A Family of Energy Minimizing Coarse Spaces for Overlapping Schwarz Preconditioners

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Summary. A simple and effective approach is presented to construct coarse spaces for overlapping Schwarz preconditioners. The approach is based on energy minimizing extensions of coarse trace spaces, and can be viewed as a generalization of earlier work by Dryja, Smith, and Widlund. The use of these coarse spaces in overlapping Schwarz preconditioners leads to condition numbers bounded by $C(1 + H/\delta)(1 + \log(H/h))$ for certain problems when coefficient jumps are aligned with subdomain boundaries. For problems without coefficient jumps, it is possible to remove the $\log(H/h)$ factor in this bound by a suitable enrichment of the coarse space. Comparisons are made with the coarse spaces of two other substructuring preconditioners. Numerical examples are also presented for a variety of problems.

1 Introduction

In order to introduce the subject of this paper, consider the linear system

$$Ax = b, \tag{1}$$

where A is a coefficient matrix, x is a vector of unknowns, and b is a known vector. The coarse space for x can be defined as the range of an interpolation matrix Φ. The vector of unknowns for overlapping subdomain i can be expressed as $R_i x$, where each row of the restriction matrix R_i has a single nonzero entry of unity. We can now express a two-level, additive, overlapping Schwarz preconditioner for A concisely as

$$M^{-1} = \Phi A_0^{-1} \Phi^T + \sum_{i=1}^{N} R_i^T A_i^{-1} R_i, \tag{2}$$

where N is the number of subdomains, and

$$A_0 = \Phi^T A \Phi, \qquad A_i = R_i A R_i^T. \tag{3}$$

Detailed introductions to overlapping Schwarz preconditioners can be found in [15] and [16]. If restriction matrices R_i are available, we see from (2) and (3) that the only missing ingredient for M^{-1} is the interpolation matrix Φ. The subject of this paper is an approach to constructing Φ.

If A in (1) originates from a finite element discretization of an elliptic partial differential equation, then Φ can be constructed using the shape functions of a coarser discretization. One obvious shortcoming of such an approach is that it requires an auxiliary finite element mesh. To address this shortcoming, *algebraic* approaches have been developed that do not require a second mesh. Examples of these include smoothed aggregation [1, 8] and partition of unity methods [12]. The approach presented here is also an algebraic approach and can be viewed as a generalization of earlier work in [6]; see also Section 5.4.3 of [16] for a description.

A common perception is that condition number bounds for iterative substructuring approaches are superior to those of their overlapping Schwarz counterparts for problems with large jumps in material properties. Although proofs are not provided here, it can be shown, under the usual assumptions for substructuring, that use of the subject coarse spaces in overlapping Schwarz preconditioners leads to condition number bounds that are competitive with iterative substructuring for certain problems. In addition, for problems with constant material properties, the coarse spaces can be enriched to give the classic bounds for two-level overlapping Schwarz preconditioners whose coarse spaces are based on coarse triangulations. We note that some other coarse spaces well suited for overlapping Schwarz preconditioners and problems with jumps in material properties can be found in [5, 13, 7, 14].

The paper is organized as follows. The subject approach for constructing coarse spaces is described in Section 2. Comparisons with two different substructuring preconditioners are given in Section 3. Some of the theoretical results available to date are summarized in Section 4. Section 5 provides numerical examples for the Poisson equation, elasticity, plate bending, and problems in $H(\text{curl}; \Omega)$.

2 Our Approach

Consider a finite element mesh, and let $\Omega_1, \ldots, \Omega_N$ denote a partitioning of its elements into nonoverlapping subdomains. Thus, each element is contained in exactly one subdomain. Decomposing a mesh into subdomains can be readily done using graph partitioning software.

Given a decomposition into nonoverlapping subdomains, the only other required input is a coarse matrix G. This matrix has the same number of rows as x in (1) and its number of columns is flexible. The most important feature of G is that its columns span the rigid body modes of each subdomain. We note that the coarse space in Algorithm 6.10 of [6] and Algorithm 5.16 of [16] is identical to the present one for the special case of scalar partial differential equations and G chosen as a vector with all entries equal to unity. Accordingly, we use the acronym GDSW for generalized Dryja, Smith, Widlund coarse space.

