p-FEM Quadrature Error Analysis on Tetrahedra

Tino Eibner[1] and Jens M. Melenk[2]

[1] Technische Universität Chemnitz, teibner@mathematik.tu-chemnitz.de
[2] Technische Universität Wien, melenk@tuwien.ac.at

1 Introduction

In the p-FEM and the closely related spectral method, the solution of an elliptic boundary value problems is approximated by piecewise (mapped) polynomials of degree p on a fixed mesh $\mathcal{T}$. In practice, the entries of the p-FEM stiffness matrix cannot be evaluated exactly due to variable coefficients and/or non-affine element maps and one has to resort to numerical quadrature to obtain a fully discrete method. Computationally, choosing shape functions that are related to the quadrature formula employed can significantly improve the computational complexity. For example, for tensor product elements (i.e., quadrilaterals, hexahedra) choosing tensor product Gauss-Lobatto quadrature with $q+1=p+1$ points in each spatial direction and taking as shape functions the Lagrange interpolation polynomials (of degree p) in the Gauss-Lobatto points effectively leads to a spectral method. The quadrature error analysis for the p-FEM/spectral method is available even for this case of minimal quadrature (see, e.g., [5, 6] and reference there). Key to the error analysis is a one-dimensional discrete stability result for the Gauss-Lobatto quadrature due to [2] (corresponding to $\alpha = 0$ in Lemma 2 below) that can readily be extended to quadrilaterals/hexahedra by tensor product arguments.

In the present paper, we show an analog of the error analysis of the above minimal quadrature for the p-FEM on tetrahedral meshes (the easier case of triangles can be treated completely analogously). Quadrature on a tetrahedron can be done by a mapping to a hexahedron via the Duffy transformation D of (3). We show in Theorem 1 that for tensor product Gauss-Lobatto-Jacobi quadrature formulas with $q+1=p+1$ points in each direction, one again has discrete stability for the fully discrete p-FEM. A complete quadrature error analysis (Theorem 2, Corollary 1) then follows from Strang's lemma and shows that the convergence rates of the Galerkin p-FEM (where all integrals are evaluated exactly) is retained by the fully discrete p-FEM. The present error analysis complements the work [3] for the p-FEM on triangles/tetrahedra where it is shown that by adapting the shape functions to the quadrature formula, the stiffness matrix can be set up in optimal complexity. However, we mention that the approximation spaces employed in [3] are no longer the classical spaces $S^{p,1}(\mathcal{T})$ of piecewise polynomials but the spaces $S^{p,1}(\mathcal{T})$ augmented by bubble shape functions for each element, which makes the static condensation more expensive.

To fix ideas, we consider

$$-\nabla \cdot (A(x)\nabla u) = f \qquad \text{on } \Omega \subset \mathbb{R}^3, \qquad u|_{\partial\Omega} = 0, \tag{1}$$

where $A \in C(\overline{\Omega}, \mathbb{R}^{3\times 3})$ is pointwise symmetric positive definite. We require A and f to be analytic on $\overline{\Omega}$ and the standard ellipticity condition

$$0 < \lambda_{min} \le A(x) \le \lambda_{max}, \qquad \forall x \in \Omega.$$

2 Quadrature Error Analysis

Notation

The reference tetrahedron $\widehat{K}$ and the reference cube $\mathcal{Q}$ are defined as

$$\widehat{K} = \{(x,y,z) \mid -1 < x,y,z \ \wedge \ x+y+z < -1\}, \qquad \mathcal{Q} := (-1,1)^3. \tag{2}$$

The Duffy transformation $D : \mathcal{Q} \to \widehat{K}$ is given by

$$D(\eta_1,\eta_2,\eta_3) := \left(\frac{(1+\eta_1)(1-\eta_2)(1-\eta_3)}{4} - 1, \frac{(1+\eta_2)(1-\eta_3)}{2} - 1, \eta_3 \right). \tag{3}$$

