

A High-Order Discontinuous Galerkin Method for the Unsteady Incompressible Navier-Stokes Equations

Khosro Shahbazi¹, Paul F. Fischer² and C. Ross Ethier¹

¹University of Toronto and ²Argonne National Laboratory

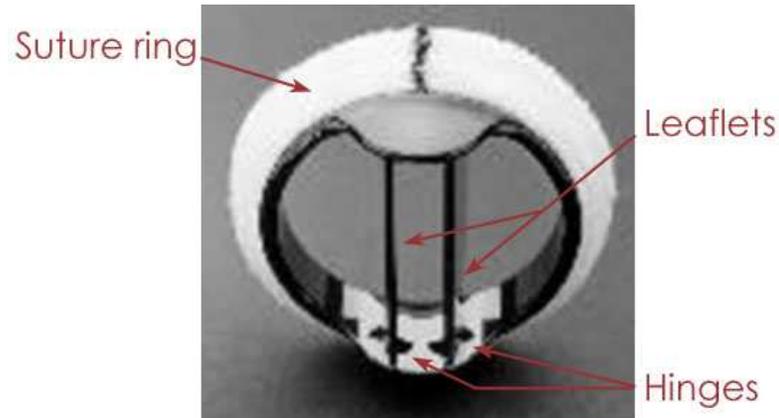
17th International Conference on Domain Decomposition Methods

St. Wolfgang/Strobl Austria, July 3-7 2006

Overview

- Introduction
- Navier-Stokes Solver
- Verification
- Conclusions

Introduction



Mechanical Heart Valve

- We are developing an efficient parallel code for the unsteady incompressible Navier-Stokes eqs. based on
 - C++, RPI's AOMD and Argonne's PETSc.
 - Complexities of the simulation of blood flow include complex geometry, pulsatility, transition and highly anisotropic and intermittent turbulence.

Introduction

The unsteady incompressible Navier-Stokes (NS) eqs:

$$\begin{aligned}\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} &= \frac{1}{Re} \nabla^2 \mathbf{u} - \nabla p && \text{in } \Omega \times [0, T], \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega \times [0, T], \\ \mathbf{u}(t = 0) &= \mathbf{u}_0 && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{u}_D && \text{on } \partial\Omega_D, \\ \nu \frac{\partial \mathbf{u}}{\partial n} - p \mathbf{n} &= 0 && \text{on } \partial\Omega_N,\end{aligned}$$

with $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$.

- We are interested in convection-dominated regimes ($Re \equiv \frac{UL}{\nu} \gg 1$).

Introduction

For the spatial discretization, we use high-order discontinuous Galerkin (DG) methods since

- Advantages in capturing features of convection-dominated flow
- Facilitate hp-adaptivity
- Yield block-diagonal mass matrix in the context of semi-explicit time integration

Introduction

- For the stationary Navier-Stokes problem, DG methods with equal and mixed polynomial order approximation for the velocity and pressure have been recently developed (Girault et al. 2005, Cockburn et al. 2005).
- However, corresponding efficient numerical solution procedures for the **unsteady** Navier-Stokes equations have not yet been proposed.

Introduction

The objective is to:

- Devise an efficient numerical scheme for the unsteady NS problem based on the high-order discontinuous Galerkin method on triangular and tetrahedral elements.

Navier-Stokes Solver

Temporal Discretization

- A simple and efficient scheme is to use a semi-explicit scheme in which the nonlinear term is treated explicitly and the Stokes operator is treated implicitly.
- We use a third-order backward differentiation (BD3) scheme and a third-order extrapolation (EX3) (Karniadakis et al. 1991) to discretize the unsteady and the nonlinear terms, respectively .

Navier-Stokes Solver

Preliminaries

The discontinuous approximate space \mathcal{V}_k is defined as

$$\mathcal{V}_k := \{v \in L^2(\Omega) \mid v|_K \in P_k(K), \forall K \in \mathcal{T}_h\},$$

and we choose $\mathbf{u}_h \in \mathcal{V}_l^d$ and $p_h \in \mathcal{V}_m$.

