Splitting methods

Idea: Decoupling of complex equations in simpler equations, solving simpler equations and re-coupling the results over the initial-conditions.

Equations: $\partial_t c = Ac + Bc$,
where the initial-conditions are $c(t^n) = c^n$, (or Variational-formulation: $(\partial_t c, v) = (Ac, v) + (Bc, v)$.)

Splitting-method of first order

$$\partial_t c^* = Ac^* \quad \text{with} \quad c^*(t^n) = c^n \ ,$$
$$\partial_t c^{**} = Bc^{**} \quad \text{with} \quad c^{**}(t^n) = c^*(t^{n+1}) \ ,$$

where the results of the methods are $c(t^{n+1}) = c^{**}(t^{n+1})$,
and there are some splitting-errors for these methods,
Literature : [Strang 68], [Karlsen et al 2001].

Splitting-Errors of the Method

The error of the splitting-method of first order is

$$\partial_t c = (B + A)c \; ,$$
$$\tilde{c} = \exp(\tau(B + A))c(t^n) \; .$$

Local error for the decomposition and the full solution

$$\begin{aligned} e(c) &= \tilde{c}(t^n + \tau) - \exp(\tau B)\exp(\tau A)c(t^n) \; , \\ &= \exp(\tau(B + A))c(t^n) - \exp(\tau B)\exp(\tau A)c(t^n) \; , \\ e(c)/\tau &= \frac{1}{2}\tau(BA - AB)c(t^n) + O(\tau^2) \; , \end{aligned}$$

$O(\tau)$ for A, B not commuting, otherwise one get exact results, where $\tau = t^{n+1} - t^n$, [Strang 68].

Higher order splitting-methods

Strang or Strang-Marchuk-Splitting, cf. [Marchuk 68, Strang68]

$$\frac{\partial c^*(t)}{\partial t} = Ac^*(t), \text{ with } t^n \leq t \leq t^{n+1/2} \text{ and } c^*(t^n) = c_{\mathrm{sp}}^n, \tag{6}$$

$$\frac{\partial c^{**}(t)}{\partial t} = Bc^{**}(t), \text{ with } t^n \leq t \leq t^{n+1} \,, c^{**}(t^n) = c^*(t^{n+1/2}),$$

$$\frac{\partial c^{***}(t)}{\partial t} = Ac^{***}(t) \,, t^{n+1/2} \leq t \leq t^{n+1} \,, c^{***}(t^{n+1/2}) = c^{**}(t^{n+1}),$$

where $t^{n+1/2} = t^n + 0.5\tau_n$ and the approximation on the next time level t^{n+1} is defined as $c_{\mathrm{sp}}^{n+1} = c^{***}(t^{n+1})$.

The splitting error of the Strang splitting is

$$\rho_n = \frac{1}{24}\tau_n^2([B,[B,A]] - 2[A,[A,B]])\, c(t^n) + O(\tau_n^3) \,, \tag{7}$$

see, e.g.[Hundsdorfer, Verwer 2003].

Combined Methods

Introduction Iterative splitting-Methods

$$\frac{\partial c_i(t)}{\partial t} = Ac_i(t) \; + \; Bc_{i-1}(t), \text{ with } \; c_i(t^n) = c_{\rm sp}^n, \tag{8}$$

$$\frac{\partial c_{i+1}(t)}{\partial t} = Ac_i(t) \; + \; Bc_{i+1}(t), \text{ with } \; c_{i+1}(t^n) = c_{\rm sp}^n, \tag{9}$$

where $c_0(t)$ is any fixed function for each iteration. (Here, as before, $c_{\rm sp}^n$ denotes the known split approximation at the time level $t = t^n$.) The split approximation at the time-level $t = t^{n+1}$ is defined as $c_{\rm sp}^{n+1} = c_{2m+1}(t^{n+1})$. (Clearly, the functions $c_k(t)$ $(k = i-1, i, i+1)$ depend on the interval $[t^n, t^{n+1}]$, too, but, for the sake of simplicity, in our notation we omit the dependence on n.)