Lemma 1. *The Duffy transformation is a bijection between the (open) cube $\mathcal{Q}$ and the (open) tetrahedron $\widehat{K}$. Additionally,*

$$D'(\eta_1,\eta_2,\eta_3) := \left[\frac{\partial \xi_i}{\partial \eta_j}\right]_{i,j=1}^{3} = \begin{bmatrix} \frac{1}{4}(1-\eta_2)(1-\eta_3) & 0 & 0 \\ -\frac{1}{4}(1+\eta_1)(1-\eta_3) & \frac{1}{2}(1-\eta_3) & 0 \\ -\frac{1}{4}(1+\eta_1)(1-\eta_2) & -\frac{1}{2}(1+\eta_2) & 1 \end{bmatrix}^{\top},$$

$$\big(D'(\eta_1,\eta_2,\eta_3)\big)^{-1} = \frac{1}{(1-\eta_2)(1-\eta_3)} \begin{bmatrix} 4 & 2(1+\eta_1) & 2(1+\eta_1) \\ 0 & 2(1-\eta_2) & 1-\eta_2^2 \\ 0 & 0 & (1-\eta_2)(1-\eta_3) \end{bmatrix},$$

$$\det D' = \left(\frac{1-\eta_2}{2}\right)\left(\frac{1-\eta_3}{2}\right)^2. \tag{4}$$

Proof. See, for example, [4].

We employ standard notation by writing $\mathcal{P}_p(\widehat{K})$ for the space of polynomials of degree p on $\widehat{K}$, and by denoting $\mathcal{Q}_p(\mathcal{Q})$ the tensor-product space of polynomials of degree p in each variable, [7]; additionally we set

$$\widetilde{\mathcal{Q}}_p := \{u \in \mathcal{Q}_p(\mathcal{Q}) \,|\, \partial_1 u = \partial_2 u = \partial_3 u = 0 \text{ on } \eta_3 = 1 \text{ and } \partial_1 u = 0 \text{ on } \eta_2 = 1\}.$$

Remark 1. The Duffy transformation D maps the face $\eta_3 = 1$ to the point $(-1,-1,1)$ and the face $\eta_2 = 1$ to a line. An important property of $\widetilde{\mathcal{Q}}_p$ is that $u \in \mathcal{P}_p(\widehat{K})$ implies $u \circ D \in \widetilde{\mathcal{Q}}_p$.

2.1 Gauss-Lobatto-Jacobi Quadrature

Gauss-Lobatto-Jacobi Quadrature in 1D

For $\alpha > -1$, $n \in \mathbb{N}$, the Gauss-Lobatto-Jacobi quadrature formula is given by

$$\mathrm{GLJ}_{(\alpha,n)}(f) := \sum_{i=0}^{n} \omega_i^{(\alpha,n)} f(x_i^{(\alpha,n)}) \approx \int_{-1}^{1} (1-x)^\alpha f(x)\mathrm{d}x; \tag{5}$$

(see, e.g., [4, App. B]): the quadrature nodes $x_i^{(\alpha,n)}$, $i = 0, \ldots, n$, are the zeros of the polynomial $x \mapsto (1-x^2)P_n^{(\alpha+1,1)}(x)$, where $P_n^{(\alpha,\beta)}$ denotes the Jacobi polynomial of degree n with respect to the weight function $(1-x)^\alpha(1+x)^\beta$. The quadrature weights $\omega_i^{(\alpha,n)}$, $i = 0, \ldots, n$, are *positive* and explicit formulas can be found, for example, in [4, App. B]. We have:

Lemma 2. *Let $\mathcal{P}_n$ be the space of polynomials of degree n. Then for $\alpha > -1$:*

1. *For all $f \in \mathcal{P}_{2n-1}$ there holds $\mathrm{GLJ}_{(\alpha,n)}(f) = \int_{-1}^{1} f(x)(1-x)^\alpha \, dx$.*
2. *For all $f \in \mathcal{P}_n$ there holds*

$$\int_{-1}^{1} f^2(x)(1-x)^\alpha \mathrm{d}x \le \mathrm{GLJ}_{\alpha,n}(f^2) \le \left(2 + \frac{\alpha+1}{n}\right) \int_{-1}^{1} f^2(x)(1-x)^\alpha \mathrm{d}x.$$

Proof. The first assertion is well-known. The second assertion follows by the same arguments as in the case $\alpha = 0$, which can be found, for example, in [2] or [1, Corollary 1.13].