As in (1), let x denote the vector of degrees of freedom (dofs) for the original problem. Similarly, let x_Γ denote the vector of dofs in x shared by two or more subdomains. We then have $x_\Gamma = R_\Gamma x$, where each row of the restriction matrix R_Γ has exactly one nonzero entry of unity. The vector x_Γ can be expressed in partitioned form as

$$x_\Gamma = \sum_{j=1}^{M} R_{\Gamma_j}^T x_{\Gamma_j}, \tag{4}$$

where $x_{\Gamma_j} = R_{\Gamma_j} x_\Gamma$. As with the other subscripted R matrices, each row of R_{Γ_j} contains exactly one nonzero entry equal to unity. The partitioning in (4) is chosen such that all dofs in x_{Γ_j} are connected and common to the same set of subdomains. Thus, the dofs in x_{Γ_j} form an equivalence class.

The coarse approximation of x_{Γ_j} is expressed as

$$x_{\Gamma_{jc}} = G_{\Gamma_j} q_j \tag{5}$$

for some q_j, where the columns of G_{Γ_j} form a basis for the columns of $R_{\Gamma_j} R_\Gamma G$. Accordingly, from (4) and (5) the coarse approximation of x_Γ is given by

$$x_{\Gamma_c} = \sum_{j=1}^{M} R_{\Gamma_j}^T G_{\Gamma_j} q_j = \Phi_\Gamma q \tag{6}$$

for some q. The coarse space for the remaining dofs, not on subdomain boundaries, is obtained from energy minimizing extensions of x_{Γ_c} into subdomain interiors. We note that these extensions require either exact or approximate (with some care) solutions of subdomain problems with nonhomogeneous essential boundary conditions. All of these problems are local to each subdomain and can be solved in parallel. Notice that the support of coarse basis functions associated with G_{Γ_j} only includes those subdomains having Γ_j a part of their boundaries. Thus, the coarse basis functions have local support.

To obtain an explicit expression for the interpolation matrix Φ, define

$$x_c = R_\Gamma^T x_{\Gamma_c} + R_I^T x_I, \tag{7}$$

where R_I is a restriction matrix to subdomain interiors and x_I is the corresponding vector of interior dofs. Substituting (6) into (7) and minimizing the potential $x_c^T A x_c$ with respect to x_I then leads to

$$x_c = (R_\Gamma^T \Phi_\Gamma + R_I^T \Phi_I) q = \Phi q,$$

where

$$\Phi_I = -(R_I A R_I^T)^{-1} R_I A R_\Gamma^T \Phi_\Gamma.$$

3 Comparisons

In this section we make some broad comparisons with the coarse spaces for the BDD, [10] and BDDC, [2, 11] approaches. The results are summarized in Table 1. Regarding Point 3, the sparsity of the coarse stiffness matrix for BDD is not as nice as the other two because coupling can occur between nonadjacent subdomains. Of the three approaches compared, notice that the present one (GDSW) is the only one not requiring individual subdomain matrices. Concerning Point 8, the problem considered is a unit cube decomposed into N cubic subdomains. Notice that the coarse problem dimension is significantly larger for GDSW than for the other two approaches. We note, however, that the $9N$ figure for BDDC would be somewhat larger to effectively handle certain problems with large material property jumps. Regarding Point 9, we comment that special considerations must be made in order for BDD and BDDC to effectively handle nearly incompressible elasticity problems. In contrast, no special considerations are needed for GDSW. The primary reason for this can be linked to the large coarse space dimension of GDSW.

Table 1. Comparisons of coarse spaces for three different approaches. Results under the heading GDSW are for the present approach.

Point		BDD	BDDC	GDSW
1	well suited for elasticity problems	yes	yes	yes
2	well suited for plate bending problems	yes	yes	yes
3	nice coarse problem sparsity	no	yes	yes
4	individual subdomain matrices required	yes	yes	no
5	null space information required	yes	no	yes
6	simple multilevel extensions	no	yes	yes
7	theory for coefficient jumps	yes	yes	yes
8	3D elasticity coarse problem dimension	$6N$	$9N$	$36N$
9	well suited for nearly incompressible elasticity	yes	yes	yes

4 Theory

Theoretical results for two-level overlapping Schwarz preconditioners which use the subject coarse spaces have been obtained for the Poisson equation and for isotropic elasticity provided the Poisson ratio ν is bounded away from $1/2$. Because of space limitations, additional details and proofs are given elsewhere [3]. Under the usual assumptions for substructuring given in Section 4.2 of [16], we have the condition number bound

$$\kappa(M^{-1}A) \leq C(1 + H/\delta)(1 + \log(H/h)), \tag{8}$$

provided the columns of the coarse matrix G of Section 2 span the rigid body modes of the problem operator. The constant C is independent of both the number of subdomains and possible jumps in material properties across subdomain boundaries. The term H/h is the ratio of the subdomain diameter to that of the elements and H/δ is the typical ratio of H and overlap widths. For problems without coefficient jumps, the $\log(H/h)$ term in (7) can be removed anytime the columns of G span all linear functions of the spatial coordinates.