Error for the Iterative splitting-method

Theorem 2. *The error for the splitting methods is given as :*

$$||e_i|| = K||B||\tau_n||e_{i-1}|| + O(\tau_n^2) \tag{10}$$

and hence

$$||e_{2m+1}|| = K_m||e_0||\tau_n^{2m} + O(\tau_n^{2m+1}), \tag{11}$$

where τ_n is the time-step, e_0 the initial error $e_0(t) = c(t) - c_0(t)$ and m the number of iteration-steps, K and K_m are constants, $||B||$ is the maximum norm of operator B and A and B are bounded, monotone operators.

Proof : Taylor-expansion and estimation of $\exp$-functions. See the work Geiser,Farago (2005).

The combined time-space iterative splitting method

Based on the iterative operator-splitting method we extend the splitting method be an embedded Schwarz-waveform-relaxation method.

We solve the following sub-problems consecutively for $i = 0, 2, \ldots 2m$ and $j = 0, 2, \ldots 2n$. In this notation i represents the iteration index for the time-splitting and j represents the iteration index for the spatial-splitting.

Initial idea:

$$\frac{\partial c_{i,j}(t)}{\partial t} = A|_{\Omega_1} c_{i,j}(t) \; + \; A|_{\Omega_2} c_{i,j-1}(t) \; + \; B|_{\Omega_1} c_{i-1,j}(t) \; + \; B|_{\Omega_2} c_{i-1,j-1}(t),$$
$$\text{with} \;\; c_{i,j}(t^n) = c^n \tag{12}$$

$$\frac{\partial c_{i+1,j}(t)}{\partial t} = A|_{\Omega_1} c_{i,j}(t) \; + \; A|_{\Omega_2} c_{i,j-1}(t) \; + \; B|_{\Omega_1} c_{i+1,j}(t) \; + \; B|_{\Omega_2} c_{i-1,j-1}(t),$$
$$\text{with} \;\; c_{i+1,j}(t^n) = c^n \tag{13}$$

$$\frac{\partial c_{i,j+1}(t)}{\partial t} = A|_{\Omega_1} c_{i,j}(t) \; + \; A|_{\Omega_2} c_{i,j+1}(t) \; + \; B|_{\Omega_1} c_{i+1,j}(t) \; + \; B|_{\Omega_2} c_{i-1,j-1}(t),$$
$$\text{with} \;\; c_{i,j+1}(t^n) = c^n \tag{14}$$

$$\frac{\partial c_{i+1,j+1}(t)}{\partial t} = A|_{\Omega_1} c_{i,j}(t) \; + \; A|_{\Omega_2} c_{i,j+1}(t) \; + \; B|_{\Omega_1} c_{i+1,j}(t) \; + \; B|_{\Omega_2} c_{i+1,j+1}(t),$$
$$\text{with} \;\; c_{i+1,j+1}(t^n) = c^n \tag{15}$$

where c^n is the known split approximation at the time level $t = t^n$.

The nonoverlapping time-space iterative splitting method

We denote for the semi-discretisation in space the variable k as the node for the point x_k and we obtain $k \in (0, \ldots, p)$, where p is the number of nodes. We have the decomposition if the space, where Ω_1 is of the points $0, \ldots, p/2$ and Ω_2 is of $p/2 + 1, \ldots, p$, we assume p is even. So we assume $\Omega_1 \cap \Omega_2 = \{\}$ and we have the following algorithm :