Gauss-Lobatto-Jacobi Quadrature on $\widehat{K}$

Using the change of variables formula $\int_{\widehat{K}} g\mathrm{d}x = \int_{\mathcal{Q}} (g \circ D)|\det D'|\mathrm{d}x$, we can introduce a quadrature formulas such that

$$\mathrm{GLJ}_{\mathcal{Q},n}(f) \approx \int_{\mathcal{Q}} f(\eta)|\det D'(\eta)|\mathrm{d}\eta, \qquad \mathrm{GLJ}_{\hat{K},n}(g) \approx \int_{\hat{K}} g(\xi)\,\mathrm{d}\xi$$

by setting

$$\mathrm{GLJ}_{\mathcal{Q},n}(f) := 1/8 \sum_{i_1,i_2,i_3=0}^{n} \omega_{i_1}^{(0,n)}\omega_{i_2}^{(1,n)}\omega_{i_3}^{(2,n)} f\left(x_{i_1}^{(0,n)}, x_{i_2}^{(1,n)}, x_{i_3}^{(2,n)}\right), \tag{6}$$

$$\mathrm{GLJ}_{\widehat{K},n}(g) := \mathrm{GLJ}_{\mathcal{Q},n}(g \circ D). \tag{7}$$

Using standard tensor product arguments one can deduce from the properties of the quadrature rules $\mathrm{GLJ}_{\alpha,n}$ and the formula (4) the following result:

Lemma 3. *Let $1 \le p \le q$ and let $\widehat{u} \in \mathcal{Q}_p(\mathcal{Q})$, $\widehat{v} \in \mathcal{Q}_{2q-1}(\mathcal{Q})$. Set $u := \widehat{u} \circ D^{-1}$, $v := \widehat{v} \circ D^{-1}$. Then the equalities $\mathrm{GLJ}_{\mathcal{Q},q}(\widehat{v}) = \int_{\mathcal{Q}} \widehat{v}|\det D'|\mathrm{d}\Omega$ and $\mathrm{GLJ}_{\widehat{K},q}(v) = \int_{\hat{K}} v\mathrm{d}\Omega$ are true and, for $\underline{C} := (2+1/p)(2+2/p)(2+3/p) \le 60$,*

$$\int_{\mathcal{Q}} |\widehat{u}|^2 |\det D'| d\Omega \le \mathrm{GLJ}_{\mathcal{Q},q}(\widehat{u}^2) \le \underline{C} \int_{\mathcal{Q}} |\widehat{u}|^2 |\det D'| d\Omega,$$

$$\|u\|^2_{L^2(\widehat{K})} \le \mathrm{GLJ}_{\widehat{K},q}(u^2) \le \underline{C}\|u\|^2_{L^2(\hat{K})}.$$

2.2 Discrete Stability

The following discrete stability result is the heart of the quadrature error analysis; its proof is deferred to Section 3.

