Robust Preconditioning in Elasticity

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System of PDEs

Linear elasticity:

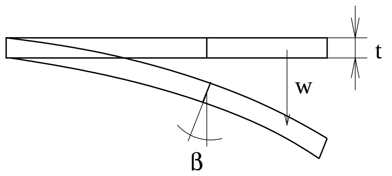
$$A(u,v) = \int \mu\,\varepsilon(u) : \varepsilon(v) + \lambda\,\mathrm{div}\; u\;\mathrm{div}\; v\,dx$$

displacement $u \in [H^1_{0,D}]^d$, strain operator $\varepsilon(u) := 0.5(\nabla u + (\nabla u)^T)$
Lamé parameters μ, λ.

Timoshenko beam model:

$$A(w,\beta;v,\delta) = \int_0^1 \beta'\delta'\,dx + t^{-2}\int_0^1 (w'-\beta)(v'-\delta)\,dx$$

vertical displacement w, rotation β, thickness t,



In principle the same as a scalar PDE

System of PDEs

Linear elasticity:

$$A(u,v) = \int \mu\, \varepsilon(u) : \varepsilon(v) + \lambda \,\mathrm{div}\; u \;\mathrm{div}\; v \, dx$$

Nearly incompressible materials: $\lambda \gg \mu$

Timoshenko beam model:

$$A(w,\beta;v,\delta) = \int_0^1 \beta'\delta' \, dx + t^{-2} \int_0^1 (w' - \beta)(v' - \delta) \, dx$$

Thin beam: $t \ll 1$

In principle the same as a scalar PDE but dependency on parameters

Parameter Dependent Problems

[Arnold 81] Find $u \in V$:

$$A^{\varepsilon}(u,v) = f(v) \qquad \forall\, v \in V$$

with

$$A^{\varepsilon}(u,v) = a(u,v) + \frac{1}{\varepsilon}\, c(\Lambda u, \Lambda v)$$

small parameter:	$\varepsilon \in (0,1]$
symmetric bilinear form:	$a(u,u) \geq 0 \qquad \forall\, u \in V$
Hilbert space:	$(Q, c(.,.))$
operator:	$\Lambda : V \rightarrow Q$
with kernel:	$V_0 :=$ kern Λ
Well posed for $\varepsilon = 1$:	$A^1(u,u) \simeq \|u\|_V^2$

A priori estimates

Univorm V-coercivity::

$$A^\varepsilon(u,u) \geq A^1(u,u) \succeq \|u\|_V^2$$

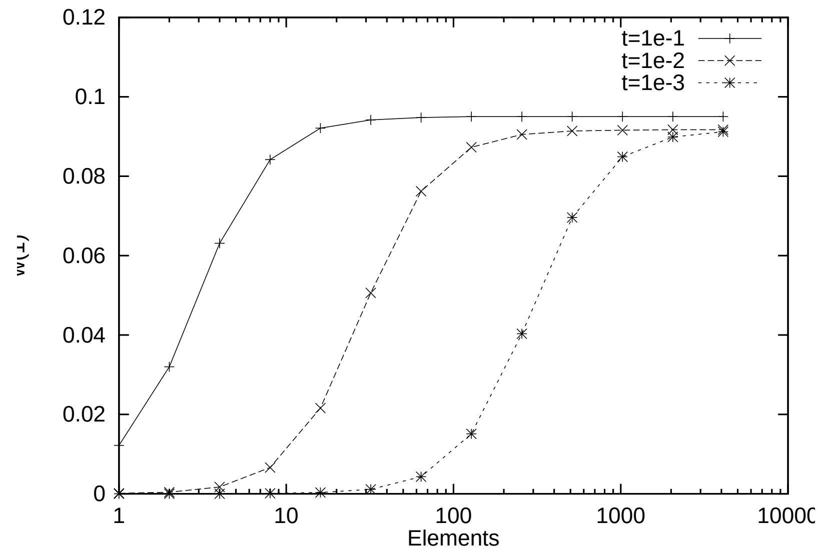
Non-uniform V-continuity:

$$A^\varepsilon(u,u) \leq \varepsilon^{-1} A^1(u,u) \preceq \varepsilon^{-1} \|u\|_V^2$$

Non-robust a priori error estimate:

$$\|u - u_h\|_V \leq \varepsilon^{-1/2} \inf_{v_h \in V_h} \|u - v_h\|_V$$

Numerical example: Timoshenko beam
Vertical load $f = 1$, compute $w(1)$:



Primal FEM with Reduction Operators

The primal FEM

$$\text{Find } u_h \in V_h \text{ s.t.:} \qquad a(u_h, v_h) + \frac{1}{\varepsilon} c(\Lambda u_h, \Lambda v_h) = f(v_h) \qquad \forall\, v_h \in V_h$$

often leads to bad results, knwon as *locking* phenomena.

(One) explanation:
This is a penalty approximation to $\Lambda u = 0$, but no FE functions fulfill $\Lambda u_h = 0$, i.e. $V_0 \cap V_h$ too small.

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Weaken the high energy term by reduction operator R_h (reduced integration, B-bar method, mixed method , EAS, ...)

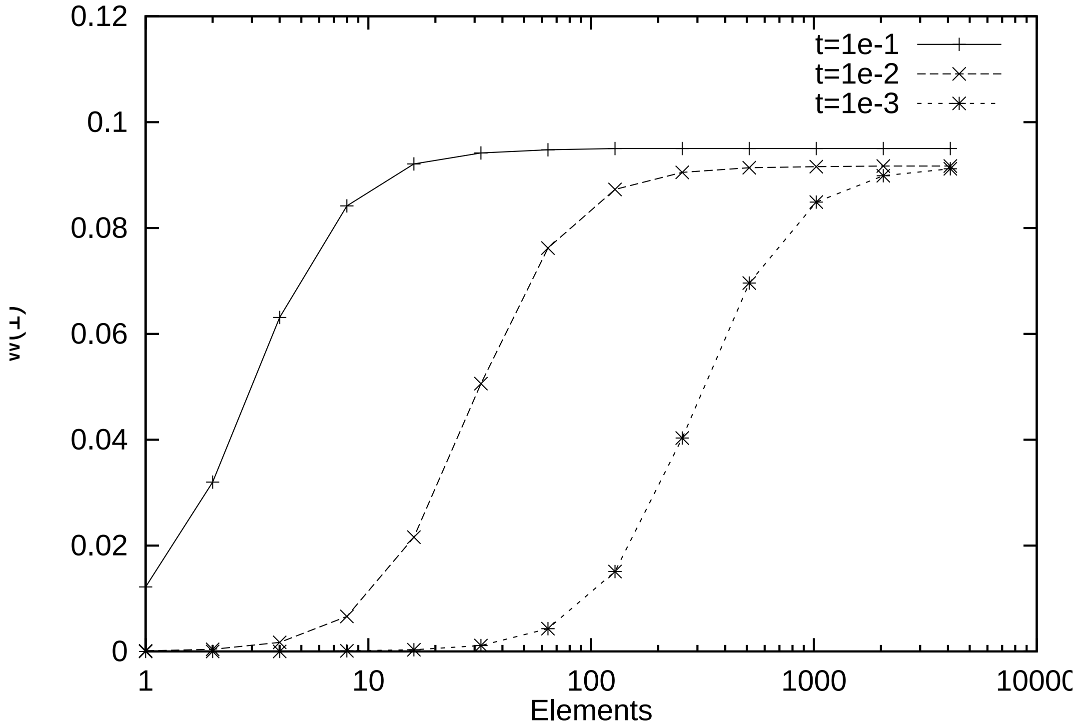
$$\text{Find } u_h \in V_h \text{ s.t.:} \qquad a(u_h, v_h) + \frac{1}{\varepsilon} c(R_h \Lambda u_h, R_h \Lambda v_h) = f(v_h) \qquad \forall\, v_h \in V_h$$

