A Robust Preconditioner for the Hessian System in Elliptic Optimal Control Problems

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Summary. We consider an elliptic optimal control problem in two dimensions, in which the control variable corresponds to the Neumann data on a boundary segment, and where the performance functional is *regularized* to ensure that the problem is well posed. A finite element discretization of this control problem yields a saddle point linear system, which can be reduced to a symmetric positive definite Hessian system for determining the *control* variables. We formulate a robust preconditioner for this reduced Hessian system, as a matrix product involving the discrete Neumann to Dirichlet map and a mass matrix, and show that it yields a condition number bound which is uniform with respect to the mesh size and regularization parameters. On a uniform grid, this preconditioner can be implemented using a fast sine transform. Numerical tests verify the theoretical bounds.

1 Introduction

Elliptic control problems arise in various engineering applications [4]. We consider a problem in which the "control" variable $u(.)$ corresponds to the Neumann data on a boundary segment, and it must be chosen so that the solution $y(.)$ to the elliptic equation with Neumann data $u(.)$ closely matches a specified "target" function $\hat{y}(.)$. To determine the "optimal" control, we employ a performance functional which measures a square norm error between $\hat{y}(.)$ and the actual solution $y(.)$, and the control variable is sought so that it minimizes the performance functional [1, 3, 5, 4]. However, this results in an *ill-posed* constrained minimization problem, which can be *regularized* by adding a small Tikhonov regularization term to the performance functional. We discretize the regularized optimal control problem using a finite element method, and this yields a saddle point system [1, 2, 7].

In this paper, we formulate a robust preconditioner for the symmetric positive definite Hessian system for the control variables, obtained by block elimination of the saddle point system. In § 2, we formulate the elliptic optimal control problem and its discretization. In § 3, we derive the Hessian system and formulate our preconditioner as a symmetric matrix product involving the discrete Neumann to Dirichlet map and a mass matrix. We show that it yields a condition number bound that is independent of the mesh size and the regularization parameters. On a uniform grid, we describe a fast sine transform (FST) implementation of it. Numerical results are presented in § 4.

2 Optimal Control Problem

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and let Γ be an edge of its boundary $\partial\Omega$. We consider the problem of determining a Neumann control data $u(\cdot)$ on Γ such that the solution $y(\cdot)$ to the following problem with forcing term $f(\cdot)$:

$$\begin{cases} -\Delta y(x) = f(x), & \text{in } \Omega \\ \frac{\partial y(x)}{\partial n} = u(x), & \text{on } \Gamma \\ y(x) = 0, & \text{on } \partial\Omega \backslash \Gamma \end{cases} \tag{1}$$

minimizes the following performance functional $J(y,u)$:

$$J(y,u) \equiv \frac{1}{2}\left(\|y-\hat{y}\|^2_{L^2(\Omega)} + \alpha_1 \|u\|^2_{L^2(\Gamma)} + \alpha_2 \|u\|^2_{H^{-1/2}(\Gamma)}\right), \tag{2}$$

where $\hat{y}(\cdot) \in L^2(\Omega)$ is a given target, and $\alpha_1,\ \alpha_2 \geq 0$ denote *regularization* parameters. Later in the paper we also consider the case where $\|y-\hat{y}\|_{L^2(\Omega)}$ in (2) is replaced by $\|y-\hat{y}\|_{L^2(\Gamma)}$. The term $\|u\|_{H^{-1/2}(\Gamma)}$ denotes the dual Sobolev norm associated with $H^{1/2}_{00}(\Gamma)$. We let $H^1_D(\Omega)$ denote the subspace of $H^1(\Omega)$ consisting of functions vanishing on $D \equiv (\partial\Omega \setminus \Gamma)$.

