Finite volume method for linear and non linear elliptic problems with discontinuities

Franck BOYER et Florence HUBERT

L.A.T.P. - Marseille, FRANCE

DD17 - 2006

Plan

PLAN

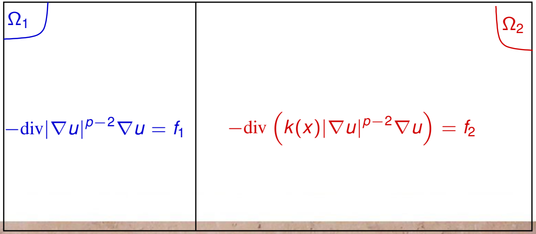
INTRODUCTION.

- DDFV scheme (DISCRETE DUALITY FINITE VOLUME) for

$$\begin{cases} -\operatorname{div}\left(\varphi(z, \nabla u_e(z))\right) = f(z), & \text{in } \Omega, \\ u_e = 0, \ \text{on } \partial\Omega, \end{cases} \tag{1}$$

  - Ω polygonal open set in $\mathbb{R}^2$.
  - $u \mapsto -\operatorname{div}(\varphi(\cdot, \nabla u))$ is coercitive, monotonous (of Leray-Lions type).
  - φ presents discontinuities with respect to the space variable z (Transmission problem).

Example :

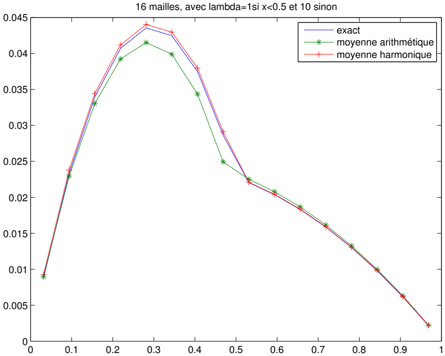


INTRODUCTION

With discontinuities the DDFV scheme converges but slowly :

In the linear case :

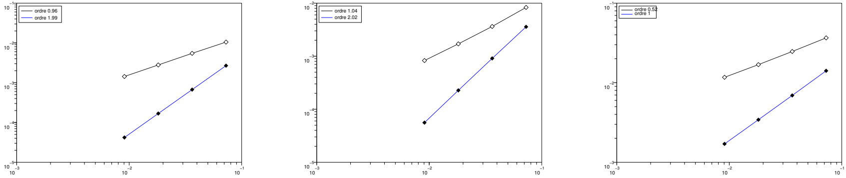
- 1D : $-(\lambda^{\pm} u_e')' = f$
  - Arythmetic mean-value : order $\frac{1}{2}$
  - Harmonic mean-value : order 1



INTRODUCTION

With discontinuities, the DDFV scheme converges but slowly :
In the linear case :

- 2D : $-\mathrm{div}(A(z)\nabla u_e) = f$
  - DDFV scheme : order $\frac{1}{2}$
  - Improved DDFV scheme (Hermeline, BH) : order 1



L^∞ error $\quad$ L^2 error $\quad$ H^1 error

$$A = \begin{pmatrix} 10 & 2 \\ 2 & 1 \end{pmatrix}.$$

BIBLIOGRAPHIE

- **Anisotropy problems with discontinuities**
  - EGH (00), Hermeline (03)
- **Gradient reconstruction problems**
  - "O-scheme", "U-scheme" Aavatsmark, Lepotier,......
  - Gradient FV schemes. Eymard, Gallouët, Herbin,...
  - Mixed FV schemes. Droniou, Eymard (06)
  - DDFV schemes. Coudière (99), Hermeline (00), Domelevo & Omnès (05), ABH (06), Pierre (06), Delcourte & al (06).......

HYPOTHÈSES SUR φ

- Let $p \in]1, \infty[$, $p' = \frac{p}{p-1}$ and $f \in L^{p'}(\Omega)$. ► $p \geq 2$ in this talk.
- $\varphi : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$ is a Caratheodory function such that :

$$(\varphi(z,\xi),\xi) \geq C_\varphi \left(|\xi|^p - 1\right), \qquad (\mathcal{H}_1)$$

$$|\varphi(z,\xi)| \leq C_\varphi \left(|\xi|^{p-1} + 1\right). \qquad (\mathcal{H}_2)$$

$$(\varphi(z,\xi) - \varphi(z,\eta), \xi - \eta) \geq \frac{1}{C_\varphi}|\xi - \eta|^p. \qquad (\mathcal{H}_3)$$

$$|\varphi(z,\xi) - \varphi(z,\eta)| \leq C_\varphi \left(1 + |\xi|^{p-2} + |\eta|^{p-2}\right)|\xi - \eta|. \qquad (\mathcal{H}_4)$$

- φ is piecewisely lipschitz in $z \Rightarrow$ Assumption $(\mathcal{H}_5)$.