Large enough kernel $V_{h,0} = \text{kern} R_h \Lambda \cap V_h$

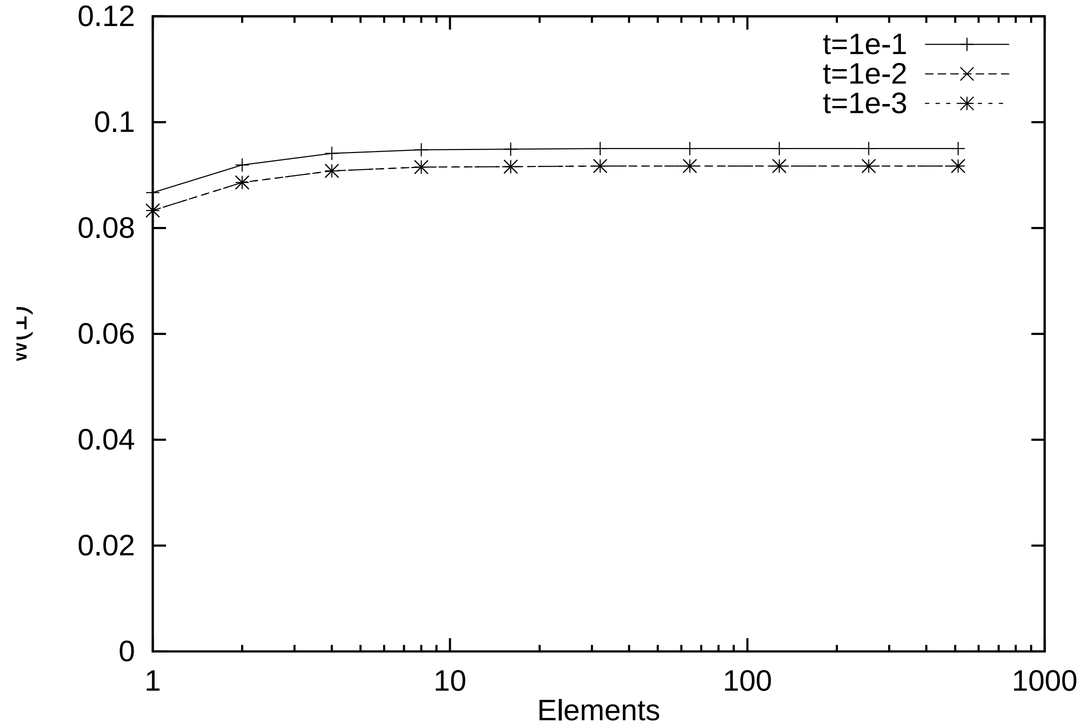
Numerical example: Timoshenko beam

Vertical load $f = 1$, compute $w(1)$:

Conforming FEM:



With reduction operator:



Analysis by mixed formulation

Primal method:

$$\text{Find } u \in V: \qquad a(u,v) + \varepsilon^{-1} c(\Lambda u, \Lambda v) = f(v) \qquad \forall\, v \in V$$

Introduce new variable $p = \varepsilon^{-1} \Lambda u \in Q$.

$$\begin{array}{rcrcll} a(u,v) & + & c(\Lambda v, p) & = & f(v) & \qquad \forall\, v \in V \\ c(\Lambda u, q) & - & \varepsilon c(p,q) & = & 0 & \qquad \forall\, q \in Q \end{array}$$

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Mixed bilinear-from $B(\cdot,\cdot) : (V \times Q) \times (V \times Q) \to \mathbb{R}$

$$B((u,p),(v,q)) = a(u,v) + c(\Lambda u, q) + c(\Lambda v, p) - \varepsilon c(p,q)$$

Mixed problem:

$$\text{Find } (u,p) \in V \times Q : \qquad B((u,p),(v,q)) = f(v) \qquad \forall\, (v,q) \in V \times Q$$

Well-posed mixed formulation

Define norm $\|.\|_{Q,0}$ such that the LBB condition is fulfilled by definition:

$$\|q\|_{Q,0} := \sup_{v \in V} \frac{c(\Lambda v, p)}{\|v\|_V}$$

Product space norm

$$\|(v,q)\|_{V \times Q}^2 = \|v\|_V^2 + \|q\|_{Q,0}^2 + \varepsilon \|q\|_c^2$$

Then $B(.,.)$ is uniformely continuous:

$$\sup_{(u,p)} \sup_{(v,q)} \frac{B((u,p),(v,q))}{\|(u,p)\|_{V \times Q} \, \|(v,q)\|_{V \times Q}} \preceq 1$$

and uniformely $\inf - \sup$ stable:

$$\inf_{(u,p)} \sup_{(v,q)} \frac{B((u,p),(v,q))}{\|(u,p)\|_{V \times Q} \, \|(v,q)\|_{V \times Q}} \succeq 1$$

Example: Nearly incompressible elasticity

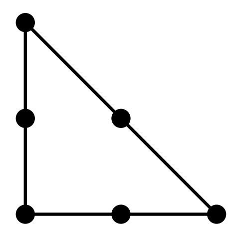
Find $u \in V = [H^1_{0,D}]^2$ and $p \in Q = L_2$ such that

$$\begin{array}{rcccl} \mu \int \varepsilon(u) : \varepsilon(v)\, dx & + & \int \operatorname{div} v\, p\, dx & = & \int f \cdot v\, dx & \quad \forall\, v \in V \\ \int \operatorname{div} u\, q\, dx & - & \lambda^{-1} \int p\, q\, dx & = & 0 & \quad \forall\, q \in Q \end{array}$$

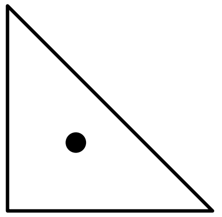
The limit problem for $\lambda \to \infty$ is a Stokes-like problem.

Mixed finite element discretization by Stokes-stable (discrete LBB !) element pairs, e.g.,

$V_h = \{v \in V : v|_T \in P^2\}$



$Q_h = \{q \in Q : q|_T \in P^0\}$.



