p-Multigrid for Fekete Spectral Element Method

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Summary. Spectral element approximations based on triangular elements and on the so-called Fekete points of the triangle have been recently developed. p-multigrid methods offer an interesting way to resolve efficiently the resulting ill-conditioned algebraic systems. For elliptic problems, it is shown that a well chosen restriction operator and a good set up of the coarse grid matrices may lead to valuable results, even with a standard Gauss-Seidel smoother.

1 Introduction

As well known, high-order approximations are highly accurate as soon as the solution is smooth and, usually, require less grid-points than low-order methods. Unfortunately, the resulting algebraic system is severely ill-conditioned. Thus, for a two-dimensional (2D) second order Partial Differential Equation (PDE), a high order Finite Element Method (FEM) usually yields a condition number proportional to N^4, where $N \equiv p$ is the (total) degree of the polynomial approximation on each triangular element. Efficient solvers are then required.

Different approaches have been investigated in our previous works. Especially, for Fekete triangular spectral elements we have focused on Overlapping Schwarz methods [7] and on Schur complement methods [9]. In both cases, the idea was to consider each element as a different subdomain and then to apply classical domain decomposition preconditioners. Similarly, here we investigate a p-multigrid method so that the roughest approximation may be the one obtained with the standard $\mathbb{P}_1$ FEM. For the usual SEM (Spectral Element Method), a multigrid spectral element approach was first proposed in [10] and more recently investigated in [3]. For standard spectral methods one can cite [13, 4] and, among others, [6] for hp-FEM.

The outline of the paper is the following. To be self contained, in Section 2 the Fekete-Gauss TSEM (Triangles based SEM) is briefly described. In Section 3 we propose different restriction algorithms and strategies for setting up the coarse-grid algebraic systems, test these different approaches and then optimize the smoother for one triangular spectral element. In Section 4, the best approach is implemented in a TSEM solver, applied to an elliptic model problem and a convergence study is carried out. We conclude and offer some perspectives in Section 5.

2 The Fekete-Gauss TSEM

The (quadrilateral-based) SEM makes use of the Gauss-Lobatto-Legendre (GLL) points, for both the approximation and the quadrature points. GLL points have indeed nice approximation *and* integration properties. Unfortunately, such a single set of points does not exist for the triangle. Thus, in its initial version the Fekete points based TSEM [11] may fail to show the "spectral accuracy" property [8].

The Fekete-Gauss TSEM makes use of two sets of points:

- The Fekete points, $\{x_i\}_{i=1}^n$, as approximation points:

$$u(x) \approx \sum_{i=0}^{n} u(x_i)\,\varphi_i(x), \quad x \in T$$

where the φ_i are the Lagrange polynomials, given by $\varphi_i(x_j) = \delta_{ij}$.

- Gauss points, $\{y_i\}_{i=1}^m$, as quadrature points:

$$\int_T uv\,dT \approx \sum_{i=0}^{m} \rho_i u(y_i) v(y_i)$$

where the ρ_i are the Gauss quadrature weights.

Let $T = \{(r,s) :\ -1 \le r,s,\ r+s \le 0\}$ and $\mathcal{P}_N(T)$ be the set of polynomials on T of total degree $\le N$. Let $n = (N+1)(N+2)/2$ and $\{\psi_j\}_{j=1}^n$ be any basis of $\mathcal{P}_N(T)$. The Fekete points $\{x_i\}_{i=1}^n$ are those which maximize over T the determinant of the Vandermonde matrix V, given by $V_{ij} = \psi_j(x_i)$, $1 \le i,j \le n$.

In Fig. 1 (top) we compare the GLL points of the quadrilateral and the Fekete points of the triangle [12], for $N = 12$ (maximum degree in each variable for the quadrilateral and total degree for the triangle). In Fig. 1 (bottom) we give the Gauss points of the triangle for $M = 19$ (maximum polynomial degree for which the quadrature is exact) and those obtained from the Gauss points, with a mapping of the quadrilateral onto the triangle. The latter set of points may be of interest for values of M for which symmetrically distributed Gauss points are unknown. As advocated in [5], GLL points mapped onto the triangle may be used for both approximation and quadrature points, but at the price of an useless accumulation of points in one vertex.

The Fekete points of the triangle show some nice properties [12, 1]: (i) Fekete points are GLL points for the cube; (ii) Fekete points of the triangle are GLL points on the sides; (iii) The Lagrange polynomials based on Fekete points are maximum at these points.

3 Multigrid Strategy for the Triangle

We assume to have two grids, a coarse grid (grid 1) and a fine grid (grid 2) and denote the polynomial approximation degree by N_j, the set of Fekete points by $\{x_i^j\}$ and the Lagrange polynomials based on these points by $\{\varphi_i^j\}$, for grid j, $1 \le j \le 2$.