PLAN

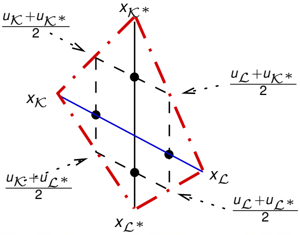
THE DDFV SCHEME

- **The discrete unknowns :**

$$u^{\mathcal{T}} = \left(u^{\mathfrak{M}}, u^{\mathfrak{M}^*}\right) \text{ where } u^{\mathfrak{M}} = (u_{\mathcal{K}})_{\mathcal{K}\in\mathfrak{M}}, u^{\mathfrak{M}^*} = (u_{\mathcal{K}^*})_{\mathcal{K}^*\in\mathfrak{M}^*}$$

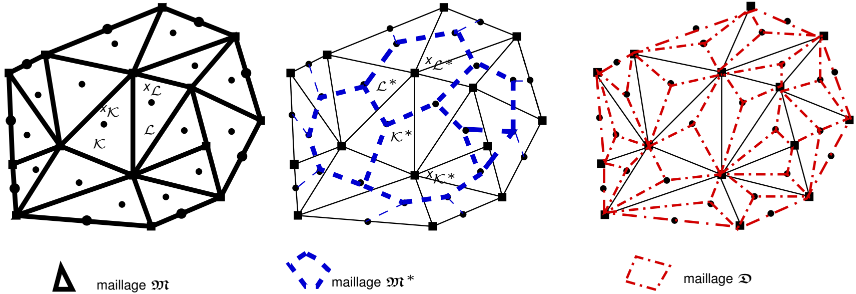
- **The discrete gradient** : $\nabla^{\mathcal{T}} u^{\mathcal{T}}$ is constant on diamond cells

$$\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}} = \frac{1}{\sin\alpha_{\mathcal{D}}}\left(\frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{|\sigma^*|}\boldsymbol{\nu} + \frac{u_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{|\sigma|}\boldsymbol{\nu}^*\right), \quad \forall \mathcal{D}.$$



THE DDFV MESHES

primal, dual and "diammond".



THE DDFV SCHEME

$$-\sum_{\mathcal{D}_{\sigma,\sigma^*}\cap\mathcal{K}\neq\emptyset} |\sigma| \left(\varphi_{\mathcal{D}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}), \boldsymbol{\nu}_{\mathcal{K}}\right) = \int_{\mathcal{K}} f(z)\,dz, \quad \forall \mathcal{K} \in \mathfrak{M},$$

$$-\sum_{\mathcal{D}_{\sigma,\sigma^*}\cap\mathcal{K}^*\neq\emptyset} |\sigma^*| \left(\varphi_{\mathcal{D}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}), \boldsymbol{\nu}_{\mathcal{K}^*}\right) = \int_{\mathcal{K}^*} f(z)\,dz, \forall \mathcal{K}^* \in \mathfrak{M}^*,$$

with

$$\varphi_{\mathcal{D}}(\xi) = \frac{1}{|\mathcal{D}|}\int_{\mathcal{D}} \varphi(z,\xi)\,dz.$$

Discrete Duality formulation :

$$2\sum_{\mathcal{D}\in\mathfrak{D}} |\mathcal{D}| \left(\varphi_{\mathcal{D}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}), \nabla_{\mathcal{D}}^{\mathcal{T}} v^{\mathcal{T}}\right) = \int_{\Omega} f v^{\mathfrak{M}} dz + \int_{\Omega} f v^{\mathfrak{M}^*} dz, \;\; \forall v^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}.$$

KNOWN RESULTS

- Case $p = 2$: Domelevo & Omnès $\Rightarrow$ Estimate in $O(h)$ for a large class of meshes.
- General case : Andreianov, Boyer & Hubert
  $\Rightarrow$ The scheme converges.

  $\Rightarrow$ If $u_e \in W^{2,p}(\Omega)$ and

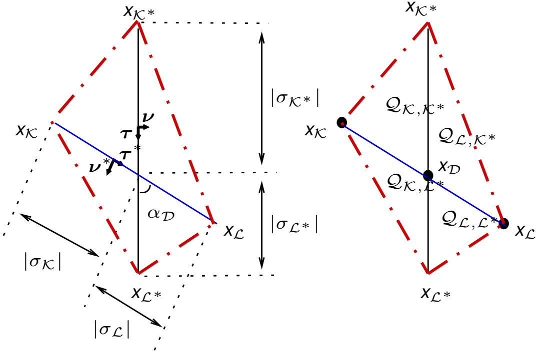
  φ Lip. on Ω, with $\left|\frac{\partial \varphi}{\partial z}(z,\xi)\right| \leq C_\varphi \left(1 + |\xi|^{p-1}\right), \quad \forall \xi \in \mathbb{R}^2. \quad (\mathcal{H}_5)$

  Then

$$\|u_e - u^{\mathfrak{M}}\|_{L^p} + \|u_e - u^{\mathfrak{M}^*}\|_{L^p} + \|\nabla u_e - \nabla^{\mathcal{T}} u^{\mathcal{T}}\|_{L^p} \leq C \,\mathrm{size}(\mathcal{T})^{\frac{1}{p-1}}.$$

Zoom on the diamond cells

- Diamond cells are supposed to be convex.
- Each diamond cell is cut into four triangles $\mathcal{Q}$.