$$
\begin{aligned}
&\frac{\partial (c_{i,j})_k(t)}{\partial t} = \tilde{A}|_{\Omega_1}(c_{i,j})_k(t) \; + \; \tilde{A}|_{\Omega_2}(c_{i,j-1})_k(t) \\
&+ \tilde{B}|_{\Omega_1}(c_{i-1,j})_k(t) \; + \; \tilde{B}|_{\Omega_2}(c_{i-1,j-1})_k(t), \\
&\text{with } \; (c_{i,j})_k(t)(t^n) = (c^n)_k \qquad (16) \\
&\frac{\partial (c_{i+1,j})_k(t)}{\partial t} = \tilde{A}|_{\Omega_1}(c_{i,j})_k(t) \; + \; \tilde{A}|_{\Omega_2}(c_{i,j-1})_k(t) \\
&+ \tilde{B}|_{\Omega_1}(c_{i+1,j})_k \; + \; \tilde{B}|_{\Omega_2}(c_{i-1,j-1})_k(t), \\
&\text{with } \; (c_{i+1,j})_k(t^n) = (c^n)_k \qquad (17) \\
&\frac{\partial (c_{i,j+1})_k(t^n)(t)}{\partial t} = \tilde{A}|_{\Omega_1}(c_{i,j})_k(t^n)(t) \; + \; \tilde{A}|_{\Omega_2}(c_{i,j+1})_k(t^n)(t) \\
&+ \tilde{B}|_{\Omega_1}(c_{i+1,j})_k(t^n)(t) \; + \; \tilde{B}|_{\Omega_2}(c_{i-1,j-1})_k(t^n)(t), \\
&\text{with } \; (c_{i,j+1})_k(t^n)(t^n) = (c^n)_k(t^n) \qquad (18) \\
&\frac{\partial (c_{i+1,j+1})_k(t^n)(t)}{\partial t} = \tilde{A}|_{\Omega_1}(c_{i,j})_k(t^n)(t) \; + \; \tilde{A}|_{\Omega_2}(c_{i,j+1})_k(t^n)(t) \\
&+ \tilde{B}|_{\Omega_1}(c_{i+1,j})_k(t^n)(t) \; + \; \tilde{B}|_{\Omega_2}(c_{i+1,j+1})_k(t^n)(t), \\
&\text{with } \; (c_{i+1,j+1})_k(t^n)(t^n) = (c^n)_k(t^n) \qquad (19)
\end{aligned}
$$

where c^n is the known split approximation at the time level $t = t^n$.

We have the operators :

$$\tilde{A}|_{\Omega_1}(c_{i,j})_k = \begin{cases} A(c_{i,j})_k & \text{for } k \in \{0, \ldots, p/2\} \\ 0 & \text{for } k \in \{p/2+1, \ldots, p\} \end{cases} \quad (20)$$

$$\tilde{A}|_{\Omega_2}(c_{i,j})_k = \begin{cases} 0 & \text{for } k \in \{0, \ldots, p/2\} \\ A(c_{i,j})_k & \text{for } k \in \{p/2, \ldots, p\} \end{cases} \quad (21)$$

Similar are the assignments for operator B.

$$\tilde{B}|_{\Omega_1}(c_{i,j})_k = \begin{cases} B(c_{i,j})_k & \text{for } k \in \{0, \ldots, p/2\} \\ 0 & \text{for } k \in \{p/2+1, \ldots, p\} \end{cases} \quad (22)$$

$$\tilde{B}|_{\Omega_2}(c_{i,j})_k = \begin{cases} 0 & \text{for } k \in \{0, \ldots, p/2\} \\ B(c_{i,j})_k & \text{for } k \in \{p/2, \ldots, p\} \end{cases} \quad (23)$$

The overlapping time-space iterative splitting method

We denote for the semi-discretisation in space the variable k as the node for the point x_k and we obtain $k \in (0, \ldots, p)$, where p is the number of nodes. Now we assume the overlapping case, so we assume $\Omega_1 \cap \Omega_2 \neq \{\}$. We have the following sets : $\Omega \backslash \Omega_2 = \{0, \ldots, p_1\}$, $\Omega_1 \cap \Omega_2 = \{p_1 + 1, \ldots, p_2\}$ and $\Omega \backslash \Omega_1 = \{p_2 + 1, \ldots, p\}$. We assume $p_1 < p_2 < p$ and can derive the following overlapping algorithm :