Theorem 1. *Let $A \in C(\overline{\widehat{K}}, \mathbb{R}^{3\times 3})$ be pointwise symmetric positive definite, $c \in C(\overline{\widehat{K}})$. Assume the existence of λ_{min}, λ_{max}, $c_{min} > 0$ with*

$$\lambda_{min} \leq A(x) \leq \lambda_{max}, \qquad c_{min} \leq c(x) \forall x \in \widehat{K}.$$

Then for $q \geq p$ there holds for all $u \in \{u \,|\, u \circ D \in \widetilde{\mathcal{Q}}_p\}$

$$\mathrm{GLJ}_{\widehat{K},q}(\nabla u \cdot A \nabla u) \geq \frac{\lambda_{min}}{10404} \|\nabla u\|^2_{L^2(\widehat{K})} \geq \frac{\lambda_{min}}{10404\lambda_{max}} \int_{\widehat{K},q} \nabla u \cdot A \nabla u \mathrm{d}\Omega, \tag{8}$$

$$\mathrm{GLJ}_{\widehat{K},q}(cu^2) \geq c_{min} \|u\|^2_{L^2(\widehat{K})}. \tag{9}$$

2.3 Convergence Analysis of Fully Discrete *p*-FEM

For the model problem (1) and given mesh $\mathcal{T}$ consisting of (curvilinear) tetrahedra with element maps $F_K : \widehat{K} \to K$, we define the discrete bilinear form a^q and right-hand side F^q by

$$a^q(u,v) := \sum_{K \in \mathcal{T}} \mathrm{GLJ}_{\widehat{K},q} \left(((\nabla u \cdot A \nabla v)|_K \circ F_K) |\det F'_K| \right),$$
$$F^q(u) := \sum_{K \in \mathcal{T}} \mathrm{GLJ}_{\widehat{K},q} \left(((fu)|_K \circ F_K) |\det F'_K| \right).$$

We let $S_0^{p,1}(\mathcal{T}) := \{u \in H_0^1(\Omega) \,|\, u|_K \circ F_K \in \mathcal{P}_p(\widehat{K}) \quad \forall K \in \mathcal{T}\}$ and consider finite dimensional spaces V_N satisfying

$$S_0^{p,1}(\mathcal{T}) \subset V_N \subset \widetilde{S}_0^{p,1}(\mathcal{T}) := \{u \in H_0^1(\Omega) \,|\, u|_K \circ F_K \circ D \in \widetilde{Q}_p \quad \forall K \in \mathcal{T}\}. \tag{10}$$

Remark 2. By Remark 1, choosing $V_N = S_0^{p,1}(\mathcal{T})$ is admissible. Taking V_N larger than $S_0^{p,1}(\mathcal{T})$ permits adapting the shape functions to the quadrature points and permits efficient ways to generate the stiffness matrix, [3].

The fully discrete problem is then:

$$\text{Find } u_N \in V_N \text{ s.t.} \qquad a^q(u_N, v) = F^q(v) \qquad \forall v \in V_N. \tag{11}$$

The discrete stability result Theorem 1 for a single element is readily extended to meshes with several elements and existence and uniqueness of solutions to (11) follows. An application of Strang's Lemma then gives error estimates:

Theorem 2. *Let the mesh $\mathcal{T}$ be fixed and the element maps F_K be analytic on $\overline{\widehat{K}}$. Assume (10) and $q \geq p$. Let u solve (1) and u_N solve (11). Then there exist C, $b > 0$ depending only on Ω, the analytic data A, f of (1), and the analytic element maps F_K such that*

$$\|u - u_N\|_{H^1(\Omega)} \leq C \left(\inf_{v \in S_0^r(\mathcal{T})} \|u - v\|_{H^1(\Omega)} + Cr^3 e^{-b(2q+p-r)} \right)$$

for arbitrary $1 \leq r \leq \min\{p, 2(q-1) - p\}$.

Proof. The proof follows along the lines of [6, Secs. 4.2, 4.3]: Theorem 1 enables us to use a Strang lemma, and the resulting consistency terms can be made exponentially small by the analyticity of A, f, and the F_K.

Remark 3. It is worth stressing that analyticity of $\partial\Omega$ is not required in Theorem 2—only analyticity of the element maps is necessary. Hence, also piecewise analytic geometries are covered by Theorem 2. The requirement that A, f be analytic can be relaxed to the condition that $A|_K$, $f|_K$ be analytic on $\overline{K}$ for all elements.