AIMS

If φ presents some discontinuities on a curve Γ

- $u_e \notin W^{2,p}(\Omega)$.
- The numerical fluxes are no more consistent along Γ .

We assume that on each $\mathcal{Q}$, φ is Lip. and saitifies $(\mathcal{H}_5)$.

- We want to get the consistency of the fluxes on Γ.
- We contruct a new approximation $\varphi_{\mathcal{D}}^{\mathcal{N}}$ of the non linearity on the diamond cell $\mathcal{D}$.

$$-\sum_{\mathcal{D}_{\sigma,\sigma^*}\cap\mathcal{K}\neq\emptyset} |\sigma| \left(\varphi_{\mathcal{D}}^{\mathcal{N}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}), \boldsymbol{\nu}_{\mathcal{K}}\right) = \int_{\mathcal{K}} f(z)\,dz, \quad \forall \mathcal{K} \in \mathfrak{M}$$

$$-\sum_{\mathcal{D}_{\sigma,\sigma^*}\cap\mathcal{K}^*\neq\emptyset} |\sigma^*| \left(\varphi_{\mathcal{D}}^{\mathcal{N}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}), \boldsymbol{\nu}_{\mathcal{K}^*}\right) = \int_{\mathcal{K}^*} f(z)\,dz, \ \forall \mathcal{K}^* \in \mathfrak{M}^*$$

PLAN

THE 1D PROBLEM

$$\Omega =]-1,1[, \ \varphi(x,\cdot) = \begin{cases} \varphi_-(\cdot), \text{if } x < 0, \\ \varphi_+(\cdot), \text{if } x > 0 \end{cases} .$$

Let $x_0 = -1 < \ldots < x_N = 0 < \ldots < x_{N+M} = 1$ a discretization of $[-1,1]$.
The 1D FV scheme reads for $i \in \{0, N+M-1\}$:

$$-F_{i+1} + F_i = \int_{x_i}^{x_{i+1}} f(x)\, dx. \qquad (2)$$

with

$$F_i = \varphi(x_i, \nabla_i u^{\mathcal{T}}), \ \ \nabla_i u^{\mathcal{T}} = \frac{u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}}, \ \ \forall i \neq N, \qquad (3)$$

QUESTION : how to define F_N ?

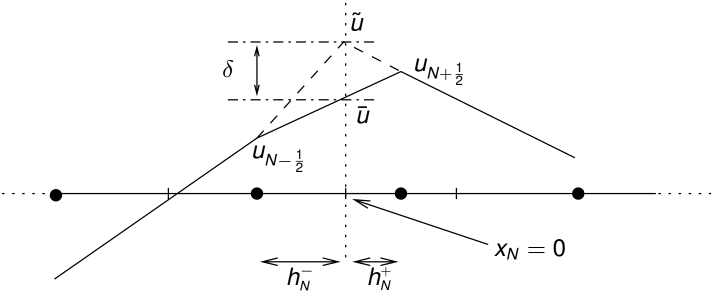
THE NEW GRADIENT

We look for $\tilde{u}$ such that

$$\nabla_N^+ u^{\mathcal{T}} = \frac{u_{N+\frac{1}{2}} - \tilde{u}}{h_N^+}, \quad \nabla_N^- u^{\mathcal{T}} = \frac{\tilde{u} - u_{N-\frac{1}{2}}}{h_N^-},$$

we have

$$\varphi_-(\nabla_N^+ u^{\mathcal{T}}) = \varphi_+(\nabla_N^- u^{\mathcal{T}}).$$