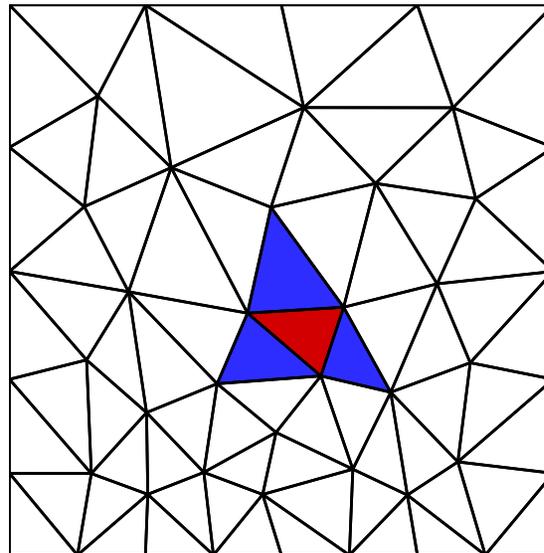
- Note that $l = m$ forms the equal-order approximation and $l = m - 1$ forms the mixed-order approximation.
- For \mathcal{V}_k , we choose a nodal high-order basis.
- The nodal basis are Lagrange polynomials calculated based on the nodal set of Hesthaven (1998) or Hesthaven and Tang (2000) defined on a standard triangle or tetrahedron, respectively.

Navier-Stokes Solver

Nonlinear Treatment

We introduce a simple method in which the nonlinear term is discretized in the divergence form using the local Lax-Friedrichs fluxes.

- Local conservativity is inherent.
- Lax-Friedrichs fluxes are suitable for unstructured meshes.
- The choice of local Lax-Friedrichs fluxes leads to a compact stencil size.



Navier-Stokes Solver

Stokes Discretization

Following Hansbo and Larson (2002), we define the discontinuous approximations \mathbf{u}_h and p_h by requiring that

$$\begin{aligned}A_h(\mathbf{u}_h, \mathbf{v}_h) + B_h(\mathbf{u}_h, \mathbf{v}_h) + D_h(\mathbf{v}_h, p_h) &= F_h(\mathbf{v}_h), \\D_h(\mathbf{u}_h, q_h) &= G_h(q_h),\end{aligned}$$

$$B_h(\mathbf{u}, \mathbf{v}) = \sum_K \int_K \frac{\beta_0}{\Delta t} \mathbf{u} \cdot \mathbf{v} d\mathbf{x},$$

$$D_h(\mathbf{v}, q) = - \sum_K \int_K q \nabla \cdot \mathbf{v} d\mathbf{x} + \sum_{\Gamma_{ID}} \int_e \{q\} [[\mathbf{v}]] \cdot \mathbf{n}_e ds,$$

Navier-Stokes Solver

Stokes Discretization

$$\begin{aligned} A_h(\mathbf{u}, \mathbf{v}) &= \sum_K \int_K \frac{1}{Re} \nabla \mathbf{u} : \nabla \mathbf{v} dx \\ &\quad - \sum_{\Gamma_{IDP}} \int_e \frac{1}{Re} \left[\mathbf{n}_e \cdot \{ \nabla \mathbf{u} \} \cdot [\mathbf{v}] + \mathbf{n}_e \cdot \{ \nabla \mathbf{v} \} \cdot [\mathbf{u}] \right] ds \\ &\quad + \sum_{\Gamma_{IDP}} \int_e \frac{\mu}{Re} [\mathbf{u}] \cdot [\mathbf{v}] ds. \end{aligned}$$

- A_h corresponds to the discretization of the Laplacian by the interior penalty (IP) method of Arnold (1982).
- We have recently derived an explicit expression for the penalty parameter μ for simplicial elements (Shahbazi 2005).