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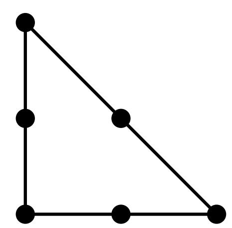

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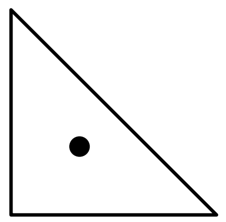

A priori estimates by stability and approximation:

$$\|(u - u_h, p - p_h)\|_{V \times Q} \preceq \inf_{v_h \in V_h, q_h \in Q_h} \|(u - v_h, p - p_h)\|_{V \times Q} \preceq h^\alpha \left(\|u\|_{H^{1+\alpha}} + \|p\|_{H^\alpha} \right)$$

Solvers for linear system

Indefinite matrix equation

$$\begin{pmatrix} A & B^T \\ B & -\varepsilon C \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

- Block Transformation:

  Inexact Uzawa, SIMPLE, GMRES

  Axelsson-Vassilevski, Bramble-Pasciak, Langer-Queck, Rusten-Winther, Bank-Welfert-Yserentant, Klawonn, Bramble-Pasciak-Vassilev, Zulehner, Benzi-Golub-Liesen, ...

  Use (standard) preconditioners for A and for Schur-complement $B^T A^{-1} B + \varepsilon C$.

- Multigrid for indefinite problem:

  Braess-Blömer, Brenner, Huang, Wittum, Braess-Sarazin, Zulehner, Schöberl-Zulehner

  Use special smoothers (squared system, Vanka, SIMPLE)

Schur complement system

Indefinite matrix equation

$$\begin{pmatrix} A & B^T \\ B & -\varepsilon C \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

Elimination of p from second line leads to the Schur complement system

$$\left(A + \frac{1}{\varepsilon} B^T C^{-1} B \right) u = f$$

Cheap if C is (block-)diagonal.

Positive definite matrix of smaller dimension, but very ill conditioned for $\varepsilon \to 0$

Goal: Design of ε-robust solver

Elimination of dual variable on the finite element level

Finite element system: Find $u_h \in V_h$ and $p_h \in Q_h$ such that

$$\begin{array}{rcrcll} a(u_h, v_h) & + & c(\Lambda u_h, p_h) & = & f(v_h) & \forall\, v_h \in V_h \\ c(\Lambda u_h, q_h) & - & \varepsilon c(p_h, q_h) & = & 0 & \forall\, q_h \in Q_h \end{array}$$

Second line defines p_h:

$$p_h = \varepsilon^{-1} P^c_{Q_h} \Lambda u_h$$

Use in first line:

$$a(u_h, v_h) + \varepsilon^{-1} c(P^c_{Q_h} \Lambda u_h, P^c_{Q_h} \Lambda p_h) = f(v_h) \qquad \forall\, v_h \in V_h$$

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Elasticity with reduction operators:

$$A_h^\varepsilon(u, v) = \int \mu \varepsilon(u) : \varepsilon(v) + \lambda\, \overline{\operatorname{div}\, u}^h\, \overline{\operatorname{div}\, v}^h\, dx$$

Discrete kernel:

$$V_{h0} = \{ v_h \in V_h : \int_T \operatorname{div}\, v_h\, dx = 0 \ \ \forall\, T \in \mathcal{T} \}$$

Timoshenko beam

Conforming bilinear form:

$$A((w,\beta),(v,\delta)) = \int \beta'\delta'\,dx + t^{-2}\int (w'-\beta)(v'-\delta)\,dx$$

has the kernel

$$V_0 = \{(v,\delta) : \delta = v'\}$$

$t \to 0$ is a penalty approximation to the 4^{th}-order Bernoulli model $A(w,v) = f(v)$ with

$$A(w,v) = \int w''v''\,dx$$

Reduction of a (stable !) mixed system with $w \in P^1, \beta \in P^1, q \in P^0$ leads to

$$A_h((w_h,\beta_h),(v_h,\delta_h)) = \int \beta_h'\delta_h'\,dx + t^{-2}\int \overline{(w_h'-\beta_h)}^h\,\overline{(v_h'-\delta_h)}^h\,dx$$

ε-Robust local preconditioner

$$A_h^\varepsilon(u,v) = a(u,v) + \varepsilon^{-1} c(R_h \Lambda u, R_h \Lambda v)$$

Space splitting $V = \sum V_i$ fulfilling the decomposition inequalities

$$\inf_{\substack{u_h = \sum u_i \\ u_i \in V_i}} \sum \|u_i\|_V^2 \leq c_1(h) \|u_h\|_V^2 \qquad \forall\, u_h \in V_h$$

$$\inf_{\substack{u_{h,0} = \sum u_i \\ u_i \in V_i \cap V_{h,0}}} \sum \|u_i\|_a^2 \leq c_2(h) \|u_{h,0}\|_V^2 \qquad \forall\, u_{h,0} \in V_{h,0}$$

Inverse inequality

$$\|q_h\|_c \leq c_3(h) \|q_h\|_{Q,0}$$

Then the (local) additive Schwarz preconditioner D_h fulfills the ε-robust spectral estimates

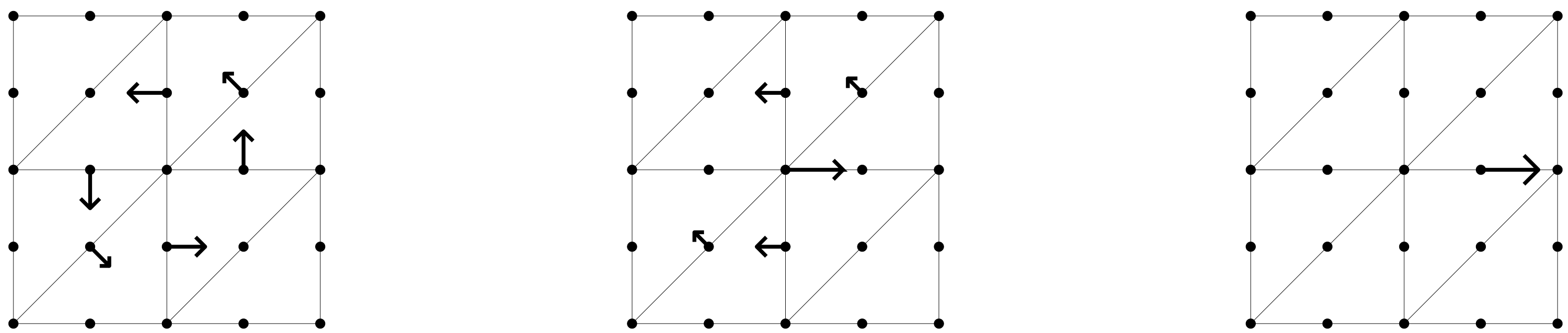
$$\{c_2(h) + c_1(h)/c_3(h)\}^{-1} D_h \preceq A_h \preceq D_h$$

Similar $H(\mathrm{div})$ and $H(\mathrm{curl})$: Vassilevski-Wang, Cai-Goldstein-Pasciak, Arnold-Falk-Winther, Hiptmair,