To obtain a weak formulation of the minimization of (2) within set (1), we employ the function space $H^1_D(\Omega)$ for $y(\cdot)$ and $H^{-1/2}(\Gamma)$ for $u(\cdot)$. Given $f \in L^2(\Omega)$, define the constraint set $\mathcal{V}_f \subset \mathcal{V} \equiv H^1_D(\Omega) \times H^{-1/2}(\Gamma)$:

$$\mathcal{V}_f \equiv \left\{(y,u) \in \mathcal{V} \,:\, \mathcal{A}(y,w) = (f,w) + < u, w >, \ \ \forall w \in H^1_D(\Omega)\right\}, \tag{3}$$

where the forms are defined by:

$$\begin{cases} \mathcal{A}(y,w) \equiv \int_\Omega \nabla y \cdot \nabla w \, dx, & \text{for } y, w \in H^1_D(\Omega) \\ (f,w) \equiv \int_\Omega f(x)\, w(x)\, dx, & \text{for } w \in H^1_D(\Omega) \\ < u, w > \equiv \int_\Gamma u(x)\, w(x)\, ds_x, & \text{for } u \in H^{-1/2}(\Gamma),\ w \in H^{1/2}_{00}(\Gamma). \end{cases} \tag{4}$$

The constrained minimization problem then seeks $(y_*, u_*) \in \mathcal{V}_f$ satisfying:

$$J(y_*, u_*) = \min_{(y,u) \in \mathcal{V}_f} J(y,u). \tag{5}$$

To obtain a saddle point formulation of (5), introduce $p(\cdot) \in H^1_D(\Omega)$ as a Lagrange multiplier function to enforce the constraints. Define the following Lagrangian functional $\mathcal{L}(\cdot,\cdot,\cdot)$:

$$\mathcal{L}(y,u,p) \equiv J(y,u) + \left(\mathcal{A}(y,p) - (f,p) - < u, p >\right), \tag{6}$$

for $(y, u, p) \in H^1_D(\Omega) \times H^{-1/2}(\Gamma) \times H^1_D(\Omega)$. Then, the constrained minimum (y_*, u_*) of $J(.,.)$ can be obtained from the saddle point (y_*, u_*, p_*) of $\mathcal{L}(\cdot,\cdot,\cdot)$, where $(y_*, u_*, p_*) \in H^1_D(\Omega) \times H^{-1/2}(\Gamma) \times H^1_D(\Omega)$ satisfies:

$$\sup_q \mathcal{L}(y_*, u_*, q) = \mathcal{L}(y_*, u_*, p_*) = \inf_{(y,u)} \mathcal{L}(y, u, p_*). \tag{7}$$

For a discussion on the well-posedness of problem (7), see [5, 4].

To obtain a finite element discretization of (5), choose a quasi-uniform triangulation $\tau_h(\Omega)$ of Ω. Let $V_h(\Omega) \subset H^1_D(\Omega)$ denote the $\mathbb{P}_1$-conforming finite element space associated with the triangulation $\tau_h(\Omega)$, and let $V_h(\Gamma) \subset L^2(\Gamma)$ denote its restriction to Γ. A finite element discretization of (5) will seek $(y_h^*, u_h^*) \in V_h(\Omega) \times V_h(\Gamma)$ such that:

$$J(y_h^*, u_h^*) = \min_{(y_h, u_h) \in \mathcal{V}_{h,f}} J(y_h, u_h) \tag{8}$$

where the discrete constraint space $\mathcal{V}_{h,f} \subset \mathcal{V}_h \equiv V_h(\Omega) \times V_h(\Gamma)$ is defined by:

$$\mathcal{V}_{h,f} = \{(y_h, u_h) \in \mathcal{V}_h \,:\, \mathcal{A}(y_h, w_h) = (f, w_h) + < u_h, w_h >, \;\; \forall w_h \in V_h(\Omega)\}.$$