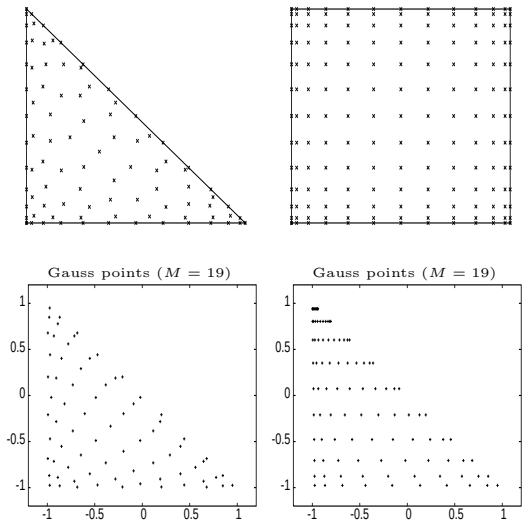

Fig. 1. Top: Triangle-Fekete and quadrilateral-GLL points ($N = 12$), Bottom: Triangle-Gauss and quadrilateral-Gauss mapped points ($M = 19$)

3.1 Prolongation / Restriction Operators and Coarse Grid System

Defining a prolongation operator is natural in the frame of spectral methods. Since the numerical approximation is everywhere defined, one has simply to express the coarse grid approximation at the Fekete points of the fine grid, to obtain:

$$u_2(x_i^2) = u_1(x_i^2) = \sum_j u_1(x_j^1)\varphi_j^1(x_i^2)$$

where $u_j \equiv u_{N_j}$ denotes the numerical approximation on grid j. In matrix form, with obvious notations:

$$\mathbf{u_2} = P\,\mathbf{u}_1\,, \quad [P]_{ij} = \varphi_j^1(x_i^2)\,.$$

Defining a restriction operator is less straightforward. We have investigated the following approaches:

- Interpolation: similarly to what is done for the prolongation operator, one can use the fine grid approximation to set up the restriction operator:

$$u_1(x_i^1) = \sum_j u_2(x_j^2)\varphi_j^2(x_i^1)\,, \quad \mathbf{u}_1 = R\mathbf{u}_2\,, \quad [R]_{ij} = \varphi_j^2(x_i^1)\,.$$

Such an approach is essentially justified for collocation methods, i.e., when the Right Hand Side (RHS) is a function and not an integral simply associated to a particular point through the corresponding Lagrange polynomial.

- Transposition (variational methods): if one takes into account the particular structure of the RHS, then

$$(f, \varphi_i^1) = (f, \sum_j \varphi_i^1(x_j^2)\varphi_j^2) = \sum_j \varphi_i^1(x_j^2)(f, \varphi_j^2) \quad \text{so that} \quad R = P^t$$

- Projection: let $\{\psi_i\}_{i=1}^{\infty}$ be an orthogonal hierarchical basis, e.g., the Koornwinder-Dubiner basis [2], then :

$$u_2(x_i^2) = \sum_{k \le n_2} \hat{u}_k \psi_k(x_i^2) \quad \text{and} \quad u_1(x_i^1) = \sum_{k \le n_1} \hat{u}_k \psi_k(x_i^1)$$

so that: $R = V_1[Id, 0]{V_2}^{-1}$ (Id, Identity matrix). Again this approach is better adapted to collocation methods.

It remains to set up the coarse grid algebraic system. On the coarse grid one has to solve $A_1\mathbf{e}_1 = \mathbf{r}_1$, with $\mathbf{r}_1 = R\,\mathbf{r}_2$ ($\mathbf{r}_2$, residual at the fine grid level; $\mathbf{e}_1$, error at the coarse grid level). One has at least the two following possibilities:

- Matrix A_1 may be set up directly, i.e., like A_2. This approach is the one used in [10].

- Matrix A_1 may be set up from: $A_1 = RA_2P$, i.e., by "aggregation" of A_2. In this case one can easily check that if $R = P^t$, then $\mathbf{e}_1$ such that $A_1\mathbf{e}_1 = R\mathbf{r}_2$ solves the constrained optimization problem: minimize

$$\phi(\mathbf{u}^*) = 0.5(A_2\mathbf{u}^*, \mathbf{u}^*) - (\mathbf{b}, \mathbf{u}^*) \quad \text{constrained by}$$
$$\mathbf{u}^* = \mathbf{u}_2 + P\mathbf{e}_1\,.$$

Numerical tests have been carried out for $-\Delta u + u = f$ in T, with the exact solution: $u_{exact} = \sin(2x + y)\,\sin(x + 1)\,\sin(1 - y)$ and the corresponding source term and Dirichlet boundary conditions.

Table 1. Number of iterations at the fine grid level / number of V-cycles. Comparison with Gauss-Seidel (GS).