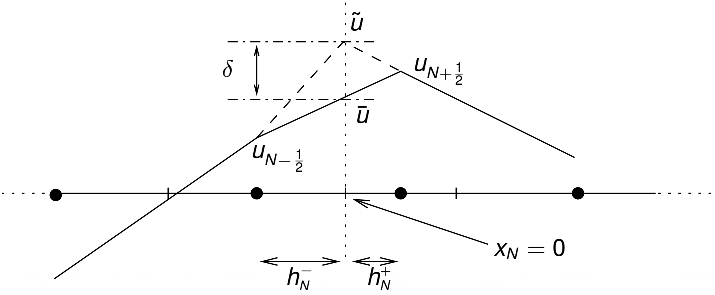
The new gradient

We look for $\tilde{u}$ in the form

$$\tilde{u} = \bar{u} + \delta, \quad \text{with } \bar{u} = \frac{h_N^- u_{N+\frac{1}{2}} + h_N^+ u_{N-\frac{1}{2}}}{h_N^- + h_N^+}.$$

that is

$$\nabla_N^+ u^{\mathcal{T}} = \nabla_N u^{\mathcal{T}} - \frac{\delta}{h_N^+}, \quad \text{and } \nabla_N^- u^{\mathcal{T}} = \nabla_N u^{\mathcal{T}} + \frac{\delta}{h_N^-}.$$



The new gradient

Theorem

- *For all* $u^{\mathcal{T}} \in \mathbb{R}^N$, *there exists a unique* $\delta_N(\nabla_N u^{\mathcal{T}})$ *such that*

$$F_N \stackrel{\text{def}}{=} \varphi_-\left(\nabla_N u^{\mathcal{T}} + \frac{\delta_N(\nabla_N u^{\mathcal{T}})}{h_N^-}\right) = \varphi_+\left(\nabla_N u^{\mathcal{T}} - \frac{\delta_N(\nabla_N u^{\mathcal{T}})}{h_N^+}\right),$$

- *The new scheme admits a unique solution.*
- *The flux* F_N *is consistent with an error in* $h^{\frac{1}{p-1}}$.

Example

For two $p-$laplacian fluxes

$$\varphi_-(\xi) = k_-|\xi + G_-|^{p-2}(\xi + G_-), \quad \text{and} \quad \varphi_+(\xi) = k_+|\xi + G_+|^{p-2}(\xi + G_+),$$

where $k_-, k_+ \in \mathbb{R}^+$ and $G_-, G_+ \in \mathbb{R}^2$. We obtain

$$F_N = \left(\frac{k_-^{\frac{1}{p-1}} k_+^{\frac{1}{p-1}} (h_N^- + h_N^+)}{h_N^+ k_-^{\frac{1}{p-1}} + h_N^- k_+^{\frac{1}{p-1}}} \right)^{p-1} \left| \nabla_N u^{\mathcal{T}} + \overline{G} \right|^{p-2} \left(\nabla_N u^{\mathcal{T}} + \overline{G} \right),$$

where $\overline{G}$ is some arythmetic mean value of G_- and G_+ defined by

$$\overline{G} = \frac{h_N^- G_- + h_N^+ G_+}{h_N^- + h_N^+}.$$

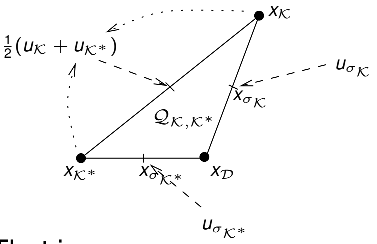
Warning : the fluxes are not explicit in general !

PLAN

THE NEW GRADIENT

- $\nabla_{\mathcal{D}}^{\mathcal{N}} u^{\mathcal{T}}$ is constant on each quarter of diamond

$$\nabla_{\mathcal{D}}^{\mathcal{N}} u^{\mathcal{T}} = \sum_{\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}} 1_{\mathcal{Q}} \nabla_{\mathcal{Q}}^{\mathcal{N}} u^{\mathcal{T}},$$



$$\nabla_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}^{\mathcal{N}} u^{\mathcal{T}} = \frac{2}{\sin \alpha_{\mathcal{D}}} \left(\frac{u_{\sigma_{\mathcal{K}^*}} - \frac{1}{2}(u_{\mathcal{K}} + u_{\mathcal{K}^*})}{|\sigma_{\mathcal{K}}|} \boldsymbol{\nu} + \frac{u_{\sigma_{\mathcal{K}}} - \frac{1}{2}(u_{\mathcal{K}} + u_{\mathcal{K}^*})}{|\sigma_{\mathcal{K}^*}|} \boldsymbol{\nu}^* \right)$$

That is

$$\nabla_{\mathcal{Q}}^{\mathcal{N}} u^{\mathcal{T}} = \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} + B_{\mathcal{Q}} \delta^{\mathcal{D}}, \ \delta^{\mathcal{D}} \in \mathbb{R}^4,$$

where $\delta^{\mathcal{D}}$ is to be determined

$$B_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}} = \frac{1}{|\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}|} \left(|\sigma_{\mathcal{K}}| \boldsymbol{\nu}^*, 0, |\sigma_{\mathcal{K}^*}| \boldsymbol{\nu}, 0 \right)$$