Let $p_h \in V_h(\Omega)$ denote discrete Lagrange multiplier variables, and let $\{\phi_1(x), \ldots, \phi_n(x)\}$ and $\{\psi_1(x), \ldots, \psi_m(x)\}$ denote the standard nodal basis functions for $V_h(\Omega)$ and $V_h(\Gamma)$, respectively. Expanding y_h, u_h and p_h with respect to its finite element basis, yields:

$$y_h(x) = \sum_{i=1}^n \mathbf{y}_i\, \phi_i(x), \quad u_h(x) = \sum_{j=1}^m \mathbf{u}_i\, \psi_i(x), \quad p_h(x) = \sum_{l=1}^n \mathbf{p}_l\, \phi_l(x), \tag{9}$$

and seeking the discrete saddle point of $\mathcal{L}(\cdot,\cdot,\cdot)$, yields the linear system:

$$\begin{bmatrix} M_\Omega & 0 & A^T \\ 0 & G & B^T \\ A & B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{bmatrix}, \tag{10}$$

where the sub-matrices M_Ω, A and Q (to be used later), are defined by:

$$\begin{cases} (M_\Omega)_{ij} \equiv \int_\Omega \phi_i(x)\, \phi_j(x)\, dx, & \text{for } 1 \le i, j \le n \\ (A)_{ij} \equiv \int_\Omega \nabla\phi_i(x) \cdot \nabla\phi_j(x)\, dx, & \text{for } 1 \le i, j \le n \\ (Q)_{ij} \equiv \int_\Gamma \psi_i(x)\, \psi_j(x)\, ds_x, & \text{for } 1 \le i, j \le m, \end{cases} \tag{11}$$

and the forcing vectors are defined by $(\mathbf{f}_1)_i = \int_\Omega \hat{y}(x)\, \phi_i(x)\, dx$, for $1 \le i \le n$ with $\mathbf{f}_2 = \mathbf{0}$, and $(\mathbf{f}_3)_i = \int_\Omega f(x)\, \phi_i(x)\, dx$ for $1 \le i \le n$. Matrix M_Ω of dimension n corresponds to a mass matrix on Ω, and matrix A to the stiffness matrix. Matrix Q of dimension m corresponds to a lower dimensional mass matrix on Γ. Matrix B will be defined in terms of Q, based on an ordering of nodal unknowns in $\mathbf{y}$ and $\mathbf{p}$ with nodes in the *interior* of Ω ordered prior to the nodes on Γ. Denote such block partitioned vectors as $\mathbf{y} = \left(\mathbf{y}_I^T, \mathbf{y}_B^T\right)^T$ and $\mathbf{p} = \left(\mathbf{p}_I^T, \mathbf{p}_B^T\right)^T$, and define B of dimension $n \times m$ as $B^T = \begin{bmatrix} 0 & Q^T \end{bmatrix}$, and define matrix G of dimension m, representing the regularizing terms as:

$$G \equiv \alpha_1\, Q + \alpha_2 \left(B^T A^{-1} B\right). \tag{12}$$

3 Preconditioned Hessian System

The algorithm we shall consider for solving (10) will be based on the solution of the following *Hessian system* for the discrete control $\mathbf{u}$. It is the Schur complement system obtained by block elimination of $\mathbf{y}$ and $\mathbf{p}$ in system (10):

$$\left(G + B^T A^{-T} M_\Omega A^{-1} B\right) \mathbf{u} = \mathbf{f}_2 - B^T A^{-T} \mathbf{f}_1 + B^T A^{-T} M_\Omega A^{-1} \mathbf{f}_3. \qquad (13)$$

The Hessian matrix $H \equiv \left(G + B^T A^{-T} M_\Omega A^{-1} B\right)$ is symmetric and positive definite of dimension m, and system (13) can be solved using a PCG algorithm. Each matrix vector product with $G + B^T A^{-T} M_\Omega A^{-1} B$ will require the action of A^{-1} twice per iteration (this can be computed iteratively, resulting in a *double iteration*). Once $\mathbf{u}$ has been determined, we obtain $\mathbf{y} = A^{-1}(\mathbf{f}_3 - B\mathbf{u})$ and $\mathbf{p} = A^{-T}\left(\mathbf{f}_1 - M_\Omega A^{-1}\mathbf{f}_3 + M_\Omega A^{-1} B\mathbf{u}\right)$.