For stable, mixed finite element formulations of elasticity that are based on continuous interpolation of displacement and discontinuous interpolation of pressure, the pressure dofs can be eliminated at the element level provided $\nu < 1/2$. Such an elimination process results in a finite element with only displacement dofs. Numerical results and initial theoretical work for problems that use such elements suggest that condition number bounds exist which are insensitive to ν being arbitrarily close to the incompressible limit of $1/2$. The bound in (8), however, has an $(H/\delta)^3$ dependence [4].

The coarse spaces considered here have also proven useful in the analysis of overlapping Schwarz [3] and iterative substructuring [9] methods on irregular subdomains in two dimensions. Efforts are underway to extend these results to irregular subdomains in three dimensions.

Numerical results in the next section suggest that the coarse spaces also work well for plate bending and $H(\text{curl};\Omega)$ problems in 2D, but we presently have no supporting theory. In addition, a suitable coarse space for $H(\text{curl};\Omega)$ problems in 3D has not yet been identified.

5 Numerical Examples

Results are presented for unit square domains with homogeneous essential boundary conditions on all four sides. The stable $\mathbb{Q}_2 - \mathbb{P}_1$ element is used in the nearly incompressible elasticity examples. This element uses continuous biquadratic interpolation of displacement and discontinuous linear interpolation of pressure. Moreover, its pressure dofs are eliminated at the element level. The standard bilinear element $\mathbb{Q}_1$ is used for all the other elasticity and Poisson equation examples. The plate bending examples use the discrete Kirchoff triangular element and the lowest-order quadrilateral edge element is used for the $H(\text{curl}; \Omega)$ examples. Except for the $H(\text{curl}; \Omega)$ examples, the columns of the coarse matrix G described in Section 2 span the rigid body modes of the problem operator.

Equation (1) is solved to a relative residual tolerance of 10^{-8} for a random vector b using preconditioned conjugate gradients. In addition to numbers of iterations, condition number estimates obtained from the conjugate gradient iterations are also reported. The overlap width δ is defined as the minimum distance between a subdomain boundary and the boundary of its overlapping extension. Unless stated otherwise, the values of the elastic modulus and Poisson ratio ν are 1 and 0.3, respectively. The elasticity results are for plane strain conditions.

5.1 Poisson Equation, Compressible Elasticity, Plate Bending

Results for fixed values of H/h, H/δ, and increasing numbers of square subdomains are shown in Table 2. Good scalability with respect to the number of subdomains N is evident for all three problem types. We now fix $N = 16$ and $H/\delta = 4$ while increasing the ratio H/h. The slow growth in iterations and condition numbers shown in Table 3 is consistent with the estimate in (8). Results for a problem with the elastic modulus equal to σ in a square centered region of length $1/2$ and equal to 1 elsewhere are shown in Table 4. Material property jumps are aligned with subdomain boundaries and there is no great sensitivity to σ in the numerical results.

Table 2. Iterations (iter) and condition number estimates (cond) for increasing numbers of subdomains N. Fixed values of $H/h = 8$ and $H/\delta = 4$ are used.

N	Poisson Equation		Linear Elasticity		Plate Bending	
	iter	cond	iter	cond	iter	cond
16	24	8.97	24	6.93	41	17.7
64	27	10.0	26	7.52	48	19.8
256	28	10.3	28	8.01	51	21.1
1024	30	10.4	29	8.28	55	21.7

5.2 Nearly Incompressible Elasticity

Results for a fixed value of H/δ are shown in Table 5 for three different values of the Poisson ratio ν. As noted earlier, the stable $\mathbb{Q}_2 - \mathbb{P}_1$ element is used. Good scalability with respect to the number of subdomains is evident for all three values

Table 3. Results for $N = 16$ and $H/\delta = 4$.

H/h	Poisson Equation		Linear Elasticity		Plate Bending	
	iter	cond	iter	cond	iter	cond
8	24	8.97	24	6.93	41	17.7
16	25	10.5	25	7.87	46	23.4
24	25	11.3	26	8.38	48	26.2
32	26	11.9	27	8.73	50	28.0
40	26	12.3	27	8.99	48	29.4

Table 4. Results for elastic modulus equal to σ in a square centered region and 1 elsewhere. Fixed values of $N = 16$, $H/h = 8$, and $H/\delta = 4$ are used.

