Mixed Plane Wave Discontinuous Galerkin Methods

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Summary. In this paper, we extend the class of plane wave discontinuous Galerkin methods for the two-dimensional inhomogeneous Helmholtz equation presented in Gittelson, Hiptmair, and Perugia [2007]. More precisely, we consider the case of numerical fluxes defined in mixed form, namely, numerical fluxes explicitly defined in terms of both the primal and the flux variable, instead of the primal variable and its gradient. In our error analysis, we rely heavily on the approximation results and inverse estimates for plane waves proved in Gittelson, Hiptmair, and Perugia [2007] and develop a new mixed duality argument.

1 Introduction

The oscillatory behavior of solutions to time harmonic wave problems, along with numerical dispersion, renders standard finite element methods inefficient already in medium-frequency regimes. As an alternative, several ways to incorporate information from the equation into the discretization spaces have been proposed in the literature, giving rise to methods based on shape functions which are solutions to either the primal or the dual problem. The so-called "ultra weak variational formulation" (UWVF) introduced by Després for the Helmholtz equation in the 1990's (see [5, 7]) belongs to this class of methods.

The UWVF was inspired by the domain decomposition approach introduced in [6], where Robin-type transmission conditions were used in order to guarantee well-posedness of the subproblems. The introduction of these impedance interelement traces as unknowns and the use of discontinuous piecewise plane wave basis functions are the basic ingredients of the UWVF. This method, which was numerically proved to be effective, has attracted new interest very recently (see, e.g., [9, 10, 11]).

From a theoretical point of view, the UWVF has been analyzed in [5], where the convergence of discrete solutions to the impedance trace of the analytical solution on the domain boundary was proved. On the other hand, numerical results showed that convergence is achieved not only at the boundary, but in the whole domain. In the recent papers [3, 8], convergence of the h–version of the UWVF was proved by recasting the UWVF in the discontinuous Galerkin (DG) framework: in [3], slightly suboptimal L^2–error estimates were derived for the homogeneous Helmholtz problem by exploiting a result by [5], while in [8], error estimates in a mesh-dependent broken H^1–norm, as well as in the L^2–norm, for the inhomogeneous Helmholtz problem were proved based on duality techniques. All these estimates require a minimal resolution of the mesh to resolve the wavelength which shows that the plane wave discontinuous Galerkin (PWDG) method is not free from the *pollution effect* (see, e.g., [2, 12]).

In order to get the stability properties necessary to develop the theoretical analysis in [8], the numerical fluxes had to be defined in a slightly different way with respect to the original UWVF, introducing mesh and wave number dependent parameters.

In this paper, we extend the class of PWDG methods for the two-dimensional inhomogeneous Helmholtz equation presented in [8] by allowing for numerical fluxes defined in mixed form, namely, numerical fluxes explicitly defined in terms of both the primal and the flux variable, instead of the primal variable and its gradient. For these *mixed* PWDG methods, we essentially prove the same results as in [8] for the *primal* PWDG methods, by exploiting the approximation results and inverse estimates for plane waves proved in [8], and by developing a new mixed duality argument.

2 Mixed Discontinuous Galerkin Approach

Consider the following model boundary value problem for the Helmholtz equation:

$$\begin{aligned} -\Delta u - \omega^2 u &= f \qquad \text{in } \Omega, \\ \nabla u \cdot n + i\omega u &= g \qquad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Here, Ω is a bounded polygonal/polyhedral Lipschitz domain in $\mathbb{R}^d$, $d = 2,3$, and $\omega > 0$ denotes a fixed wave number (the corresponding wavelength is $\lambda = 2\pi/\omega$). The right hand side f is a source term in $H^{-1}(\Omega)$, n is the outer normal unit vector to $\partial\Omega$, and i is the imaginary unit. Inhomogeneous first order absorbing boundary conditions in the form of impedance boundary conditions are used in (1), with boundary data $g \in H^{-1/2}(\partial\Omega)$.