N-Grids	I-D	T-D	P-D	T-A	GS
(6,12)	48 / 6	88 / 11	48 / 6	40 / 5	78
(3,6,12)	48 / 6	92 / 12	48 / 6	40 / 5	78
(6,12,18)	92 / 12	356 / 45	84 / 11	72 / 9	203
(3,6,12,18)	92 / 12	364 / 46	84 / 11	72 / 9	203

Depending on (i) the restriction strategy: Interpolation, Transposition or Projection and (ii) the setting up of the coarse matrix: Direct or Aggregation, four cases are considered: I-D, T-D, P-D and T-A. In these numerical tests, the number of grids is not restricted to 2, we use a V-cycle and at the smoothing grid levels 4 Gauss-Seidel iterations.

The number of iterations at the fine grid and the number of V-cycles required to get a residual less than 10^{-6} are given in Table 1. The multigrid results are compared to those obtained with the Gauss-Seidel (GS) method. Clearly, the transposition-aggregation (T-A) strategy gives the best results. Moreover, one observes that the number of iterations at the fine grid level is nearly independent of the number of grids involved in the V-cycle.

3.2 Analysis of the Smoother

On the basis of the following Successive Over Relaxation (SOR) decomposition of the matrix A_2, associated to the fine grid

$$A_2 = \frac{1}{\omega}(D + \omega L) - \frac{1}{\omega}[(1-\omega)D - \omega U] \equiv N - M$$

with D, L, U: the Diagonal, strictly Lower and Upper triangular parts of A_2, we want to optimize the relaxation coefficient ω and the number of iterations m of each SOR smoothing. Note that the GS smoothing is recovered for $\omega = 1$ and that, to obtain a stable algorithm, $0 < \omega < 2$.

We follow here an approach similar to the one proposed in [10]. Let n be the iteration index, defined as the sum of the number of iterations on grid 2 and the number of coarse grid corrections, $\mathbf{e}^n$ the error and $\mathbf{r}^n$ the residual.

- Pre-smoothing: after m iterations:

$$\mathbf{e}^{n+m} = (N^{-1}M)^m \mathbf{e}^n \qquad \mathbf{r}^{n+m} = A_2 \mathbf{e}^{n+m} = A_2 (N^{-1}M)^m A_2^{-1} \mathbf{r}^n$$

- After the coarse grid correction:

$$\begin{aligned} \mathbf{e}^{n+m+1} &= \mathbf{e}^{n+m} - P A_1^{-1} R\, \mathbf{r}^{n+m} \\ \mathbf{r}^{n+m+1} &= (Id - A_2 P A_1^{-1} R)\, \mathbf{r}^{n+m} \end{aligned}$$

- Post-smoothing: after m iterations:

$$\begin{aligned} \mathbf{r}^{n+2m+1} &= A_2 (N^{-1}M)^m A_2^{-1}\, \mathbf{r}^{n+m+1} = T\, \mathbf{r}^n \\ T &= A_2 (N^{-1}M)^m A_2^{-1} (Id - A_2 P A_1^{-1} R) A_2 (N^{-1}M)^m A_2^{-1}\,. \end{aligned}$$

Then:

$$\|\mathbf{r}^{2m+1+n}\| = \|T\mathbf{r}^n\| \le \|T\| \|\mathbf{r}^n\| \equiv \rho^{2m+1} \|\mathbf{r}^n\|, \quad \rho(\omega, m) = \|T\|^{1/(2m+1)}\,.$$

The parameter $\rho(\omega, m)$ that we have introduced may constitute a good indicator of the smoothing efficiency and so it allows an optimization of the relaxation parameter and of the number of iterations.

From the conclusion of Section 3.1, the restriction is achieved by transposition and the coarse grid matrix A_1 is set up by aggregation of the fine grid matrix A_2. Figure 2 shows isolines of ρ in the (m, ω) plane. Clearly, choosing $\omega = 1$ appears satisfactory and increasing the number of iterations beyond 4 appears useless, since this does not allow to really decrease the value of ρ.

4 Application to a Model Problem

The present multigrid method has been implemented in a TSEM solver using the T-A strategy and an arbitrary number (≥ 2) of grids. The matrix A_i, associated to the level i, is computed from :