Consistency of the flux

We note

$$\varphi_{\mathcal{Q}}(\xi) = \int_{\mathcal{Q}} \varphi(z,\xi)\, d\mu_{\bar{\mathcal{Q}}}(z).$$

and choose $\delta^{\mathcal{D}} \in \mathbb{R}^4$ such that

$$\begin{aligned}
\left(\varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}\delta^{\mathcal{D}}), \boldsymbol{\nu}^*\right) &= \left(\varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}\delta^{\mathcal{D}}), \boldsymbol{\nu}^*\right)\\
\left(\varphi_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}\delta^{\mathcal{D}}), \boldsymbol{\nu}^*\right) &= \left(\varphi_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}\delta^{\mathcal{D}}), \boldsymbol{\nu}^*\right)\\
\left(\varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}\delta^{\mathcal{D}}), \boldsymbol{\nu}\right) &= \left(\varphi_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}\delta^{\mathcal{D}}), \boldsymbol{\nu}\right)\\
\left(\varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}\delta^{\mathcal{D}}), \boldsymbol{\nu}\right) &= \left(\varphi_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}\delta^{\mathcal{D}}), \boldsymbol{\nu}\right)
\end{aligned}$$

$\Rightarrow$ For all $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$, and all diamond cell $\mathcal{D}$, there exists a unique $\delta^{\mathcal{D}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}) \in \mathbb{R}^4$ that ensures such equalities.

THE NEW SCHEME

$$\varphi_{\mathcal{D}}^{\mathcal{N}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}) = \frac{1}{|\mathcal{D}|} \sum_{\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}} |\mathcal{Q}| \varphi_{\mathcal{Q}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} + B_{\mathcal{Q}} \delta^{\mathcal{D}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}})), \qquad (5)$$

$$\varphi_{\mathcal{Q}}(\xi) = \int_{\mathcal{Q}} \varphi(z, \xi)\, d\mu_{\bar{\mathcal{Q}}}(z).$$

FV FORMULATION

$$\begin{aligned} -\sum_{\mathcal{D}_{\sigma,\sigma^*} \cap \mathcal{K} \neq \emptyset} |\sigma| \left(\varphi_{\mathcal{D}}^{\mathcal{N}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}), \boldsymbol{\nu}_{\mathcal{K}}\right) &= \int_{\mathcal{K}} f(z)\, dz, \quad \forall \mathcal{K} \in \mathfrak{M} \\ -\sum_{\mathcal{D}_{\sigma,\sigma^*} \cap \mathcal{K}^* \neq \emptyset} |\sigma^*| \left(\varphi_{\mathcal{D}}^{\mathcal{N}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}), \boldsymbol{\nu}_{\mathcal{K}^*}\right) &= \int_{\mathcal{K}^*} f(z)\, dz, \ \forall \mathcal{K}^* \in \mathfrak{M}^* \end{aligned} \qquad (6)$$

THE NEW SCHEME

$$\varphi_{\mathcal{D}}^{\mathcal{N}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}) = \frac{1}{|\mathcal{D}|} \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} |\mathcal{Q}| \varphi_{\mathcal{Q}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} + B_{\mathcal{Q}} \delta^{\mathcal{D}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}})), \qquad (5)$$

$$\varphi_{\mathcal{Q}}(\xi) = \int_{\mathcal{Q}} \varphi(z,\xi)\, d\mu_{\bar{\mathcal{Q}}}(z).$$

DISCRETE DUALITY FORMULATION :

$$2 \sum_{\mathcal{D}\in\mathfrak{D}} |\mathcal{D}| \left(\varphi_{\mathcal{D}}^{\mathcal{N}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}), \nabla_{\mathcal{D}}^{\mathcal{T}} v^{\mathcal{T}}\right) = 2 \sum_{\mathcal{Q}\in\mathfrak{Q}} |\mathcal{Q}| \left(\varphi_{\mathcal{Q}}(\nabla_{\mathcal{Q}}^{\mathcal{N}} u^{\mathcal{T}}), \nabla_{\mathcal{Q}}^{\mathcal{N}} v^{\mathcal{T}}\right)$$

$$= \int_{\Omega} f v^{\mathfrak{M}} dz + \int_{\Omega} f v^{\mathfrak{M}^*} dz, \;\; \forall v^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}.$$

EXAMPLE

If φ is **linear** ($\varphi(z,\xi) = A(z)\xi$) and **constant on each primal cells**, we find Hermeline schemes for which the numerical fluxes can be **explicited.**