$$\frac{\partial (c_{i,j})_k(t)}{\partial t} = \tilde{A}|_{\Omega\setminus\Omega_2}(c_{i,j})_k(t) \ + \ \tilde{A}|_{\Omega_1\cap\Omega_2}(c_{i,j}, c_{i,j-1})_k(t) \ + \ \tilde{A}|_{\Omega\setminus\Omega_1}(c_{i,j-1})_k(t)$$
$$+ \tilde{B}|_{\Omega\setminus\Omega_2}(c_{i-1,j})_k(t) \ + \ \tilde{B}|_{\Omega_1\cap\Omega_2}(c_{i-1,j}, c_{i-1,j-1})_k(t) + \ \tilde{B}|_{\Omega\setminus\Omega_1}(c_{i-1,j-1})_k(t),$$
$$\text{with} \ \ (c_{i,j})_k(t)(t^n) = (c^n)_k \qquad (24)$$
$$\frac{\partial (c_{i+1,j})_k(t)}{\partial t} = \tilde{A}|_{\Omega\setminus\Omega_2}(c_{i,j})_k(t) \ + \ \tilde{A}|_{\Omega_1\cap\Omega_2}(c_{i,j}, c_{i,j-1})_k(t) \ + \ \tilde{A}|_{\Omega\setminus\Omega_1}(c_{i,j-1})_k(t)$$
$$+ \tilde{B}|_{\Omega\setminus\Omega_2}(c_{i+1,j})_k(t) \ + \ \tilde{B}|_{\Omega_1\cap\Omega_2}(c_{i+1,j}, c_{i-1,j-1})_k(t) + \ \tilde{B}|_{\Omega\setminus\Omega_1}(c_{i-1,j-1})_k(t),$$
$$\text{with} \ \ (c_{i+1,j})_k(t^n) = (c^n)_k \qquad (25)$$
$$\frac{\partial (c_{i,j+1})_k(t)}{\partial t} = \tilde{A}|_{\Omega\setminus\Omega_2}(c_{i,j})_k(t) \ + \ \tilde{A}|_{\Omega_1\cap\Omega_2}(c_{i,j+1}, c_{i,j})_k(t) \ + \ \tilde{A}|_{\Omega\setminus\Omega_1}(c_{i,j+1})_k(t)$$
$$+ \tilde{B}|_{\Omega\setminus\Omega_2}(c_{i+1,j})_k(t) \ + \ \tilde{B}|_{\Omega_1\cap\Omega_2}(c_{i+1,j}, c_{i-1,j-1})_k(t) + \ \tilde{B}|_{\Omega\setminus\Omega_1}(c_{i-1,j-1})_k(t),$$
$$\text{with} \ \ (c_{i,j+1})_k(t^n)(t^n) = (c^n)_k(t^n) \qquad (26)$$
$$\frac{\partial (c_{i+1,j+1})_k(t)}{\partial t} = \tilde{A}|_{\Omega\setminus\Omega_2}(c_{i,j})_k(t) \ + \ \tilde{A}|_{\Omega_1\cap\Omega_2}(c_{i,j+1}, c_{i,j})_k(t) \ + \ \tilde{A}|_{\Omega\setminus\Omega_1}(c_{i,j+1})_k(t)$$
$$+ \tilde{B}|_{\Omega\setminus\Omega_2}(c_{i+1,j})_k(t) \ + \ \tilde{B}|_{\Omega_1\cap\Omega_2}(c_{i+1,j}, c_{i+1,j+1})_k(t) + \ \tilde{B}|_{\Omega\setminus\Omega_1}(c_{i+1,j+1})_k(t),$$
$$\text{with} \ \ (c_{i+1,j+1})_k(t^n)(t^n) = (c^n)_k(t^n) \ . \qquad (27)$$

We have the operators :

$$\tilde{A}|_{\Omega\setminus\Omega_2}(c_{i,j})_k = \begin{cases} A(c_{i,j})_k & \text{for } k \in \{0,\dots,p_1\} \\ 0 & \text{for } k \in \{p_1+1,\dots,p\} \end{cases} \quad (28)$$

$$\tilde{A}|_{\Omega_1\cap\Omega_2}(c_{i,j}, c_{i,j+1})_k = \begin{cases} A((c_{i,j}+c_{i,j+1})/2)_k & \text{for } k \in \{p_1+1,\dots,p_2\} \\ 0 & \text{for } k \in \{p_2+1,\dots,p\} \end{cases} \quad (29)$$

$$\tilde{A}|_{\Omega\setminus\Omega_1}(c_{i,j})_k = \begin{cases} 0 & \text{for } k \in \{0,\dots,p_2\} \\ A(c_{i,j})_k & \text{for } k \in \{p_2+1,\dots,p\} \end{cases} \quad (30)$$

Similar are the assignments for operator B.