$$\mu = \frac{(k+1)(k+d)}{d} \max_K \left(\frac{(Area)_K}{(Volume)_K} \right).$$

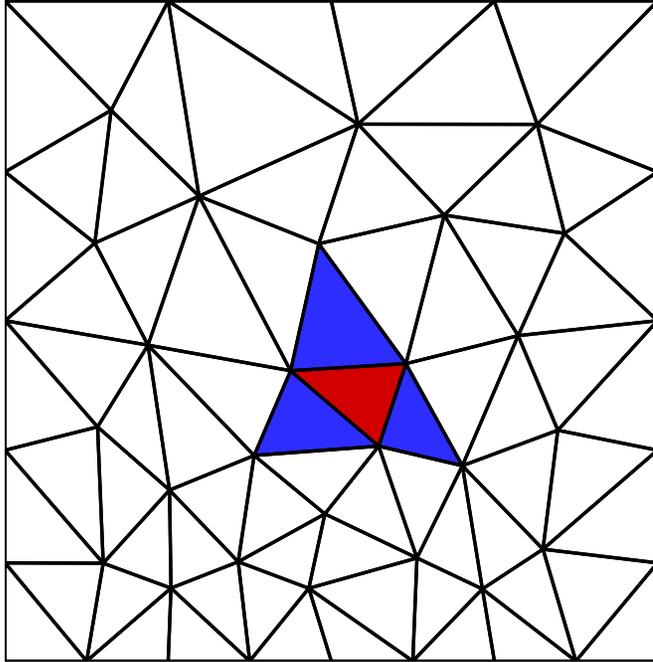
Navier-Stokes Solver

Stokes Discretization

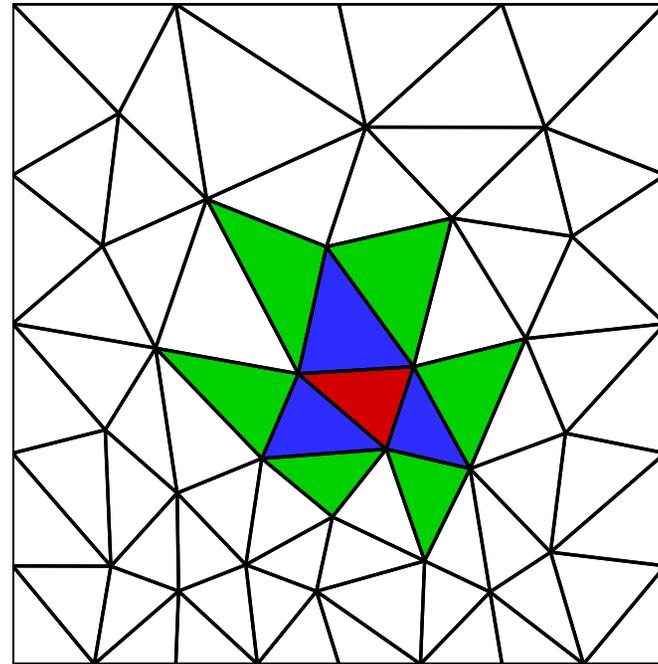
- Why do we prefer IP over other types of DG methods?
- Because the IP method offers:
 - Simplicity, minimum stencil size, symmetry, stability, local conservativity, optimal rate of convergence.

Navier-Stokes Solver

Stokes Discretization



Minimum Stencil
IP method



Extended Stencil
Local DG method
(Cockburn and Shu, 1998)

Navier-Stokes Solver

Stokes Discretization

- The algebraic form of the Stokes system after the application of the nodal high-order representation is:

$$\begin{bmatrix} \mathbf{H} & \mathbf{D}^T \\ \mathbf{D} & 0 \end{bmatrix} \begin{bmatrix} \underline{\mathbf{u}}^{n+1} \\ \underline{p}^{n+1} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{f}}^{n+1} \\ \underline{g}^{n+1} \end{bmatrix}.$$

- $\mathbf{H} = (1/Re)\mathbf{A} + (\beta_0/\Delta t)\mathbf{B}$.
- \mathbf{A} represents the Laplacian operator, and \mathbf{B} denotes the block diagonal mass matrix.
- \mathbf{D} represents the divergence operator.