$$A_i = {\sum_{k=1}^{K}}' R_i\, A_{k,i+1}\, P_i \qquad R_i = P_i^t$$

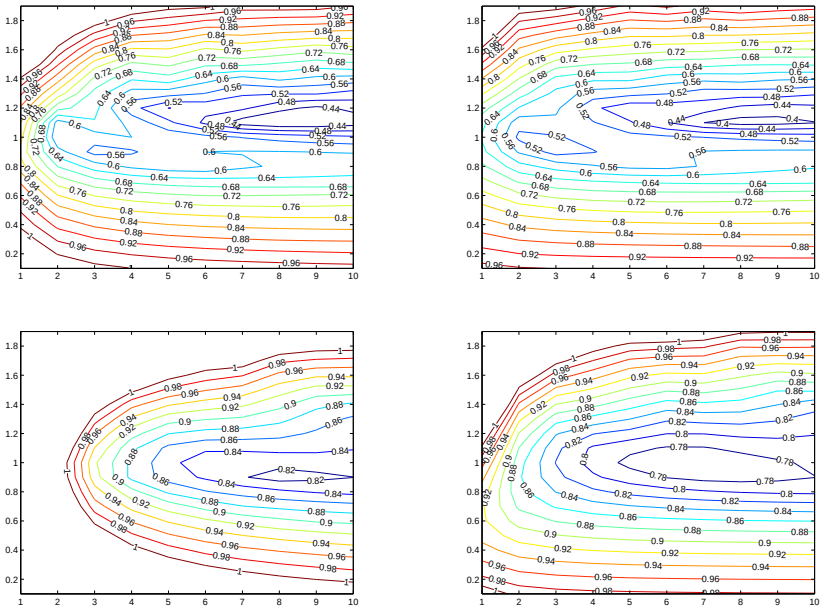

Fig. 2. $\rho(\omega, m)$ for the $\|\cdot\|_\infty$ (left) and $\|\cdot\|_2$ norms (right); $N_1 = 3$, $N_2 = 6$ (top) and $N_1 = 6$, $N_2 = 12$ (bottom)

where R_i, P_i are the restriction and prolongation operators between grids i (coarse) and $i+1$ (fine), $\sum'$ is the stiffness sum and $A_{k,i+1}$ is the element matrix associated to the element $k \leq K$ at the grid level $(i+1)$. Note that the restriction and prolongation operators are set up on the reference element, where the polynomial approximation holds, and so they do not depend on the element index k.

Convergence tests have been made for the elliptic PDE, $-\Delta u + u = f$ in $\Omega = (-1, 1)^2$, with the exact solution $u_{exact} = \sin(\pi x)\sin(\pi y)$ and corresponding Dirichlet boundary conditions and source term.

The computational domain $\Omega = (-1, 1)^2$ has been discretized using $K = 10 \times 10 \times 2 = 200$ triangular elements and $N = 12$. One has then 14641 degrees of freedom and the condition number of the system matrix equals 55345.

In Fig. 3 are shown convergence results for different configurations involving from 2 to 4 grids and comparisons are provided with the Conjugate Gradient and the Gauss-Seidel algorithms. Clearly the multigrid technique appears very efficient. Moreover, just like for one triangular element, the results obtained with $N = 12$ for the fine grid show that the convergence rate is nearly independent of the number of grids, so that the exact solve is only required on a very coarse grid. The convergence result given for the finer grid $N = 18$ shows that the convergence rate only slightly deteriorates, consistently with the results obtained for one element.

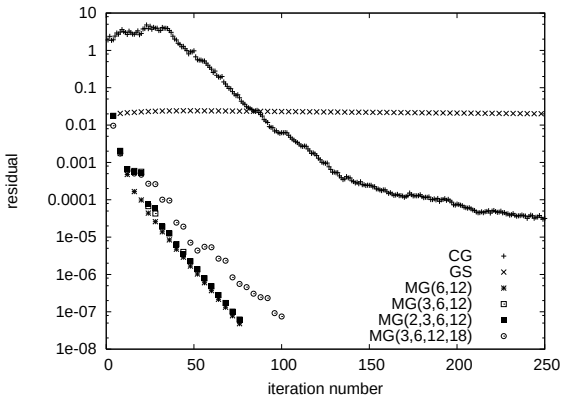

Fig. 3. MG convergence for $N = (6, 12)$, $N = (3, 6, 12)$, $N = (2, 3, 6, 12)$ (10 cycles) and $N = (3, 6, 12, 18)$ (13 cycles). Comparisons with CG and GS for $N = 12$.

5 Conclusion and Perspectives

A multigrid approach has been investigated for the TSEM approximation of elliptic problems. In particular,

- For one triangular Fekete-Gauss spectral element, different formulations of the restriction operator and of the coarse grid matrix have been compared. The best results are obtained when the restriction operator is defined by transposition and the coarse matrices by aggregation.

- An analysis of the influence of the control parameters (ω and m) of the SOR-smoother has been carried out. Good properties are obtained for $\omega = 1$ and $m = 4$.

- This multigrid approach has been implemented in a TSEM solver and tests have been carried out for a model problem.

Many points have not yet been investigated, e.g., (i) influence of a deformation of the mesh, (ii) comparisons with standard (quadrilateral based) SEM and (iii) improvement of the smoother.

Beyond that, it would be interesting to provide extensions to $3D$ geometries and also to more realistic problems, like fluid flows in complex geometries.

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