We note that choosing $r = \lfloor p/2 \rfloor$ in Theorem 2 implies that the *rate of convergence* of the fully discrete p-FEM is typically the same as the Galerkin p-FEM in which all quadratures are performed exactly:

Corollary 1. *Assume the hypotheses of Theorem 2. Then:*

1. *If* $\inf_{v \in S_0^{p,1}(\mathcal{T})} \|u - v\|_{H^1(\Omega)} = O(p^{-\alpha})$, *then* $\|u - u_N\|_{H^1(\Omega)} = O(p^{-\alpha})$.
2. *If* $\inf_{v \in S_0^{p,1}(\mathcal{T})} \|u - v\|_{H^1(\Omega)} = O(e^{-bp})$ *for some* $b > 0$, *then there exists* $b' > 0$ *such that* $\|u - u_N\|_{H^1(\Omega)} = O(e^{-b'p})$.

3 Proof of Theorem 1

The heart of the proof of Theorem 1 consists in the assertion that for the Duffy transformation D, the matrix $(D')^{-1}(D')^{-\top}$ is equivalent to its diagonal. To that end, we recall for square matrices $A, B \in \mathbb{R}^{n\times n}$ the standard notation $A \le B$ which expresses $v^\top A v \le v^\top B v$ for all $v \in \mathbb{R}^n$. We have:

Lemma 4. *Let* $E(\eta) := (D'^{-1}D'^{-\top})(\eta)$ *and denote by* $\operatorname{diag} E(\eta) \in \mathbb{R}^{3\times 3}$ *the diagonal of* $E(\eta)$. *Then*

$$\frac{1}{3468}\operatorname{diag} E(\eta) \le E(\eta) \le 3 \operatorname{diag} E(\eta) \qquad \forall \eta \in \mathcal{Q}. \tag{12}$$

Proof. One easily shows for any invertible matrix $G \in \mathbb{R}^{n\times n}$

$$B \le A \quad \Longleftrightarrow \quad G^\top B G \le G^\top A G. \tag{13}$$

In order to prove (12), we define the diagonal matrix

$$B(\eta) := \operatorname{diag}\left[(1-\eta_2)(1-\eta_3), (1-\eta_3), 1\right]$$

and in view of (13) we are led to showing

$$\frac{1}{3468}(B^\top (\operatorname{diag} E) B)(\eta) \le (B^\top E B)(\eta) \le 3 (B^\top (\operatorname{diag} E) B)(\eta) \quad \forall \eta \in \mathcal{Q}. \tag{14}$$

Explicitly computing

$$(B^\top E B)(\eta) = \begin{pmatrix} 8(1+\eta_1)^2 + 16 & (1+\eta_1)\{4 + 2(1+\eta_2)\} & 2(1+\eta_1) \\ \text{sym.} & 4 + (1+\eta_2)^2 & (1+\eta_2) \\ \text{sym.} & \text{sym.} & 1 \end{pmatrix}$$

and applying the three estimates

$$2(1+\eta_1)\{4+2(1+\eta_2)\}v_1v_2 \leq 8(1+\eta_1)^2v_1^2 + [4+(1+\eta_2)^2]v_2^2,$$
$$4(1+\eta_1)v_1v_3 \leq 4(1+\eta_1)^2v_1^2 + v_3^2, \qquad 2(1+\eta_2)v_2v_3 \leq (1+\eta_2)^2v_2^2 + v_3^2$$

for all $\eta \in \mathcal{Q}, v_1, v_2, v_3 \in \mathbb{R}$, we conclude for any vector $v = (v_1, v_2, v_3)^\top \in \mathbb{R}^3$

$$v^\top (B^\top EB)(\eta)v \leq v^\top \operatorname{diag}\left[20(1+\eta_1)^2 + 16, 8 + 3(1+\eta_2)^2, 3\right] v.$$