Navier-Stokes Solver

Stokes Discretization

- Since $(Re/\Delta t) \gg 1$, an efficient scheme is to decouple velocity and pressure via an approximate algebraic splitting (e.g., Perot 1993).

$$1. \quad \mathbf{H}\underline{\mathbf{u}}^{*n+1} = \underline{\mathbf{f}}^{n+1} - \mathbf{D}^T \underline{\hat{p}}^n$$

$$2. \quad (-\mathbf{D}\mathbf{H}_I\mathbf{D}^T) (\underline{\hat{p}}^{n+1} - \underline{\hat{p}}^n) = (-\mathbf{D}\underline{\mathbf{u}}^{*n+1} + \underline{\mathbf{g}}^{n+1}) \frac{\beta_0}{\Delta t}$$

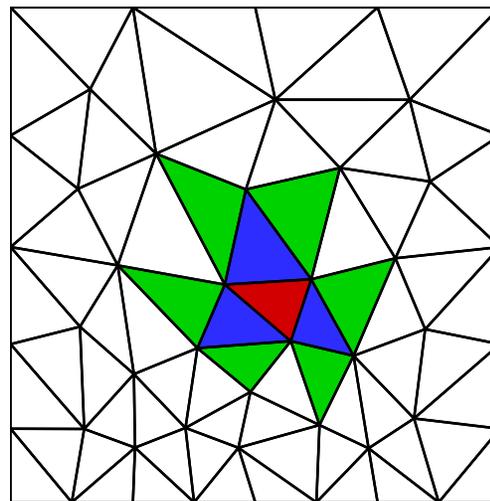
$$3. \quad \underline{\hat{\mathbf{u}}}^{n+1} = \underline{\mathbf{u}}^{*n+1} - \frac{\Delta t}{\beta_0} \mathbf{H}_I \mathbf{D}^T (\underline{\hat{p}}^{n+1} - \underline{\hat{p}}^n)$$

- $\mathbf{H}_I = \mathbf{B}^{-1}$.
- $(-\mathbf{D}\mathbf{H}_I\mathbf{D}^T)$ is called the consistent Poisson operator.
- The scheme is second order accurate in time.

Navier-Stokes Solver

Stokes Discretization

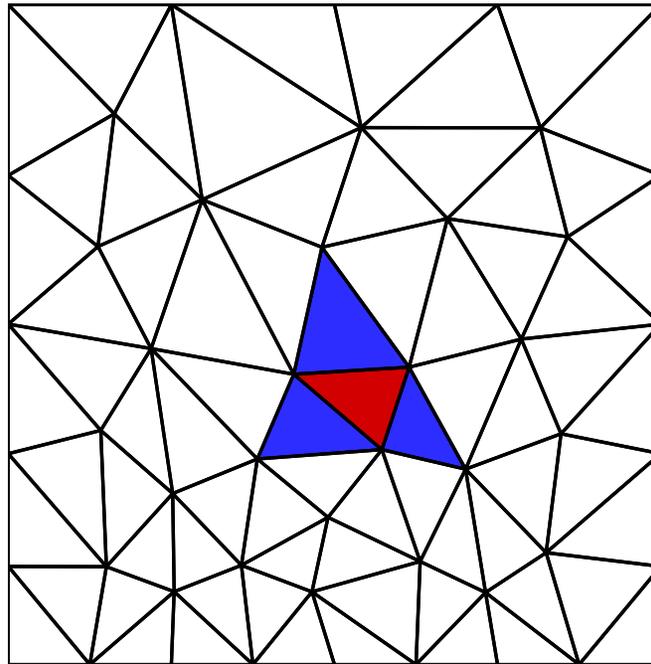
- The first and the second steps are solved iteratively using the Conjugate Gradient method.
- Since $(Re/\Delta t) \gg 1$, the Helmholtz solves are easily preconditioned by the block diagonal mass matrix.
- The pressure solve is the dominant computation.
- The pressure operator $(-\mathbf{D}\mathbf{H}_I\mathbf{D}^T)$ has extended stencil.