The task of finding an effective preconditioner for the Hessian matrix H is complicated by the presence of the parameters $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$. As noted in [5], when α_1 or α_2 is large (or equivalently, when $\lambda_{\min}(G)$ is sufficiently large), then G is spectrally equivalent to H and therefore G will be an effective preconditioner for H, while when both α_1 and α_2 are small (or equivalently, when $\lambda_{\max}(G)$ is sufficiently small), then the matrix $(B^T A^{-T} M_\Omega A^{-1} B)$ will be an effective preconditioner for H. For intermediate values of α_i, however, neither limiting approximation may be effective. In the special case when we replace $\|y - \hat{y}\|_{L^2(\Omega)}$ in (2) by $\|y - \hat{y}\|_{L^2(\Gamma)}$, then matrix M_Ω is replaced by $M_\Gamma \equiv \text{blockdiag}(0, Q)$ and we shall indicate a preconditioner for H, uniformly effective with respect to $\alpha_1 > 0$ or $\alpha_2 > 0$.

The preconditioner we shall formulate for H will be based on spectrally equivalent representations of G and $(B^T A^{-T} M A^{-1} B)$, for special choices of the matrix M. Lemma 1 below describes uniform spectral equivalences between G, $(B^T A^{-1} B)$, $(B^T A^{-T} M_\Omega A^{-1} B)$ and one or more of the matrices Q and S^{-1}, where $S = \left(A_{\Gamma\Gamma} - A_{I\Gamma}^T A_{II}^{-1} A_{I\Gamma}\right)$ denotes the discrete Dirichlet to Neumann map. Properties of S have been studied extensively in the domain decomposition literature [8].

Lemma 1. *Let $\Omega \subset \mathbb{R}^2$ be a convex domain. Then, the following equivalences:*

$$\begin{aligned} (B^T A^{-1} B) &= Q\, S^{-1}\, Q \\ (B^T A^{-T} M A^{-1} B) &= Q\, S^{-1}\, Q\, S^{-1}\, Q \qquad\quad \text{when } M = M_\Gamma \\ (B^T A^{-T} M A^{-1} B) &\asymp Q\, S^{-1}\, Q\, S^{-1}\, Q\, S^{-1}\, Q \text{ when } M = M_\Omega, \end{aligned} \qquad (14)$$

will hold with constants independent of h, where $S = (A_{\Gamma\Gamma} - A_{I\Gamma}^T A_{II}^{-1} A_{I\Gamma})$, $M_\Gamma =$ blockdiag$(0, Q)$ and M_Ω is the mass matrix on Ω.

Proof. The first statement is a trivial calculation. To prove the second, use:

$$A^{-1} = \begin{bmatrix} A_{II}^{-1} + A_{II}^{-1} A_{I\Gamma} S^{-1} A_{I\Gamma}^T A_{II}^{-1} & -A_{II}^{-1} A_{I\Gamma} S^{-1} \\ -S^{-1} A_{I\Gamma}^T A_{II}^{-1} & S^{-1} \end{bmatrix}.$$

Employing this and using the block matrix structure of B yields:

$$A^{-1} B\mathbf{u} = \begin{bmatrix} -A_{II}^{-1} A_{I\Gamma} S^{-1} Q\mathbf{u} \\ S^{-1} Q\mathbf{u} \end{bmatrix}.$$