$$\tilde{B}|_{\Omega\setminus\Omega_2}(c_{i,j})_k = \begin{cases} B(c_{i,j})_k & \text{for } k \in \{0,\dots,p_1\} \\ 0 & \text{for } k \in \{p_1+1,\dots,p\} \end{cases} \quad (31)$$

$$\tilde{B}|_{\Omega_1\cap\Omega_2}(c_{i,j}, c_{i,j+1})_k = \begin{cases} B((c_{i,j}+c_{i,j+1})/2)_k & \text{for } k \in \{p_1+1,\dots,p_2\} \\ 0 & \text{for } k \in \{p_2+1,\dots,p\} \end{cases} \quad (32)$$

$$\tilde{B}|_{\Omega\setminus\Omega_1}(c_{i,j})_k = \begin{cases} 0 & \text{for } k \in \{0,\dots,p_2\} \\ B(c_{i,j})_k & \text{for } k \in \{p_2+1,\dots,p\} \end{cases} \quad (33)$$

Dicretisation of the operators

The discretization of the operators is given as :

$$\begin{aligned} A(c_{i,j})_k &= D/(\Delta x)^2(-(c_{i,j})_{k+1} + 2(c_{i,j})_k - (c_{i,j})_{k-1}) \\ &-v/\Delta x((c_{i,j})_k - (c_{i,j})_{k-1}) \end{aligned} \quad (34)$$

$$B(c_{i,j})_k = \lambda(c_{i,j})_k \ , \quad (35)$$

Consistency and stability analysis of the combined method

Theorem 3. *Let us consider the nonlinear operator-equation in a Banach space* **X**

$$\partial_t c(t) = A_1(c(t)) + A_2(c(t)) + B_1(c(t)) + B_2(c(t)), \quad 0 < t \leq T \; ,$$
$$c(0) = c_0 \; , \tag{36}$$

where $A_1, A_2, B_1, B_2, A_1 + A_2 + B_1 + B_2 : \mathbf{X} \to \mathbf{X}$ *are given linear operators being generators of the* C_0*-semigroup and* $c_0 \in \mathbf{X}$ *is a given element. Then the iteration process (12)–(15) is convergent and the rate of the convergence is of second order.*

We obtain the iterative result :

we obtain

$$\|e_{i,j}\| = K \tau_n \|e_{i-1,j-1}\| + \mathcal{O}(\tau_n^2), \tag{37}$$

and hence

$$\|e_{i+1,j+1}\| = K_1 \tau_n^2 \|e_{i-1,j-1}\| + \mathcal{O}(\tau_n^3), \quad (38)$$

which proves our statement.

Proof see [Geiser & Kravvaritis 2006]

Let us consider the iteration (12)–(15) on the sub-interval $[t^n, t^{n+1}]$. For the error function $e_i(t) = c(t) - c_i(t)$ we have the relations

$$\partial_t e_{i,j}(t) = A_1(e_{i,j}(t)) + A_2(e_{i,j-1}(t)) \quad (39)$$

$$+B_1(e_{i-1,j}(t)) + B_2(e_{i-1,j-1}(t)),$$

$$t \in (t^n, t^{n+1}],\ e_{i,j}(t^n) = 0\ , \quad (40)$$

and

$$\partial_t e_{i+1,j}(t) = A_1(e_{i,j}(t)) + A_2(e_{i,j-1}(t)) \tag{41}$$

$$+B_1(e_{i+1,j}(t)) + B_2(e_{i-1,j-1}(t)),$$

$$t \in (t^n, t^{n+1}],\ e_{i+1,j}(t^n) = 0\ , \tag{42}$$

and

$$\partial_t e_{i,j+1}(t) = A_1(e_{i,j}(t)) + A_2(e_{i,j+1}(t)) \tag{43}$$

$$+B_1(e_{i+1,j}(t)) + B_2(e_{i-1,j-1}(t)),$$

$$t \in (t^n, t^{n+1}],\ e_{i,j+1}(t^n) = 0\ , \tag{44}$$

and

$$\partial_t e_{i,j}(t) = A_1(e_{i,j}(t)) + A_2(e_{i,j+1}(t)) \tag{45}$$

$$+B_1(e_{i+1,j}(t)) + B_2(e_{i+1,j+1}(t)),$$

$$t \in (t^n, t^{n+1}],\ e_{i,j}(t^n) = 0\ , \tag{46}$$

for $i, j = 0, 2, 4, \ldots,$ with $e_{0,0}(0) = 0$ and $e_{-1,0} = e_{0,-1} = e_{-1,-1}(t) = c(t)$.