- $A(z) = \lambda(z)\mathrm{Id}$, λ constant on the primal cells, $\alpha_{\mathcal{D}} = \frac{\pi}{2}$

$$
\begin{aligned}
(\varphi_{\mathcal{D}}^{\mathcal{N}}, \boldsymbol{\nu}) &= \frac{\lambda_{\mathcal{K}}\lambda_{\mathcal{L}}}{\frac{|\sigma_{\mathcal{K}}|}{|\sigma_{\mathcal{K}}|+|\sigma_{\mathcal{L}}|}\lambda_{\mathcal{K}} + \frac{|\sigma_{\mathcal{L}}|}{|\sigma_{\mathcal{K}}|+|\sigma_{\mathcal{L}}|}\lambda_{\mathcal{L}}} \frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{|\sigma_{\mathcal{K}}| + |\sigma_{\mathcal{L}}|}, \\
(\varphi_{\mathcal{D}}^{\mathcal{N}}, \boldsymbol{\nu}^*) &= \left(\frac{|\sigma_{\mathcal{K}^*}|}{|\sigma_{\mathcal{K}^*}| + |\sigma_{\mathcal{L}^*}|}\lambda_{\mathcal{K}} + \frac{|\sigma_{\mathcal{L}^*}|}{|\sigma_{\mathcal{K}^*}| + |\sigma_{\mathcal{L}^*}|}\lambda_{\mathcal{L}} \right) \frac{u_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{|\sigma_{\mathcal{K}^*}| + |\sigma_{\mathcal{L}^*}|}.
\end{aligned}
$$

MAIN RESULT

THEOREM

We assume that φ satisfies $(\mathcal{H}_5)$ on each diamond cell.

- *The scheme* (6), (5) *admits a* **unique** *solution $u^{\mathcal{T}}$.*
- *More over if $u_{e|_{\mathcal{Q}}} \in W^{2,p}(\mathcal{Q}), \forall \mathcal{Q}$, we have*

$$\|u_e - u^{\mathfrak{M}}\|_{L^p} + \|u_e - u^{\mathfrak{M}^*}\|_{L^p} + \|\nabla u_e - \nabla^{\mathcal{N}} u^{\mathcal{T}}\|_{L^p} \leq C \,\mathrm{size}(\mathcal{T})^{\frac{1}{(p-1)^2}}.$$

- *If φ is dicontinuous along some curve Γ in Ω and if we use $\varphi_{\mathcal{D}}^{\mathcal{N}}$ only in a neighbourhood V of Γ and if $u_{e|_{\mathcal{Q}}} \in W^{2,p(p-1)^2}(\mathcal{Q}), \forall \mathcal{Q} \subset V$, we have*

$$\|u_e - u^{\mathfrak{M}}\|_{L^p} + \|u_e - u^{\mathfrak{M}^*}\|_{L^p} + \|\nabla u_e - \nabla^{\mathcal{N}} u^{\mathcal{T}}\|_{L^p} \leq C \,\mathrm{size}(\mathcal{T})^{\frac{1}{(p-1)}}.$$

Plan

POTENTIAL CASE

If φ derives from a potentiel Φ

$$\begin{cases} \varphi(z,\xi) &= \nabla_\xi \Phi(z,\xi), \quad \text{for all } \xi \in \mathbb{R}^2 \ \text{ a.e. } z \in \Omega, \\ \Phi(z,0) &= 0, \quad \text{a.e. } z \in \Omega. \end{cases}$$

PROPOSITION

The solution $u^{\mathcal{T}}$ of the scheme (6) is the unique minimum of

$$J^{\mathcal{T}}(v^{\mathcal{T}}) = \sum_{\mathcal{D}\in\mathfrak{D}} \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} |\mathcal{Q}| \Phi_{\mathcal{Q}}(\nabla_{\mathcal{Q}}^{\mathcal{N}} v^{\mathcal{T}}) - \sum_{\mathcal{K}} |\kappa| f_{\mathcal{K}} v_{\mathcal{K}} - \sum_{\mathcal{K}^*} |\kappa^*| f_{\mathcal{K}^*} v_{\mathcal{K}^*}, \ \forall v^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}} \tag{6}$$

with $\Phi_{\mathcal{Q}}(\cdot) = \int_{\mathcal{Q}} \Phi(z,\cdot) d\mu_{\mathcal{Q}}(z)$.