In view of $(B^\top(\operatorname{diag} E)B)(\eta) = \operatorname{diag}\left[8(1+\eta_1)^2+16, 4+(1+\eta_2)^2, 1\right]$ we arrive at $(B^\top EB)(\eta) \leq 3(B^\top(\operatorname{diag} E)B)(\eta)$. In order to prove the lower bound of (14) we observe that $(B^\top EB)(\eta)$ is symmetric positive definite for all $\eta \in \mathcal{Q}$; denoting by $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3$ the three eigenvalues of $(B^\top EB)(\eta)$, we conclude from the Gershgorin circle theorem $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq 68$ for all $\eta \in \mathcal{Q}$. Moreover, a direct calculation shows $\det(B^\top EB)(\eta) = 64$. Thus, $\lambda_1 \geq \det(B^\top EB)/\lambda_2^2 \geq 4/289$ for all $\eta \in \mathcal{Q}$. Hence for all $\eta \in \mathcal{Q}$

$$\begin{aligned}(B^\top EB)(\eta) &\geq \frac{4}{289} I \geq \frac{4}{289} \operatorname{diag}\left[\frac{8(1+\eta_1)^2+16}{48}, \frac{4+(1+\eta_2)^2}{8}, 1\right] \\ &\geq \frac{1}{3468}(B^\top(\operatorname{diag} E)B)(\eta).\end{aligned}$$

P*roof of Theorem 1.* We will only show (8) as (9) follows easily from Lemma 3. Let u be such that $\widehat{u} := u \circ D \in \widetilde{\mathcal{Q}}_p$. In view of the positivity of the quadrature weights and Lemma 4 we get for $\widetilde{E} := \operatorname{diag}((D')^{-1}(D')^{-\top})$

$$\begin{aligned}&\mathrm{GLJ}_{\widehat{K},q}(\nabla u \cdot A\nabla u) \geq \lambda_{min}\, \mathrm{GLJ}_{\widehat{K},q}(|\nabla u|^2) \\ &= \lambda_{min}\, \mathrm{GLJ}_{\mathcal{Q},q}(\nabla\widehat{u} \cdot (D')^{-1}(D')^{-\top}\nabla\widehat{u}) \geq \frac{\lambda_{min}}{3468}\, \mathrm{GLJ}_{\mathcal{Q},q}(\nabla\widehat{u} \cdot \widetilde{E}\nabla\widehat{u}).\end{aligned}$$

A calculation reveals $\widetilde{E} = \left(E^{(1)}\right)^2 + \left(E^{(2)}\right)^2$ if we introduce

$$\begin{aligned}E^{(1)} &:= \operatorname{diag}\left\{\frac{\sqrt{8}(1+\eta_1)}{(1-\eta_2)(1-\eta_3)}, \frac{1+\eta_2}{1-\eta_3}, 1\right\}, \\ E^{(2)} &:= \operatorname{diag}\left\{\frac{4}{(1-\eta_2)(1-\eta_3)}, \frac{2}{1-\eta_3}, 0\right\}.\end{aligned}$$

The assumption $\widehat{u} \in \widetilde{\mathcal{Q}}_p$ implies that the components of $E^{(1)}\nabla\widehat{u}$ and $E^{(2)}\nabla\widehat{u}$ are in $\mathcal{Q}_p(\mathcal{Q})$; hence, from Lemma 3

$$\begin{aligned}\mathrm{GLJ}_{\mathcal{Q},q}(\nabla\widehat{u} \cdot \widetilde{E}\nabla\widehat{u}) &= \mathrm{GLJ}_{\mathcal{Q},q}(|E^{(1)}\nabla\widehat{u}|^2) + \mathrm{GLJ}_{\mathcal{Q},q}(|E^{(2)}\nabla\widehat{u}|^2) \\ &\geq \int_{\mathcal{Q}} |E^{(1)}\nabla\widehat{u}|^2 |\det D'| d\Omega + \int_{\mathcal{Q}} |E^{(2)}\nabla\widehat{u}|^2 |\det D'| d\Omega \\ &= \int_{\mathcal{Q}} (\nabla\widehat{u})^\top \widetilde{E}\nabla\widehat{u}\, |\det D'| d\Omega \\ &\geq \frac{1}{3}\int_{\mathcal{Q}} (\nabla\widehat{u})^\top (D')^{-1}(D')^{-\top}\nabla\widehat{u}\, |\det D'| d\Omega = \frac{1}{3}\int_{\widehat{K}} |\nabla u|^2 d\Omega,\end{aligned}$$