In the following we derive the linear system of equations. We use the notations $\mathbf{X}^2$ for the product space $\mathbf{X} \times \mathbf{X}$ enabled with the norm $\|(u,v)\| = \max\{\|u\|, \|v\|\}$ $(u, v \in \mathbf{X})$. The elements $\mathcal{E}_i(t)$, $\mathcal{F}_i(t) \in \mathbf{X}^2$ and the linear operator $\mathcal{A} : \mathbf{X}^2 \rightarrow \mathbf{X}^2$ are defined as follows

$$\mathcal{E}_{i,j}(t) = \begin{bmatrix} e_{i,j}(t) \\ e_{i+1,j}(t) \\ e_{i,j+1}(t) \\ e_{i+1,j+1}(t) \end{bmatrix}; \quad \mathcal{A} = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ A_1 & A_2 & 0 & 0 \\ A_1 & A_2 & B_1 & 0 \\ A_1 & A_2 & B_1 & B_2 \end{bmatrix}, \tag{47}$$

$$\mathcal{F}_{i,j}(t) = \begin{bmatrix} A_2(e_{i,j-1}(t)) + B_1(e_{i-1,j}(t)) + B_2(e_{i-1,j-1}) \\ B_1(e_{i-1,j}(t)) + B_2(e_{i-1,j-1}) \\ B_2(e_{i-1,j-1}) \\ 0 \end{bmatrix}. \tag{48}$$

Then, using the notations (48), the relations (40)–(46) can be written in the form

$$\begin{aligned} \partial_t \mathcal{E}_{i,j}(t) &= \mathcal{A}\mathcal{E}_{i,j}(t) + \mathcal{F}_{i,j}(t), \quad t \in (t^n, t^{n+1}], \\ \mathcal{E}_{i,j}(t^n) &= 0. \end{aligned} \tag{49}$$

Due to our assumptions, $\mathcal{A}$ is a generator of the one-parameter C_0 semigroup $(\mathcal{A}(t))_{t\geq 0}$.

Hence using the variations of constants formula, the solution of the abstract Cauchy problem (49) with homogeneous initial condition can be written as

$$\mathcal{E}_{i,j}(t) = \int_{t^n}^{t} \exp(\mathcal{A}(t-s))\mathcal{F}_{i,j}(s)ds, \quad t \in [t^n, t^{n+1}]. \tag{50}$$

Hence, using the denotation

$$\|\mathcal{E}_{i,j}\|_\infty = \sup_{t\in[t^n,t^{n+1}]} \|\mathcal{E}_{i,j}(t)\| \quad , \tag{51}$$

We could estimate the right hand side $\mathcal{F}_i(t)$ and $\exp(\mathcal{A}(t))$

We could then estimate the $\mathcal{F}_i(t)$ as

$$||\mathcal{F}_{i,j}(t)|| \leq C||e_{i-1,j-1}|| \ . \tag{52}$$

and

$$\int_{t^n}^{t^{n+1}} \|\exp(\mathcal{A}(t-s))\| ds \leq K_\omega(t), \quad t \in [t^n, t^{n+1}], \tag{53}$$