Navier-Stokes Solver

Stokes Discretization

- Can we have a minimum stencil size for the pressure equation?



Navier-Stokes Solver

Stokes Discretization

- Careful inspection of the pressure operator $(-\mathbf{D}\mathbf{H}_I\mathbf{D}^T)$ reveals that this operator results from the application of the Local DG method to a Laplacian with the following BCs:

$$\begin{aligned} -\nabla^2 v & \quad \text{in } \Omega \\ \nabla v \cdot \mathbf{n} & = 0 \quad \text{on } \partial\Omega_D \\ v & = 0 \quad \text{on } \partial\Omega_N \end{aligned}$$

- We propose to replace the pressure operator $(-\mathbf{D}\mathbf{H}_I\mathbf{D}^T)$ with the operator arising from the IP discretization of the Laplacian with the above BCs.

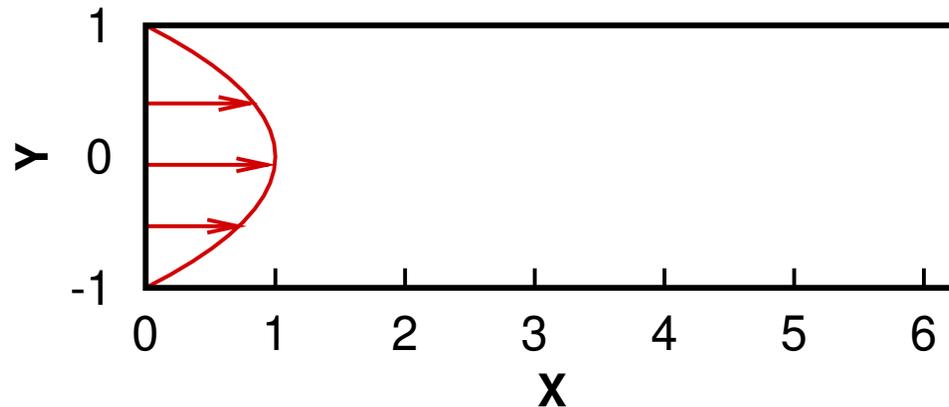
Navier-Stokes Solver

Stokes Discretization

- The justification is that the IP method and the Local DG method are very similar for stability, boundedness and the optimal rate of convergence as shown by Arnold et al. (2002) in a unified analysis of the DG methods for elliptic problems.
- Note that since the replacement is applied in the algebraic level, no unphysical BCs have been introduced.
- This not only simplifies the scheme, but also enhances the overall efficiency of the scheme.

Verification

2D Orr-Sommerfeld Stability Problem



- A parabolic base flow are sustained in x direction using a constant body force.
- $Re = 7500$ based on the maximum velocity and half channel height.
- Periodic and Dirichlet boundary conditions are imposed in the streamwise and spanwise directions, respectively.

Verification

2D Orr-Sommerfeld Stability Problem

- The base flow is disturbed with small-magnitude Tollmien-Schlichting waves, i.e, the initial condition is

$$\mathbf{u}(x, y, 0) = \mathbf{U}_0 + \epsilon \hat{\mathbf{u}}.$$

- \mathbf{U}_0 is the parabolic profile,
- $\hat{\mathbf{u}}$ corresponds to the only unstable eigensolution of the Orr-Sommerfeld equation with wave number unity at $Re = 7500$, and $\epsilon = 10^{-4}$.
- According to the linear stability theory, the perturbation energy