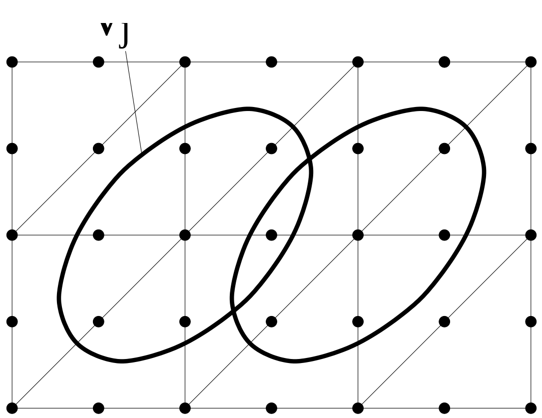
Local sub-spaces for nearly incompressible materials

$$R_h \operatorname{div} u_h = 0 \Leftrightarrow \int_T \operatorname{div} u_h = 0 \Leftrightarrow \int_{\partial T} n^T u \, ds = 0 \qquad \forall T \in \mathcal{T}_h$$

Discrete divergence-free base functions:

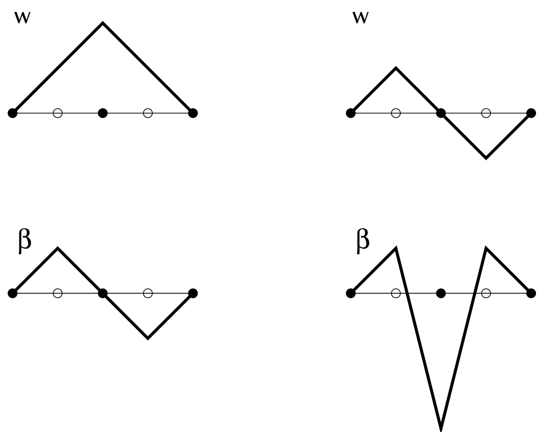


Sub-space covering:



Timoshenko beam splitting

Discrete Kernel: $\int_T w_h' - \beta_h \, dx = 0.$



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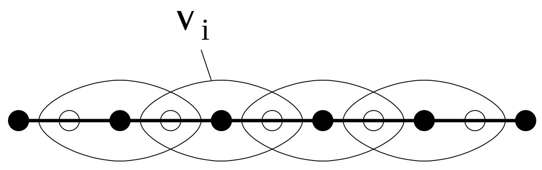
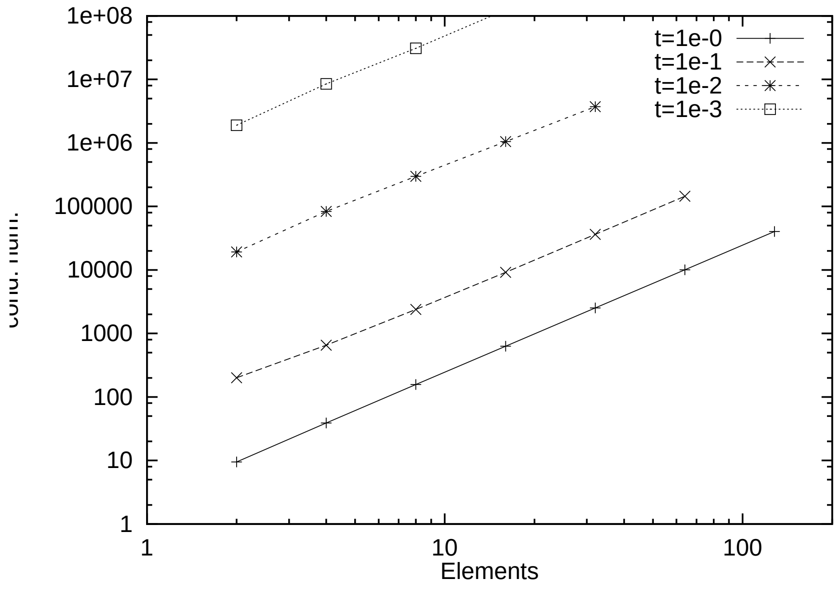
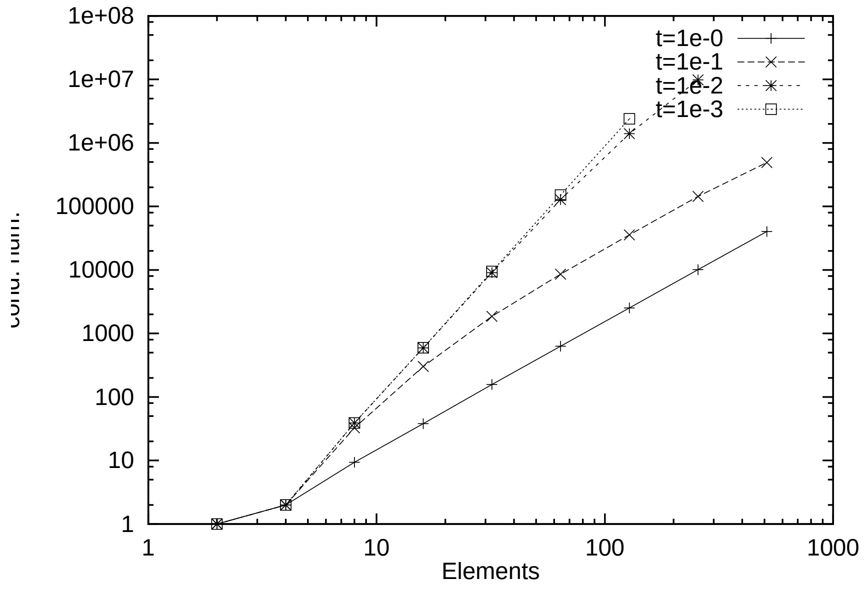


Point Jacobi:



Block Jacobi:



Two-level preconditioner

2-level norm:

$$\|v_h\|_C^2 = \inf_{v_h = E_H v_H + \sum v_i} \left\{ \|v_H\|_{A_H}^2 + \sum \|v_i\|_{A_h}^2 \right\}$$

Norm equivalence $C \simeq A_h$ requires:

- Continuous prolongation operator $E_H : (V_H, \|.\|_{A_H}) \to (V_h, \|.\|_{A_h})$
- Existence of continuous interpolation operator $\Pi_H : (V_h, \|.\|_{A_h}) \to (V_H, \|.\|_{A_H})$

ε-Robust two-Level preconditioner

Coarse grid bilinear form:

$$A_H^\varepsilon(u_H, v_H) = a(u_H, v_H) + \varepsilon^{-1} c(R_H \Lambda u_H, R_H \Lambda v_H)$$

$$V_{H0} = \text{kern } R_H \Lambda$$

Fine grid bilinear form:

$$A_h^\varepsilon(u_h, v_h) = a(u_h, v_h) + \varepsilon^{-1} c(R_h \Lambda u_h, R_h \Lambda v_h)$$
$$V_{h0} = \text{kern } R_h \Lambda$$