and hence

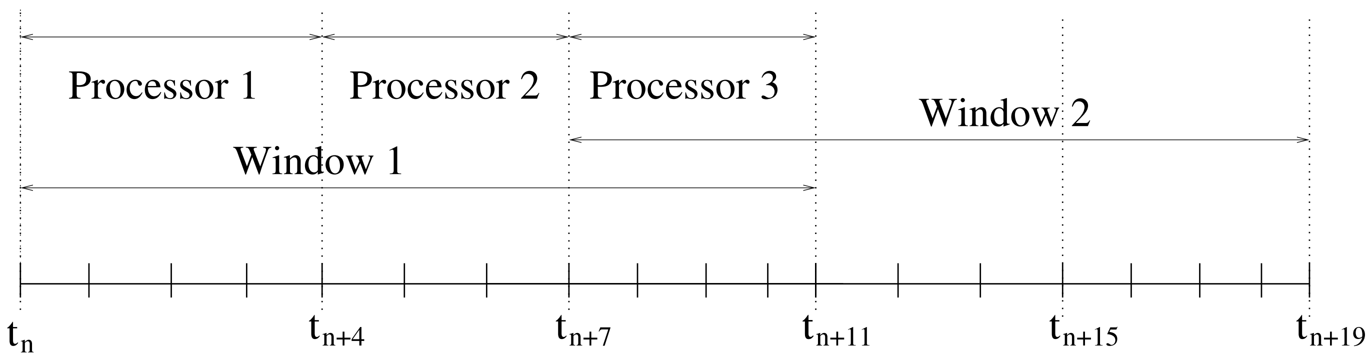
$$K_\omega(t) \leq \frac{K}{\omega} \left(\exp(\omega \tau_n) - 1\right) = K\tau_n + \mathcal{O}(\tau_n^2) \ , \tag{54}$$

We obtain the a priori error-estimates

$$\|e_{i,j}\| = K\tau_n \|e_{i-1,j-1}\| + \mathcal{O}(\tau_n^2) \ . \tag{55}$$

Parallelization of the Time-Decomposition method : Windowing

The idea for parallelization in time are the windowing, that the processors has an amount of time-steps to compute and to share the end-result of the computation as an initial-condition for the next processor.



Numerical Experiments

We consider the one-dimensional convection-reaction-diffusion equation

$$\partial_t u + v\partial_x u - \partial_x D \partial_x u = -\lambda u \ , \ \text{in } \Omega \times (T_0, T_f) \ , \quad (56)$$

$$u(x, 0) = u_{ex}(x, 0) \ , \ \text{(Initial-Condition)} \ , \quad (57)$$

$$u(x, t) = u_{ex}(x, t) \ , \text{on } \partial\Omega \times (T_0, T_f) \ , \quad (58)$$

where $\Omega \times [T_0, T_f] = [0, 150] \times [100, 10^5]$.

The exact solution is given as

$$u_{ex}(x, t) = \frac{u_0}{2\sqrt{D\pi t}} \exp(-\frac{(x - vt)^2}{4Dt}) \exp(-\lambda t) \ . \quad (59)$$

The initial condition and the Dirichlet boundary conditions are defined using the exact solution (59) at starting time $T_0 = 100$ and with $u_0 = 1.0$. We have $\lambda = 10^{-5}$, $v = 0.001$ and $D = 0.0001$.

First example : A-B splitting combined with Schwarz wave form relaxation method

In order to solve the model problem using overlapping Schwarz wave form relaxation method, we divide the domain Ω in two overlapping sub-domains $\Omega_1 = [0, L_2]$ and $\Omega_2 = [L_1, L]$, where $L_1 < L_2$, and $\Omega_1 \bigcap \Omega_2 = [L_1, L_2]$ is the overlapping region for Ω_1 and Ω_2.

For the sequential operator splitting method (A-B splitting). For this purpose we divide each of these two equations in terms of the operators $A = D\frac{\partial^2}{\partial x^2} - \nu\frac{\partial}{\partial x}$ and $B = -\lambda$.

For the discretization of equation (6) we apply the finite-difference method for the spatial discretization and the implicite Euler method for the time discretization.

We provide a variety of results for several sizes of space- and time-partition, and also for various overlap sizes.

Precisely, we treat the cases $h = 1,\ 0.5,\ 0.25$ as spatial step-size, $\Delta t = 5,\ 10,\ 20$ as time step.

The considered subdomains are $\Omega_1 = [0, 80]$ and $\Omega_2 = [70, 150]$, $\Omega_1 = [0, 60]$ and $\Omega_2 = [30, 150]$ and $\Omega_1 = [0, 100]$ and $\Omega_2 = [30, 150]$, with overlap sizes 10, 30 and 70, respectively.