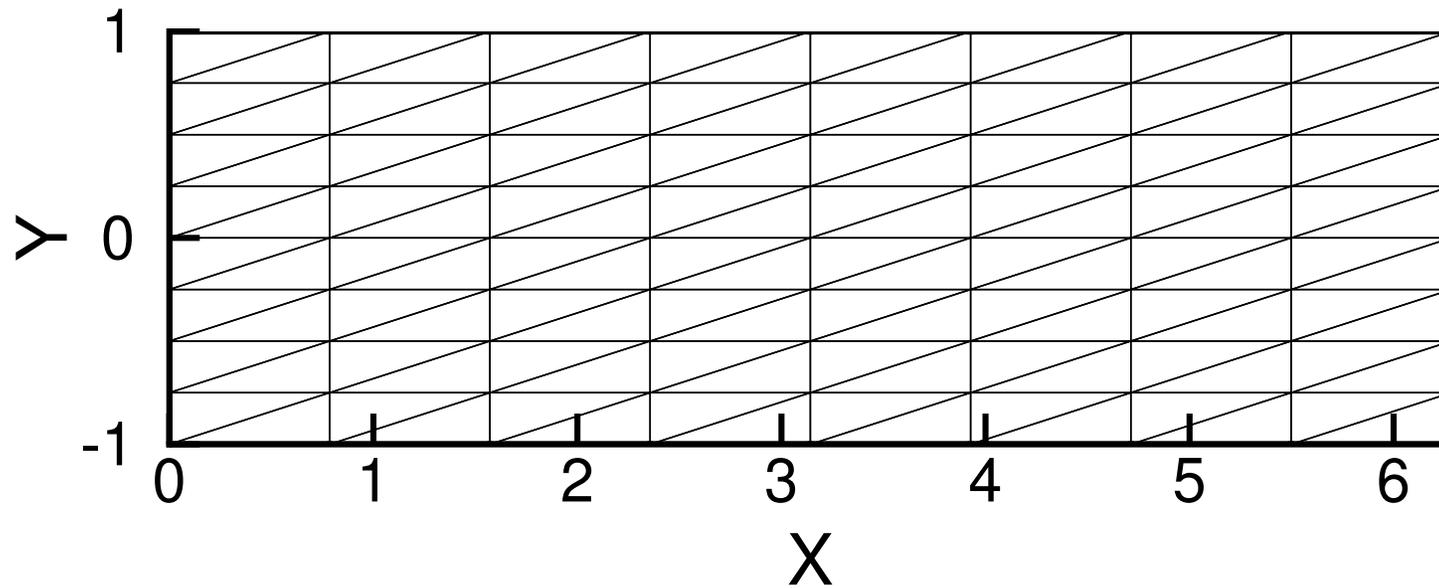
$$E(t) = \int_0^{2\pi} \int_{-1}^1 (\mathbf{u} - \mathbf{U}_0)^2 dy dx$$

should grow as $e^{2\omega_i t}$, where $\omega_i = 0.002234976$.