Prolongation operator $E_H : V_H \rightarrow V_h$ has to map

$$E_H : V_{H0} \rightarrow V_{h0}$$

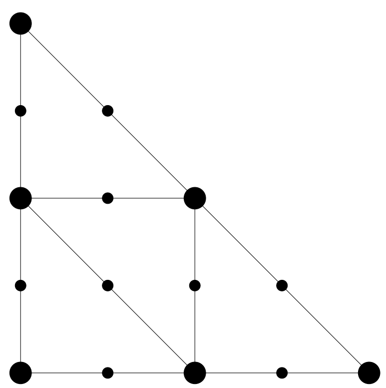
to be uniformely bounded. Since for $u_H \in V_{H0}$

$$\|E_H u_H\|^2_{A_h^\varepsilon} = \|E_H u_H\|^2_a + \frac{1}{\varepsilon}\|R_h \Lambda E_H u_H\|^2_c \qquad \text{and} \qquad \|u_H\|^2_{A_H^\varepsilon} = \|u_H\|^2_a$$

Robust prolongation for nearly incompressible materials

$$u_H \in kern(\Lambda_H) \quad \Leftrightarrow \quad \int_{\partial T} n^T u_H \, \mathsf{ds} = 0$$

$$E_H u_H \in kern(\Lambda_h) \quad \Leftrightarrow \quad \int_{\partial t_i} n^T (E_H u_H) \, \mathsf{ds} = 0, \qquad i = 1 \ldots 4$$

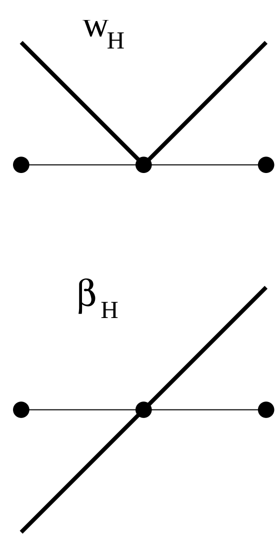


$$T = \bigcup_{i=1}^{4} t_i$$

1. Conforming (quadratic) prolongation at ∂T
2. Adjust inner nodes by solving local Dirichlet problems

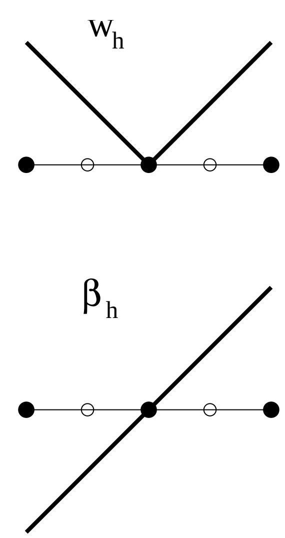
Robust prolongation for the Timoshenko beam

Coarse grid kernel function:



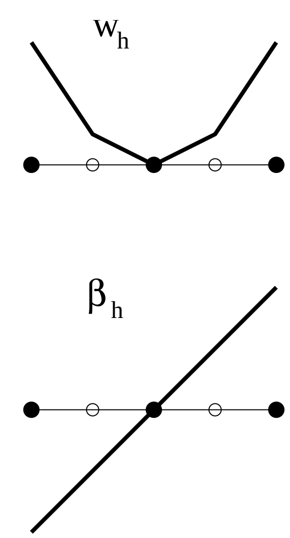
$$w'_H = \overline{\beta_H}^H$$

simply prolongated coarse grid kernel function:



$$w'_h = \overline{\beta_h}^H$$

locally adjusted to fine grid kernerl:



$$w'_h = \overline{\beta_h}^h$$

Fortin operator

Error estimates are based on equivalent mixed formulations. Discrete LBB condition is usually verified by the Fortin operator $\Pi^F : V \rightarrow V_h$:

$$\begin{array}{rl} \text{Continuous:} & \|\Pi^F\|_V \preceq 1 \\ \text{Preserves weak constraints:} & R_h \Lambda v = R_h \Lambda \Pi^F v \end{array}$$

This is a robust interpolation operator from $(V, \|\cdot\|_{A^\varepsilon})$ to $(V_h, \|\cdot\|_{A_h^\varepsilon})$:

$$\begin{aligned} \|\Pi_h^F v\|_{A_h^\varepsilon}^2 &= \|\Pi_h^F v\|_a^2 + \varepsilon^{-1} \|R_h \Lambda \Pi_h^F v\|_c^2 \preceq \|v\|_V^2 + \varepsilon^{-1} \|R_h \Lambda v\|_c^2 \\ &\preceq \|u\|_{A^1}^2 + \varepsilon^{-1} \|\Lambda v\|_c^2 \preceq \|u\|_{A^\varepsilon}^2 \end{aligned}$$

Such operators are used to define the coarse grid function in the 2-level decomposition

History

- J. S.: Proceedings to EMG 96:
  Multigrid method with 2-level analysis for nearly incompressible materials and Timoshenko
- J. S.: Numer. Math. 99:
  Multigrid analysis for nearly incompressible materials
- J. S.: Thesis, 99:
  Multigrid method and analysis for Reissner Mindlin plates
- J. S. and W. Zulehner, 03:
  Iteration in mixed variables (Vanka smoother)

In preparation:

- J. S. and R. Stenberg: Multigrid for MITC and stabilized MITC

Unit square model problem

$$A_h(u_h, u_h) = \int_\Omega \varepsilon(u_h) : \varepsilon(u_h)\,\mathsf{dx} + \frac{1}{\varepsilon}\int_\Omega (\overline{\operatorname{div}\ u_h})^2\,\mathsf{dx}$$

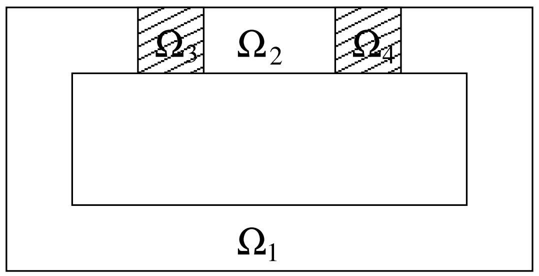
Multigrid preconditioner C with

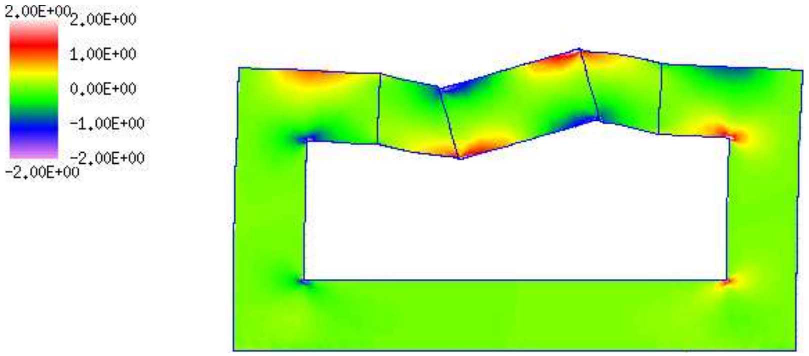
- Symmetric V-1-1 cycle
- Block - Gauss - Seidel smoother
- Robust prolongation