Potential case

Proposition

The couple $(u^{\mathcal{T}}, (\delta^{\mathcal{D}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}))_{\mathcal{D}})$ *is the unique minimum of*

$$J^{\mathcal{T},\Delta}(v^{\mathcal{T}}, \tilde{\delta}) = \sum_{\mathcal{D}\in\mathfrak{D}} \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} |\mathcal{Q}| \Phi_{\mathcal{Q}}(\nabla_{\mathcal{D}}^{\mathcal{T}} v^{\mathcal{T}} + B_{\mathcal{Q}} \tilde{\delta}^{\mathcal{D}}) - \sum_{\mathcal{K}} |\kappa| f_{\mathcal{K}} v_{\mathcal{K}} - \sum_{\mathcal{K}^*} |\kappa^*| f_{\mathcal{K}^*} v_{\mathcal{K}^*}, \quad \forall v^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}, \ \forall \tilde{\delta} \in \Delta. \quad (7)$$

SADDLE-POINT FORMULATION

We want to solve this non quadratic minimization problem with a saddle-point formulation (See Glowinsky & al.)

$$L_r^{\mathcal{T},\Delta}(v^{\mathcal{T}},\tilde{\delta},g,\lambda) = \sum_{\mathcal{Q}\in\mathfrak{Q}} |\mathcal{Q}|\Phi_{\mathcal{Q}}(g_{\mathcal{Q}}) + \sum_{\mathcal{Q}\in\mathfrak{Q}} |\mathcal{Q}|(\lambda_{\mathcal{Q}}, g_{\mathcal{Q}} - \nabla_{\mathcal{D}}^{\mathcal{T}} v^{\mathcal{T}} - B_{\mathcal{Q}}\tilde{\delta}^{\mathcal{D}})$$
$$+ \frac{r}{2}\sum_{\mathcal{Q}\in\mathfrak{Q}} |\mathcal{Q}| \left| g_{\mathcal{Q}} - \nabla_{\mathcal{D}}^{\mathcal{T}} v^{\mathcal{T}} - B_{\mathcal{Q}}\tilde{\delta}^{\mathcal{D}}\right|^2 - \sum_{\mathcal{K}} |\kappa| f_{\mathcal{K}} v_{\mathcal{K}} - \sum_{\mathcal{K}^*} |\kappa^*| f_{\mathcal{K}^*} v_{\mathcal{K}^*},$$
$$\forall v^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}, \forall\tilde{\delta}\in\Delta, \forall g,\lambda \in (\mathbb{R}^2)^{\mathfrak{Q}}.$$

SADDLE-POINT FORMULATION

The lagragian $L_r^{\mathcal{T},\Delta}$ admits a unique saddle-point given by

$$\varphi_{\mathcal{Q}}(g_{\mathcal{Q}}) + \lambda_{\mathcal{Q}} + r(g_{\mathcal{Q}} - \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} - B_{\mathcal{Q}}\delta^{\mathcal{D}}) = 0, \ \ \forall \mathcal{Q} \in \mathfrak{Q},$$

$$r \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} |\mathcal{Q}|{}^t B_{\mathcal{Q}}(B_{\mathcal{Q}}\delta^{\mathcal{D}} + \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} - g_{\mathcal{Q}}) - \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} |\mathcal{Q}|{}^t B_{\mathcal{Q}}\lambda_{\mathcal{Q}} = 0, \ \ \forall \mathcal{D} \in \mathfrak{D},$$

$$g_{\mathcal{Q}} - \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} - B_{\mathcal{Q}}\delta^{\mathcal{D}} = 0, \ \ \forall \mathcal{Q} \in \mathfrak{Q},$$

$$r \sum_{\mathcal{Q}\in\mathfrak{Q}} |\mathcal{Q}|(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} + B_{\mathcal{Q}}\delta^{\mathcal{D}} - g_{\mathcal{Q}}, \nabla_{\mathcal{D}}^{\mathcal{T}} v^{\mathcal{T}}) = \sum_{\mathcal{K}} |\kappa| f_{\mathcal{K}} v_{\mathcal{K}} + \sum_{\mathcal{K}^*} |\kappa^*| f_{\mathcal{K}^*} v_{\mathcal{K}^*}$$
$$+ \sum_{\mathcal{Q}\in\mathfrak{Q}} |\mathcal{Q}|(\lambda_{\mathcal{Q}}, \nabla_{\mathcal{D}}^{\mathcal{T}} v^{\mathcal{T}}), \ \ \forall v^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}.$$

DECOMPOSITION-COORDINATION ALGORITHM

Let $r > 0$.

- *Step 1 :* Find $(u^{\mathcal{T},n}, \delta_{\mathcal{D}}^n)$ solution of

$$r \sum_{\mathcal{Q}\in\mathfrak{Q}} |\mathcal{Q}|(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T},n} + B_{\mathcal{Q}}\delta_{\mathcal{D}}^n - g_{\mathcal{Q}}^{n-1}, \nabla_{\mathcal{D}}^{\mathcal{T}} v^{\mathcal{T}})$$

$$= \sum_{\mathcal{K}} |\kappa| f_{\mathcal{K}} v_{\mathcal{K}} + \sum_{\mathcal{K}^*} |\kappa^*| f_{\mathcal{K}^*} v_{\mathcal{K}^*} + \sum_{\mathcal{Q}\in\mathfrak{Q}} |\mathcal{Q}|(\lambda_{\mathcal{Q}}^{n-1}, \nabla_{\mathcal{D}}^{\mathcal{T}} v), \quad \forall v^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}.$$

$$r \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} |\mathcal{Q}| {}^t B_{\mathcal{Q}}(B_{\mathcal{Q}}\delta_{\mathcal{D}}^n + \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T},n} - g_{\mathcal{Q}}^{n-1}) - \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} |\mathcal{Q}| {}^t B_{\mathcal{Q}} \lambda_{\mathcal{Q}}^{n-1} = 0, \quad \forall \mathcal{D} \in \mathfrak{D}.$$