Verification

2D Orr-Sommerfeld Stability Problem

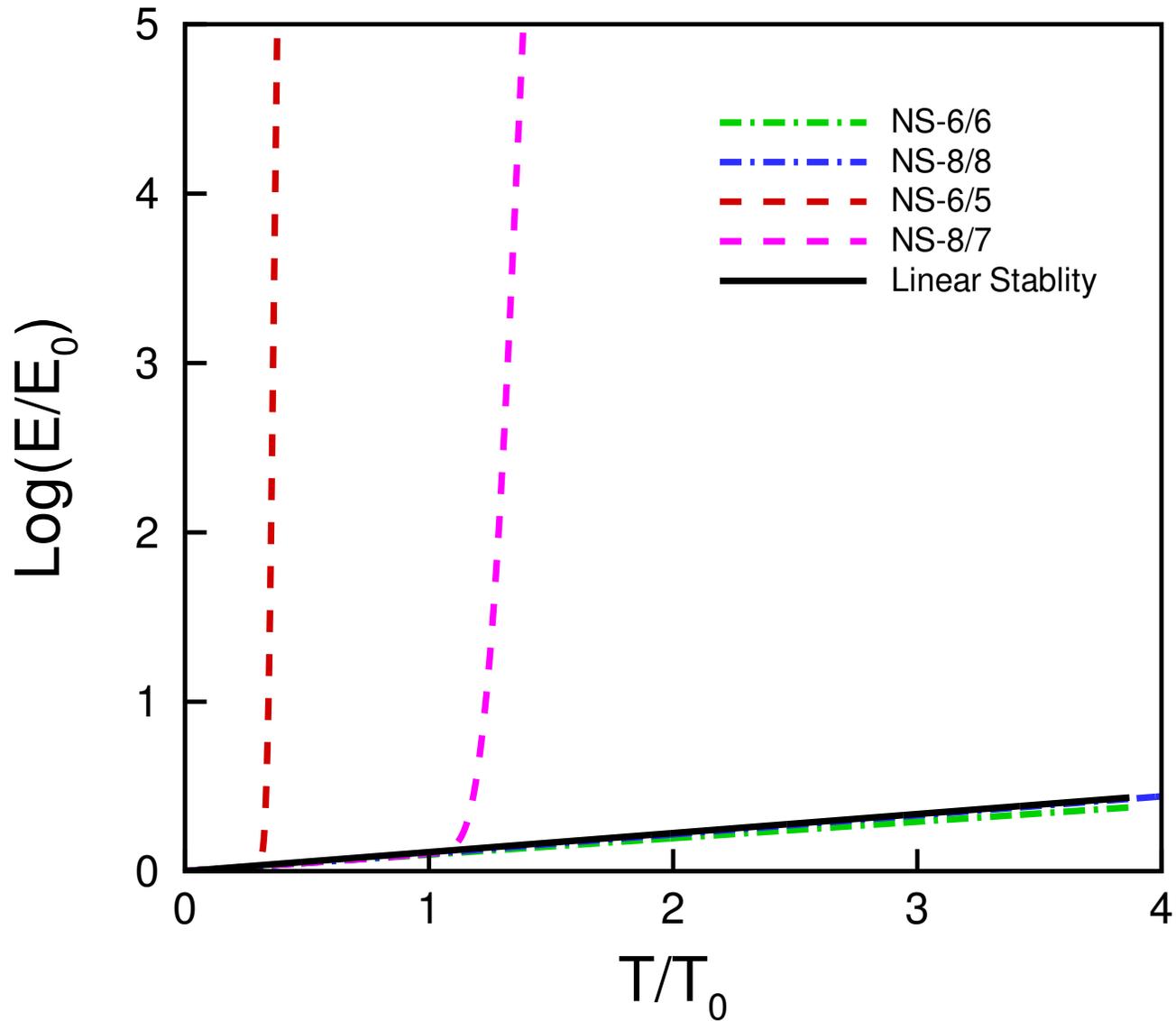
128 Triangles



Verification

2D Orr-Sommerfeld Stability Problem

$Re = 7500$, 128 Triangles, $\Delta t = 10^{-3}$



Verification

2D Orr-Sommerfeld Stability Problem

- The analysis of a similar instability in the $Q_k - Q_{k-2}$ spectral element discretization of the Navier-Stokes equations was reported by Wilhelm and Kleiser (2001).
- This instability may remain hidden at lower Re number (Shahbazi et al. 2006).

Conclusions

- We have presented an efficient numerical scheme for the unsteady incompressible Navier-Stokes equations in convection-dominated regimes.
- Our scheme is based on the high-order discontinuous Galerkin spatial discretization and approximate algebraic splitting of the velocity and pressure calculations.
- An important feature of our method is to discretize the nonlinear term, velocity and pressure equations with minimum stencil size; thus, enhancing simplicity and overall efficiency of the scheme.
- We have verified the accuracy and stability of our method by solving popular benchmarking tests, including Orr-Sommerfeld stability problem.

Acknowledgments

- Natural Sciences and Engineering Research Council of Canada (NSERC).
- Canada Research Chairs Program.
- U.S. Department of Energy, the Office of Advanced Scientific Computing Research.