σ	Poisson Equation		Linear Elasticity		Plate Bending	
	iter	cond	iter	cond	iter	cond
10^{-4}	23	6.93	24	6.02	38	13.7
10^{-2}	23	7.05	23	6.05	40	15.4
1	24	8.97	24	6.93	41	17.7
10^{2}	24	10.5	26	7.72	39	17.3
10^{4}	24	10.5	27	7.74	38	15.3

of ν. Table 5 also shows results for 16 subdomain and different values of H/h. As in the previous examples, the number of iterations and condition number estimates grow slowly as H/h increases. Notice in all the examples that the ratio H/δ has been fixed. Although the relevant numerical results are not presented here, we have observed a stronger dependence on H/δ than in (8) for problems with ν very close to 1/2.

Table 5. Plane strain results for $H/\delta = 4$.

$H/h = 8$							$N = 16$						
N	$\nu = 0.3$		$\nu = 0.4999$		$\nu = 0.49999$		H/h	$\nu = 0.3$		$\nu = 0.4999$		$\nu = 0.49999$	
	iter	cond	iter	cond	iter	cond		iter	cond	iter	cond	iter	cond
16	25	8.05	31	10.6	33	10.6	8	25	8.05	31	10.6	33	10.6
64	29	8.93	32	11.2	34	11.2	16	27	8.89	33	12.3	34	12.3
256	32	9.67	34	11.6	35	11.7	24	28	9.35	34	13.4	36	13.4
1024	33	10.1	34	11.7	35	11.7	32	28	9.67	35	14.1	36	14.1
4096	34	10.3	34	11.7	35	11.8	40	28	9.90	34	14.6	36	14.7

5.3 $H(\mathbf{curl}; \Omega)$ Examples

We now consider examples for the bilinear form

$$a(\mathbf{u}, \mathbf{v}) = \int_{\Omega} (\alpha(\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) + \beta \mathbf{u} \cdot \mathbf{v}) \, \mathbf{dx},$$

where $\alpha \geq 0$, $\beta > 0$, and $\nabla \times \mathbf{u}$ denotes the curl of $\mathbf{u}$. We assume that edge element shape functions are scaled so that the integral of the tangential component along each edge of an element is unity. Assuming a consistent sign convention for each element edge of a subdomain edge, the matrix G is chosen as a vector with all entries equal to unity.

To simplify the computer implementation, an overlapping subdomain is chosen to include all edges a graph distance m or less from the edges of the nonoverlapping subdomain. Results for fixed values of β, H/h, and m are shown in Table 6 for different values of α and N. Similar results for increasing values of H/h are shown in Table 7. In contrast to the previous examples, monotonic growth of condition number estimates with H/h is not evident. This may be caused by our choice of overlapping subdomains, but the results are quite acceptable.

Table 6. $H(\text{curl}; \Omega)$ results for $H/h = 8$, $m = 1$, and $\beta = 1$.

N	$\alpha = 0.0$		$\alpha = 10^{-2}$		$\alpha = 1$		$\alpha = 10^{2}$		$\alpha = 10^{4}$	
	iter	cond	iter	cond	iter	cond	iter	cond	iter	cond
16	6	3.01	20	5.28	25	7.38	28	7.48	30	7.49
32	6	3.01	22	5.96	26	7.47	28	7.53	31	7.54
64	6	3.01	23	6.43	26	7.52	29	7.56	31	7.57
100	6	3.01	24	6.77	27	7.58	30	7.61	32	7.62

Table 7. $H(\text{curl}; \Omega)$ results for $N = 16$, $H/(mh) = 8$, and $\beta = 1$.

H/h	$\alpha = 0.0$		$\alpha = 10^{-2}$		$\alpha = 1$		$\alpha = 10^{2}$		$\alpha = 10^{4}$	
	iter	cond	iter	cond	iter	cond	iter	cond	iter	cond
8	6	3.01	20	5.28	25	7.38	28	7.48	30	7.49
16	4	3.00	21	5.61	25	7.46	28	7.52	30	7.53
24	4	3.00	21	5.76	25	7.39	27	7.41	29	7.45
32	4	3.00	21	5.85	25	7.47	26	7.52	29	7.53
40	3	3.00	21	5.88	25	7.30	27	7.45	29	7.49

6 Conclusions

A simple and effective approach to constructing coarse spaces for overlapping Schwarz preconditioners has been presented. Initial numerical and theoretical results suggest that it could be a viable approach for a variety of problem types. There remain several opportunities for future discovery and development from both a theoretical and practical point of view.

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