Condition number $\kappa(C^{-1}A)$ for different choices of the Poisson ration $\nu \approx 0.5 - \varepsilon$

Level	Nodes	$\nu = 0.3$	$\nu = 0.49$	$\nu = 0.4999$	$\nu = 0.499999$
2	25	1.05	1.14	1.16	1.16
3	81	1.37	2.27	2.60	2.61
4	289	1.46	2.51	2.88	2.89
5	1089	1.49	2.59	2.99	2.99
6	4225	1.49	2.61	3.02	3.02
7	16641	1.49	2.63	3.03	3.03
8	66049	1.49	2.64	3.04	3.04

Nearly incompressible sub-domains





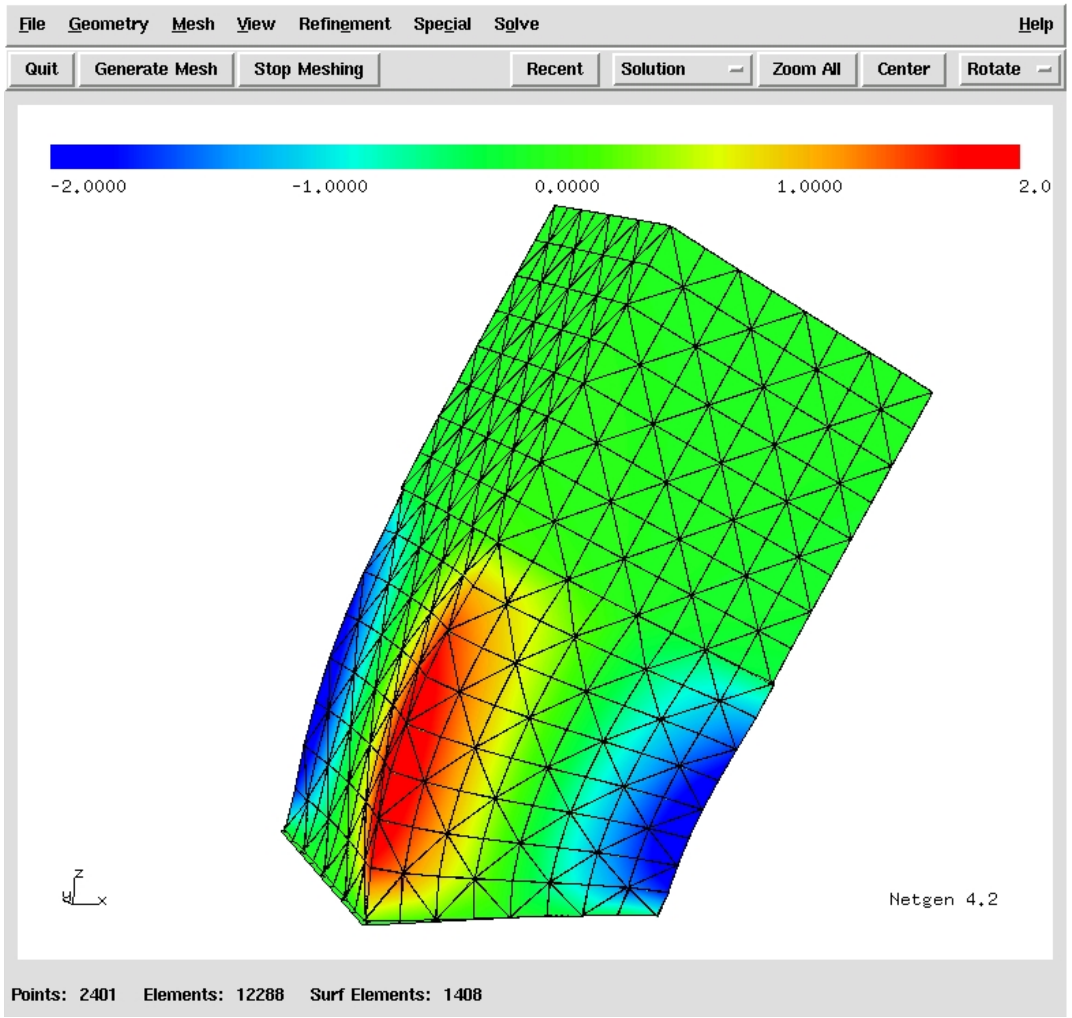
$$\Omega_1, \Omega_2 \quad : \quad E = 100, \nu = 0.3$$

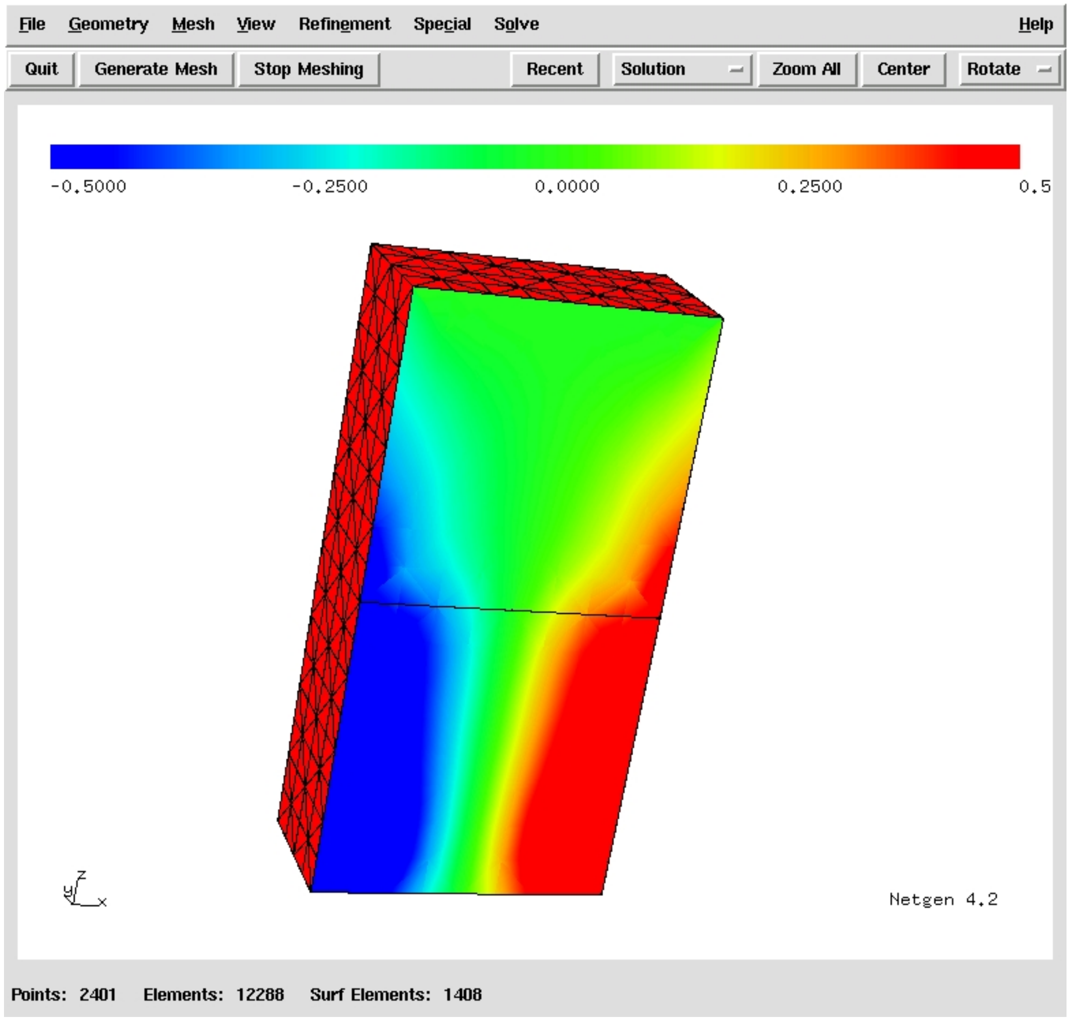
$$\Omega_3, \Omega_4 \quad : \quad E = 1, \nu = 0.49999$$