- *Step 2 :* On each $\mathcal{Q}$, find $g_{\mathcal{Q}}^n$ solution de

$$\varphi_{\mathcal{Q}}(g_{\mathcal{Q}}^n) + \lambda_{\mathcal{Q}}^{n-1} + r(g_{\mathcal{Q}}^n - \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T},n} - B_{\mathcal{Q}}\delta_{\mathcal{D}}^n) = 0.$$

- *Step 3 :* On each $\mathcal{Q}$ solve $\lambda_{\mathcal{Q}}^n$ define by

$$\lambda_{\mathcal{Q}}^n = \lambda_{\mathcal{Q}}^{n-1} + r(g_{\mathcal{Q}}^n - \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T},n} - B_{\mathcal{Q}}\delta_{\mathcal{D}}^n).$$

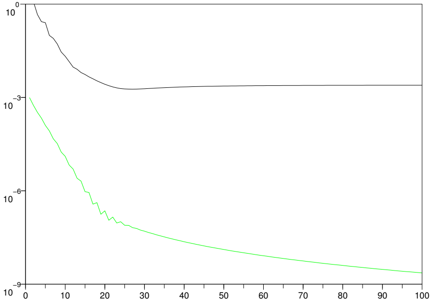
CONVERGENCE OF THE ALGORITHM

Remark : it works even in the non potential case.

THEOREM

$\forall r > 0$, *the algorithm converges towards the unique solution of the improved DDFV scheme.*

Example $\Rightarrow$ Converges in 25 iterations.



PLAN

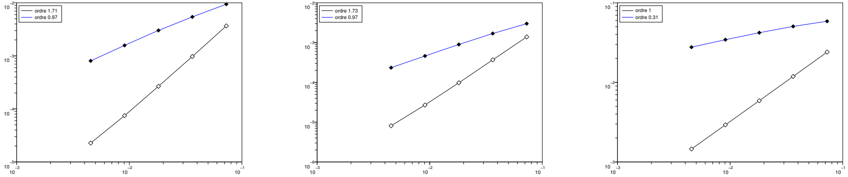
EXEMPLE

$\Omega =]0,1[\times]0,1[$, triangular mesh, u_e piecewise quadratic, $p = 3$

$$\text{if } z_1 < 0.5, \quad \varphi(z,\xi) = |\xi|^{p-2}\xi,$$

$$\text{if } z_1 > 0.5, \quad \varphi(z,\xi) = (A\xi,\xi)^{\frac{p-2}{2}} A\xi, \text{ with } A = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}.$$

- DDFV (♦) and improved DDFV (◇)



L^∞ error | L^p error | $W^{1,p}$ error

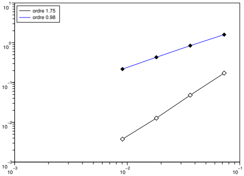
EXAMPLE

$\Omega =]0,1[\times]0,1[$, triangular mesh, u_e piecewise quadratic, $p = 5$

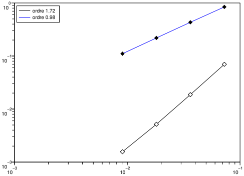
$$\text{if } z_1 < 0.5, \quad \varphi(z,\xi) = |\xi|^{p-2}\xi,$$

$$\text{if } z_1 > 0.5, \quad \varphi(z,\xi) = (A\xi,\xi)^{\frac{p-2}{2}} A\xi, \text{ with } A = 10\, Id.$$

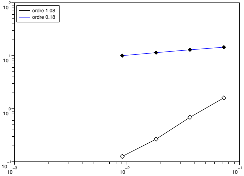
- DDFV scheme (♦) and improved DDFV scheme (◊)



L^∞ error



L^p error



$W^{1,p}$ error

CONCLUDING REMARKS

Summary :

- The variational structure of the equation is preserved.
- The discontinuities are handled in a good way (order 1 for $p = 2$).
- The scheme can be solved by a efficient saddle-point algorithm.

Remarks :

- It works in the same way if $1 < p \leq 2$.
- We can couple nonlinearity whose order depends on the subdomain. For example : Darcy / Darcy-Fochheimer.
- We plan to couple the iterative method with domain decomposition method.