where we also appealed to Lemma 4. Collecting our findings, we arrive at

$$\mathrm{GLJ}_{\widehat{K},q}(\nabla u \cdot A\nabla u) \geq \frac{\lambda_{min}}{3468}\frac{1}{3}\|\nabla u\|^2_{L^2(\widehat{K})} \geq \frac{\lambda_{min}}{10404\lambda_{max}} \int_{\widehat{K}} \nabla u \cdot A\nabla u d\Omega.$$

4 Numerical Example

Corollary 1 states that the fully discrete p-FEM converges at the same rate as a Galerkin p-FEM where all integrals are evaluated exactly. We illustrate this behavior for the following example:

$$-\nabla \cdot (A\nabla u) = 1 \quad \text{on } \Omega := \widehat{K} \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega, \tag{15}$$

$$A(x_1, x_2, x_3) := \operatorname{diag}\left[\frac{1}{r^2+1}, \exp\left(r^2\right), \cos\left(\frac{1}{r^2+1}\right)\right], \tag{16}$$

where $r^2 = x_1^2 + x_2^2 + x_3^2$. We base the p-FEM on a single element on two different sets of shape functions: Φ^{KS} is the set of shape functions proposed by Karniadakis and Sherwin [4] and spans $\mathcal{P}_p(\widehat{K}) \cap H_0^1(\widehat{K})$; the set Φ^{Lag} is, roughly, speaking, the set of Lagrange interpolation points in the quadrature points (on $\mathcal{Q}$); it spans a space that contains $\mathcal{P}_p(\widehat{K}) \cap H_0^1(\widehat{K})$ and we refer to [3] for details. In both cases the stiffness matrix is set up using the minimal quadrature, i.e., $q = p$. Figure 1 shows the relative energy norm error $(\frac{E_{exact} - a^q(u_N, u_N)}{E_{exact}})^{1/2}$ for both cases, where $E_{exact} = \int_\Omega \nabla u \cdot A\nabla \mathrm{d}\Omega$. To illustrate that the optimal rate of convergence is not affected by the quadrature, we include in Fig. 1 a calculation (based on Φ^{KS}) that corresponds to (15) with $A = I$; in this case the linear system of equations can be set up without quadrature errors. We observe indeed that the rate of convergence is the same as in the case of quadrature.

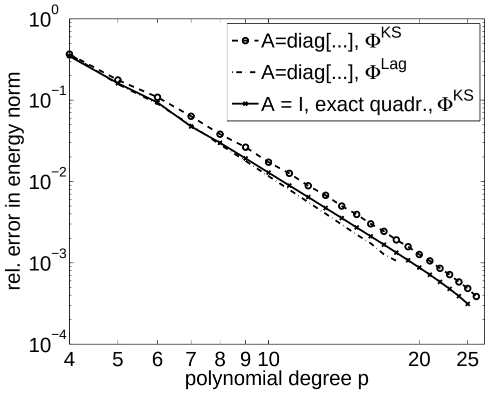

Fig. 1. Relative energy norm error

We close by pointing out that the shape functions in Φ^{Lag} are adapted to the quadrature rule. While the number of functions in Φ^{Lag} is (asymptotically for large p) 6 times that of Φ^{KS}, setting up the stiffness matrix is not slower than setting up the stiffness matrix based on Φ^{KS}. We refer to [3] for a detailed study.

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