Level	Nodes	its
2	196	2
3	672	11
4	2464	14
5	9408	15
6	36736	16
7	145152	16

3D Nearly Incompressible Elasticity

Two cubes, one nearly incompressible ($\nu = 0.4999$)
Hybrid elements based on a stabilized Hellinger Reissner formulation, BDM_1 elements





12288 tets, 28930 faces, 260370 unknowns

Iteration numbers

Robust Multigrid (V-3-3):

level	unknowns	$\nu = 0.3$	$\nu = 0.49$	$\nu = 0.4999$
1	0.5k			
2	4.3k	20	26	30
3	33k	20	29	36
4	260k	21	32	42

Robust Smoother (3-3):

level	unknowns	$\nu = 0.3$	$\nu = 0.49$	$\nu = 0.4999$
1	0.5k			
2	4.3k	55	74	140
3	33k	98	148	351

Standard Multigrid (V-3-3):

level	unknowns	$\nu = 0.3$	$\nu = 0.49$	$\nu = 0.4999$
1	0.5k			
2	4.3k	62	181	1721
3	33k	64	271	2000+

CG iteration, error reduction 10^{-10}

Reissner Mindlin Plate

The unknown variables are:

- vertical displacement $w \in H^1_{0,D}(\Omega)$
- rotation vector $\beta \in [H^1_{0,D}(\Omega)]^2$

Inner energy consisting of bending and shear term:

$$A(w,\beta;w,\beta) = \int_\Omega D\varepsilon(\beta) : \varepsilon(\beta) + \frac{1}{t^2}\int |\nabla w - \beta|^2 \, dx$$

Stabilized mixed method by Chapelle and Stenberg in primal variables:

$$A_h(w,\beta;w,\beta) = \int_\Omega D\varepsilon(\beta) : \varepsilon(\beta) + \int \frac{1}{(h+t)^2}|\nabla w - \beta|^2 \, dx + \int \left(\frac{1}{t^2} - \frac{1}{(h+t)^2}\right) |\overline{\nabla w - \beta}|^2 \, dx$$

Numerical results for Reissner Mindlin

Dirichlet problem on $[0,1]^2$, $E = 1$, $\nu = 0.2$:

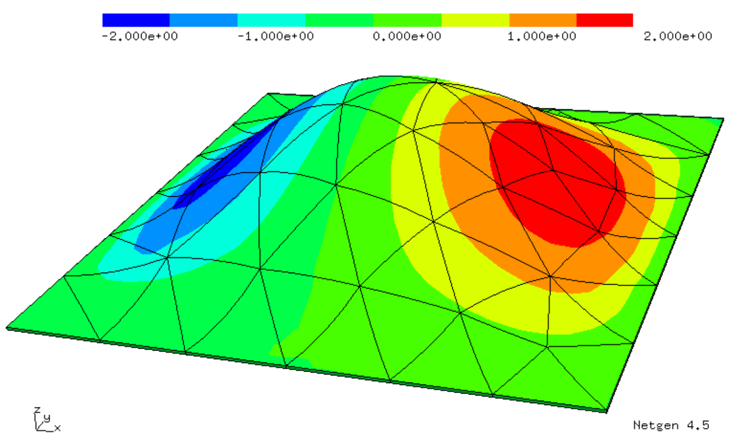
Multigrid preconditioner with Symmetric V-1-1 cycle, Block - Gauss - Seidel smoother, Robust prolongation.

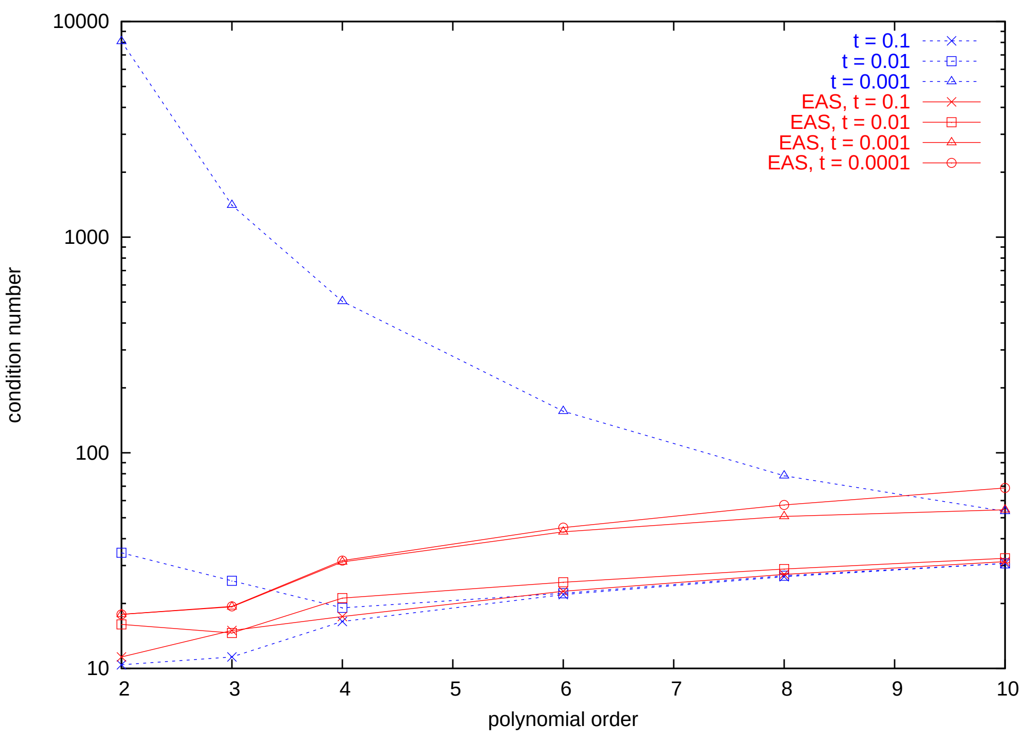
Condition number $\kappa(C^{-1}A)$:

Level	h	Nodes	$t = 10^{-1}$	$t = 10^{-2}$	$t = 10^{-3}$	$t = 10^{-4}$
2	$1/2$	33	1.0	1.1	1.1	1.1
3	$1/4$	113	1.5	5.4	6.2	6.2
4	$1/8$	417	1.6	6.1	9.1	9.1
5	$1/16$	1601	1.9	4.5	11.5	11.8
6	$1/32$	6273	2.0	3.8	11.5	12.6
7	$1/64$	24633	2.1	3.7	9.5	12.4

Thin structures with high order EAS reduction operators

Comparison of relative condition numbers for standard and EAS elements:

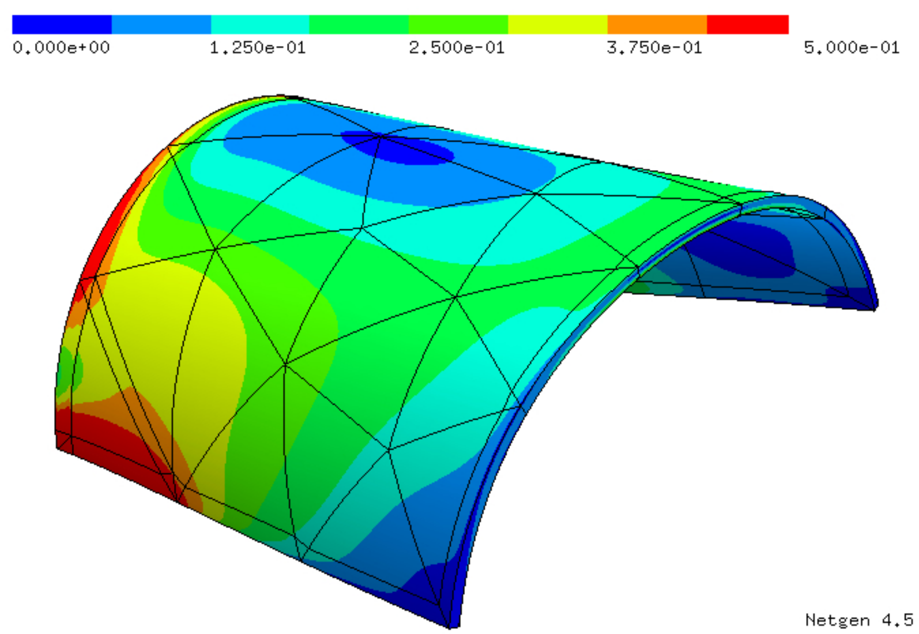




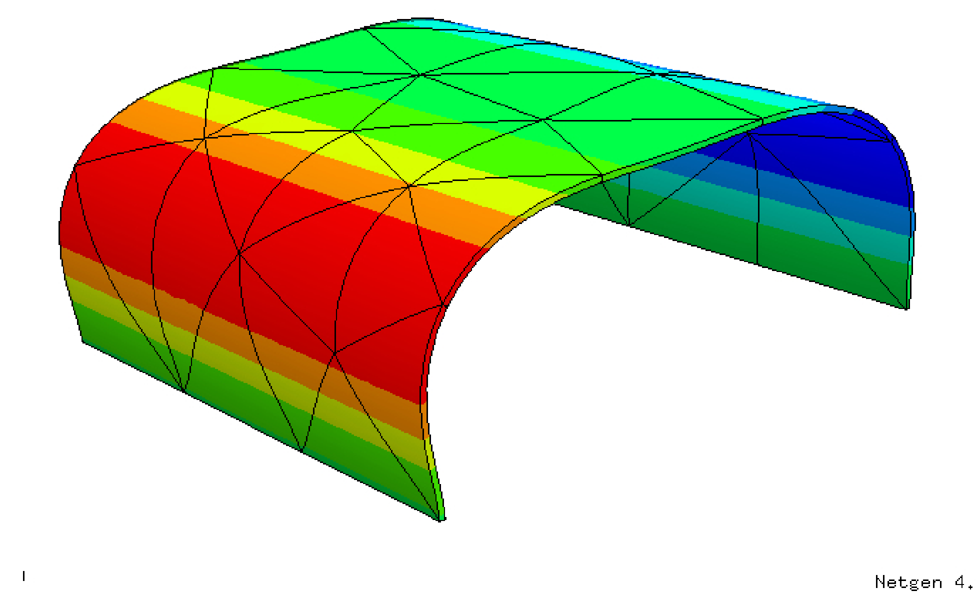
[A. Becirovic + J.S., Proc. to IASS Salzburg, 2005]

Computations on cylindrical shells

Tensor product elements, anisotropic polynomial order



membrane dominated case



bending dominated case

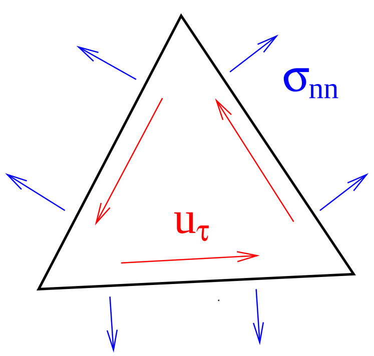
R = 0.5, t = 0.01, h = 0.25
$p = 6$, $p_z = 2$: 144 its, $\kappa = 118$
$p = 8$, $p_z = 2$: 175 its, $\kappa = 223$

New Mixed Finite Elements

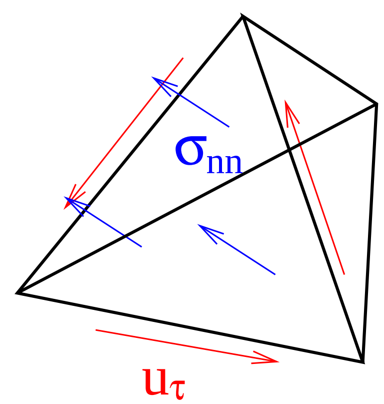
Mixed elements for approximating displacements and stresses.

- tangential components of displacement vector
- normal-normal component of stress tensor

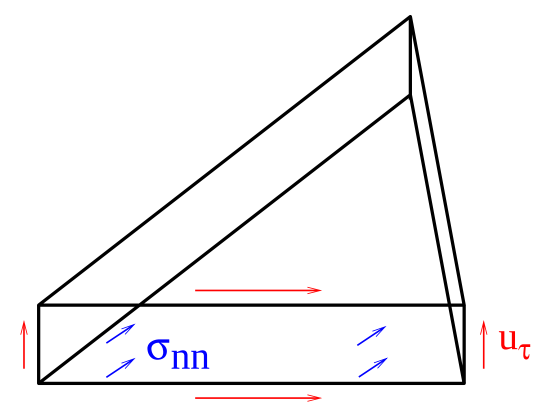
Triangular Finite Element:



Tetrahedral Finite Element:



Prismatic Finite Element:



Robust with respect to volume and shear locking

[J.S. and Astrid Sinwel]

Conclusion

We have considered

- Robust discretization methods for parameter dependent problems
- Robust preconditioners for the arising matrix equations

Ongoing work

- Construction of locking free 3D elements
- High order elements and p-version preconditioning