Both the approximated and the exact solution are evaluated at the end-time $t = 10^5$. The errors given in Table 2 are the maximum errors that occurred over the whole space domain, i.e. they are calculated using the $\infty-$norm for vectors.

time-step	err	err	err	err	err	err
$\Delta t = 5$	$2.85e-3$	$2.24e-3$	$1.28e-3$	$2.66e-4$	$2.21e-4$	$2.20e-4$
$\Delta t = 10$	$3.94e-3$	$2.61e-3$	$2.56e-3$	$3.03e-4$	$3.02e-4$	$3.01e-4$
$\Delta t = 20$	$5.03e-3$	$2.81e-3$	$2.73e-3$	$8.51e-4$	$5.22e-4$	$5.14e-4$
overlap	10	30	70	10	30	70
space-step	$h = 1$			$h = 0.5$		

Table 1: Error for the scalar convection diffusion reaction-equation using the Schwarz waveform relaxation method for three different sizes of overlapping 10,30 and 70.

time-step	err	err	err
$\Delta t = 5$	$2.09e-5$	$1.99e-5$	$1.97e-5$
$\Delta t = 10$	$4.55e-5$	$4.34e-5$	$4.29e-5$
$\Delta t = 20$	$8.10e-4$	$5.66e-4$	$4.88e-4$
overlap	10	30	70
space-step	$h = 0.25$		

Table 2: Error for the scalar convection diffusion reaction-equation using the Schwarz waveform relaxation method for three different sizes of overlapping 10,30 and 70.

Second Example : Combined method : Time-Space iterative operator splitting method

For the solution of (56) with the combined time-space iterative splitting method we divide again the equation in terms of the operators

$$A = D\frac{\partial^2}{\partial x^2} - \nu\frac{\partial}{\partial x}$$

and

$$B = -\lambda.$$

The index $k = 0, 1, \dots p$ is associated with the subdomains, i.e. for $k = 0, \dots, p/2$ we are working on Ω_1 and for $k = p/2 + 1, \dots, p$ on Ω_2. For the first set of values for k we have actually only the effect of the restrictions of the operators A and B on Ω_1. Similarly, the second set of values for k indicates the action of the restrictions of both operators on Ω_2.

The indices i and j are related to the time- and space-discretization, respectively. For every $k = 0, \ldots, p/2$ and for every interval of the space-discretization we solve the appropriate problems on Ω_1, for every interval of the time-discretization. Similarly for $k = p/2+1, \ldots, p$ on Ω_2.

By a closer examination of the scheme (24)–(27), taking into account the definitions (32)–(23), we observe that the problems to be solved in the innermost loop are of the form $\partial_t c = Ac + Bc,\ c(x, t^n) = c^n$, where c appears with appropriate indices i and j.

Both the approximated and the exact solution are evaluated at the end-time $t = 10^5$. The errors given in the following tables are the maximum errors that occurred over the whole space domain, i.e. they are calculated using the $\infty-$norm for vectors.

The results are given in Table 4.

time-step	err	err	err	err	err	err
$\Delta t = 5$	$4.38e-2$	$1.47e-2$	$3.49e-3$	$2.59e-4$	$2.13e-4$	$1.54e-4$
$\Delta t = 10$	$5.12e-2$	$2.26e-2$	$7.46e-3$	$2.45e-4$	$2.22e-4$	$2.15e-4$
$\Delta t = 20$	$6.14e-2$	$4.39e-2$	$1.20e-2$	$7.43e-4$	$5.21e-4$	$4.53e-4$
overlap	10	30	70	10	30	70
space-step	$h = 1$			$h = 0.5$		

Table 3: Error for the scalar convection diffusion reaction-equation using the Schwarz waveform relaxation method for three different sizes of overlapping 10,30 and 70.

time-step	err	err	err
$\Delta t = 5$	$7.23e-6$	$6.49e-6$	$8.29e-6$
$\Delta t = 10$	$3.49e-5$	$3.47e-5$	$3.37e-5$
$\Delta t = 20$	$5.23e-4$	$5.42e-4$	$3.21e-4$
overlap	10	30	70
space-step	$h = 0.25$		

Table 4: Error for the scalar convection diffusion reaction-equation using the Schwarz waveform relaxation method for three different sizes of overlapping 10,30 and 70.

Future Work

1. Theory for the Stability of the time-space iterative splitting methods.
2. Commutative, non-commutative theory : How to decouple
3. Degenerated problems and non-smooth problems
4. Numerical examples