Substituting this expression yields that $B^T A^{-T} M_\Gamma A^{-1} B = Q S^{-1} Q S^{-1} Q$. To prove the third equivalence, let u_h denote a finite element control function defined on Γ with associated nodal vector $\mathbf{u}$. Let v_h denote the Dirichlet data associated with the Neumann data u_h, i.e. with associated nodal vector $\mathbf{v} = S^{-1} Q \mathbf{u}$. When $M = M_\Omega$, then $\mathbf{u}^T (B^T A^{-T} M A^{-1} B) \mathbf{u}$ will be equivalent to $\|E v_h\|^2_{L^2(\Omega)}$, where $E v_h$ denotes the *discrete harmonic* extension of the Dirichlet boundary data v_h into Ω with associated nodal vector $A^{-1} B \mathbf{u}$. When Ω is convex, $H^2(\Omega)$ elliptic regularity will hold for (1) and a result from [6] shows that $\|E v_h\|^2_{L^2(\Omega)}$ is spectrally equivalent to $\|v_h\|^2_{H^{-1/2}(\Gamma)}$. In matrix terms, the nodal vector associated with the discrete Dirichlet data v_h will be $\mathbf{v} = S^{-1} Q \mathbf{u}$, given by the discrete Neumann to Dirichlet map. For $v_h \in H^{-1/2}(\Gamma)$, it will hold that $\|v_h\|^2_{H^{-1/2}(\Gamma)}$ is spectrally equivalent to $\mathbf{v}^T Q^T S^{-1} Q \mathbf{v}$, and in turn equivalent to $\mathbf{u}^T Q^T S^{-1} Q^T S^{-1} Q S^{-1} Q \mathbf{u}$ and the third equivalence follows, since $Q^T = Q$ and $S^{-T} = S^{-1}$.

As a consequence, we obtain the following uniform spectral equivalences.

Lemma 2. *Let $\Omega \subset R^2$ be a convex domain. Then, the following equivalences will hold for the Hessian matrix $H \equiv \left(G + B^T A^{-T} M A^{-1} B\right)$:*

$$\begin{aligned} H = H_0 \;&\equiv\; \alpha_1 Q + \alpha_2 Q S^{-1} Q + Q S^{-1} Q S^{-1} Q, && \text{when } M = M_\Gamma \\ H \asymp H_0 \;&\equiv\; \alpha_1 Q + \alpha_2 Q S^{-1} Q + Q S^{-1} Q S^{-1} Q S^{-1} Q, && \text{when } M = M_\Omega, \end{aligned} \tag{15}$$

with constants independent of h, α_1 and α_2.

H_0 will be our model preconditioner for H. To obtain an efficient solver for H_0, in applications we shall replace Q and S by $Q_0 \asymp Q$ and $S_0 \asymp S$. However, since a product of matrices is involved, caution must be exercised in the choice of Q_0 and S_0. Bounds independent of h and α_i will be retained only under additional regularity assumptions or the commutativity of Q, S, Q_0 and S_0.

3.1 An FST Based Preconditioner $\tilde{H} \asymp H_0$ for H

If $\Omega \subset \mathbb{R}^2$ is rectangular and the grid is uniform, and Γ is one of the four edges forming $\partial\Omega$, then the Dirichlet to Neumann map S (hence S^{-1}) and the mass matrix Q will be diagonalized by the discrete Sine Transform F, where:

$$(F)_{ij} = \sqrt{\frac{2}{m+1}}\,\sin(\frac{i\,j\,\pi}{m+1}) \quad \text{for} \quad 1 \le i,\, j \le m,$$

see [8]. Regularity theory shows that the Dirichlet to Neumann map S satisfies $S \asymp S_0 \equiv Q^{1/2}\left(Q^{-1/2} L Q^{-1/2}\right)^{1/2} Q^{1/2} \asymp \|\cdot\|^2_{H^{1/2}_{00}(\Gamma)}$, where L denotes a discretization of the *Laplace-Beltrami* operator $L_B = -\frac{d^2}{ds_x^2}$ on Γ with homogeneous Dirichlet conditions, see [8]. For a uniform grid, the Laplace-Beltrami matrix is $L = h^{-1}\,\mathrm{tridiag}(-1, 2, -1)$, and it is diagonalized by the sine transform F with $L = F \Lambda_L F^T$, where the diagonal matrix Λ_L has entries $\Lambda_L(ii) = 4\,(m+1)\,\sin^2(\frac{i\,\pi}{2\,(m+1)})$. For a uniform grid, the mass matrix satisfies $Q = Q_0 \equiv \frac{h}{6}\,\mathrm{tridiag}(1, 4, 1)$ and it is also diagonalized by F, satisfying $Q_0 = F \Lambda_{Q_0} F^T$ for $\Lambda_{Q_0}(ii) = \frac{1}{3\,(m+1)}\,(3 - 2\,\sin^2(\frac{i\,\pi}{2\,(m+1)}))$. Thus, we obtain:

$$
\begin{aligned}
S \asymp S_0 &\equiv F\Lambda_{S_0}F^T = F\left(\Lambda_{Q_0}^{1/4}\Lambda_L^{1/2}\Lambda_{Q_0}^{1/4}\right)F^T \\
Q = Q_0 &= F\Lambda_{Q_0}F^T\,.
\end{aligned}
$$

Since matrices S, Q, S_0 and Q_0 are diagonalized by F on a uniform grid, these matrices *commute*. As a result, it can be verified that $\tilde{H} \asymp H_0 \asymp H$:

$$
\begin{aligned}
\tilde{H} &\asymp F\left(\alpha_1\,\Lambda_{Q_0} + \alpha_2\,\Lambda_{Q_0}^2\Lambda_S^{-1} + \Lambda_{Q_0}^3\,\Lambda_S^{-2}\right)F^T, \text{ when } M = M_\Gamma \\
\tilde{H} &\asymp F\left(\alpha_1\,\Lambda_{Q_0} + \alpha_2\,\Lambda_{Q_0}^2\,\Lambda_S^{-1} + \Lambda_{Q_0}^4\,\Lambda_S^{-3}\right)F^T, \text{ when } M = M_\Omega,
\end{aligned}
\tag{16}
$$

with bounds independent of h and α_i. The eigenvalues of $\tilde{H}^{-1}$ can be found analytically, and the action of $\tilde{H}^{-1}$ can be computed at low cost using FST's.

4 Numerical Experiments

We present numerical tests of control problem (2) on the two-dimensional unit square $(0,1)\times(0,1)$. Neumann conditions are imposed on $\Gamma = (0,1)\times\{0\}$, and homogeneous Dirichlet conditions are imposed on the remaining sides of $\partial\Omega$, with forcing term $f(x,y) = 0$ in Ω. We consider a structured triangulation on Ω with mesh parameter $h = 2^{-N}$, where N is an integer denoting the number of refinements. We test different values for the relaxation parameters α_1 and α_2, for the mesh size h, and for mass matrix M. In all numerical experiments, we run PCG until the preconditioned l_2 initial residual is reduced by a factor of 10^{-9}. We use the FST based preconditioner described in (16).

Table 1. Number of PCG iterations and (condition) for $\alpha_2 = 0$ and $M = M_\Omega$.

$N \setminus \alpha_1$	1	$(0.1)^2$	$(0.1)^4$	$(0.1)^6$	0
3	3 (1.02)	5 (1.65)	7 (1.60)	6 (1.44)	7 (1.54)
4	3 (1.02)	5 (1.63)	9 (1.95)	6 (1.29)	7 (1.56)
5	3 (1.02)	5 (1.63)	8 (2.00)	7 (1.50)	7 (1.56)
6	3 (1.02)	5 (1.64)	8 (2.01)	6 (1.86)	6 (1.55)
7	3 (1.02)	5 (1.64)	8 (2.00)	6 (1.96)	5 (1.51)

Tables 1 and 2 list results on runs with $M = M_\Omega$ and target function $\hat{y}(x,y) = 1$ on $[1/4,3/4]\times[0,3/4]$ and equal to zero otherwise. We list the number of PCG iterations and in parenthesis the condition number estimate for the preconditioned system. As expected from the analysis, the number of iterations and the condition number remain bounded, and when no preconditioning is used, the problem becomes very ill-conditioned for small regularization α_i; see Table 3. In Tables 4 and 5 we report the results for $M = M_\Gamma$ with target function $\hat{y}(x,0) = 1$ on $[1/4,3/4]\times\{0\}$, and equal to zero otherwise. As before, the number of iterations and the condition number remain bounded.

Table 2. Number of PCG iterations and (condition) for $\alpha_1 = 0$ and $M = M_\Omega$.

$N \setminus \alpha_2$	1	$(0.1)^2$	$(0.1)^4$	$(0.1)^6$	0
3	7 (2.15)	6 (1.45)	7 (1.50)	7 (1.53)	7 (1.54)
4	8 (2.26)	7 (1.71)	7 (1.45)	7 (1.56)	7 (1.56)
5	7 (2.24)	7 (1.84)	6 (1.32)	7 (1.56)	7 (1.56)
6	5 (2.03)	7 (1.95)	5 (1.33)	6 (1.52)	6 (1.55)
7	4 (1.82)	6 (1.76)	5 (1.40)	5 (1.44)	5 (1.51)

Table 3. Number of CG iterations and (condition) for $\alpha_2 = 0$ and $M = M_\Omega$.

$N \setminus \alpha_1$	1	$(0.1)^2$	$(0.1)^4$	$(0.1)^6$	0
3	7 (2.75)	7 (6.80)	8 (351)	8 (2.2+3)	8 (2.3+3)
4	9 (2.97)	9 (7.47)	15 (448)	23 (1.6+4)	23 (2.4+4)
5	7 (3.03)	8 (7.64)	16 (468)	35 (3.8+4)	53 (2.0+5)
6	6 (3.04)	6 (7.69)	12 (472)	39 (4.6+4)	106 (1.6+6)
7	4 (3.05)	5 (7.70)	11 (473)	34 (4.7+4)	162 (1.3+7)

Table 4. Number of PCG iterations and (condition) for $\alpha_2 = 0$ and $M = M_\Gamma$.

$N \setminus \alpha_1$	1	$(0.1)^2$	$(0.1)^4$	$(0.1)^6$	0
3	3 (1.01)	4 (1.17)	4 (3.96)	4 (5.08)	4 (5.09)
4	2 (1.00)	4 (1.07)	7 (2.73)	8 (5.64)	8 (5.72)
5	2 (1.00)	3 (1.02)	7 (1.76)	11 (5.44)	11 (5.75)
6	2 (1.00)	3 (1.00)	5 (1.29)	12 (4.69)	13 (5.78)
7	2 (1.00)	3 (1.01)	4 (1.10)	8 (3.14)	10 (5.65)

Table 5. Number of PCG iterations and (condition) for $\alpha_1 = 0$ and $M = M_\Gamma$.

$N \setminus \alpha_2$	1	$(0.1)^2$	$(0.1)^4$	$(0.1)^6$	0
3	4 (2.29)	4 (3.99)	4 (5.08)	4 (5.09)	4 (5.09)
4	8 (2.41)	8 (3.81)	8 (5.68)	8 (5.72)	8 (5.72)
5	8 (2.37)	9 (3.25)	11 (5.66)	11 (5.75)	11 (5.75)
6	7 (2.33)	8 (2.84)	12 (5.57)	13 (5.78)	13 (5.78)
7	5 (2.09)	6 (2.45)	9 (5.24)	10 (5.64)	10 (5.65)

5 Conclusions

We have introduced a robust preconditioner for the Hessian matrix in a class of elliptic optimal control problems. We have shown that the Hessian matrix is spectrally equivalent to a composition of the discrete *Laplace-Beltrami* and mass matrices.

For a uniform grid, these matrices are simultaneously diagonalized by a fast sine transform. The resulting preconditioner is optimal with respect to the mesh size and relaxation parameters. Numerical results confirm the robustness of the preconditioner.

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