Denoting by $(\cdot,\cdot)$ the standard complex $L^2(\Omega)$–inner product, namely, $(u,v) = \int_\Omega u\overline{v}\,\mathrm{d}V$, the variational formulation of (1) reads as follows: find $u \in H^1(\Omega)$ such that, for all $v \in H^1(\Omega)$,

$$(\nabla u, \nabla v) - \omega^2(u,v) + i\omega \int_{\partial\Omega} u\overline{v}\,\mathrm{d}S = (f,v) + \int_{\partial\Omega} g\overline{v}\,\mathrm{d}S\,. \tag{2}$$

Existence and uniqueness of solutions of (2) is well establish, see, e.g., [13, sec. 8.1].

Introduce the auxiliary variable $\sigma := \nabla u/i\omega$ and write problem (1) as a first order system:

$$\begin{aligned} i\omega\,\sigma &= \nabla u && \text{in } \Omega\ , \\ i\omega\,u - \nabla\cdot\sigma &= \frac{1}{i\omega} f && \text{in } \Omega\ , \\ i\omega\sigma\cdot n + i\omega\,u &= g && \text{on } \partial\Omega\ . \end{aligned} \tag{3}$$

Now, introduce a partition $\mathcal{T}_h$ of Ω into subdomains K, and proceed as in [4]. By multiplying the first and second equations of (3) by smooth test functions τ and v, respectively, and integrating by parts on each K, we obtain

$$\begin{aligned} &\int_K i\omega\,\sigma\cdot\overline{\tau}\,\mathrm{d}V + \int_K u\,\overline{\nabla\cdot\tau}\,\mathrm{d}V - \int_{\partial K} u\,\overline{\tau\cdot n}\,\mathrm{d}S = 0 && \forall\tau\in\mathbf{H}(\mathrm{div};K) \\ &\int_K i\omega\,u\,\overline{v}\,\mathrm{d}V + \int_K \sigma\cdot\overline{\nabla v}\,\mathrm{d}V - \int_{\partial K}\sigma\cdot n\,\overline{v}\,\mathrm{d}S = \frac{1}{i\omega}\int_K f\,\overline{v}\,\mathrm{d}V && \forall v\in H^1(K)\ . \end{aligned} \tag{4}$$

Introduce discontinuous *discrete* function spaces Σ_h and V_h; replace σ,τ by $\sigma_h,\tau_h\in\Sigma_h$ and u,v by $u_h,v_h\in V_h$. Then, approximate the traces of u and σ across interelement boundaries by the so-called *numerical fluxes* denoted by $\widehat{u}_h$ and $\widehat{\sigma}_h$, respectively (see, e.g., [1] for details) and obtain

$$\begin{aligned} &\int_K i\omega\,\sigma_h\cdot\overline{\tau}_h\,\mathrm{d}V + \int_K u_h\,\overline{\nabla\cdot\tau_h}\,\mathrm{d}V - \int_{\partial K}\widehat{u}_h\,\overline{\tau_h\cdot n}\,\mathrm{d}S = 0 && \forall\tau_h\in\Sigma_h(K) \\ &\int_K i\omega\,u_h\,\overline{v}_h\,\mathrm{d}V + \int_K \sigma_h\cdot\overline{\nabla v_h}\,\mathrm{d}V - \int_{\partial K}\widehat{\sigma}_h\cdot n\,\overline{v}_h\,\mathrm{d}S = \frac{1}{i\omega}\int_K f\,\overline{v}_h\,\mathrm{d}V && \forall v_h\in V_h(K)\ . \end{aligned} \tag{5}$$

At this point, in order to complete the definition of classical DG methods, one "simply" needs to choose the numerical fluxes $\widehat{u}_h$ and $\widehat{\sigma}_h$ (notice that only the normal component of $\widehat{\sigma}_h$ is needed).

In order to define the numerical fluxes, we first introduce the following standard notation (see, e.g., [1]): let u_h and σ_h be a piecewise smooth function and vector field on $\mathcal{T}_h$, respectively. On $\partial K^-\cap\partial K^+$, we define

$$\begin{aligned} &\text{the averages:} \quad \{\{u_h\}\} := \tfrac{1}{2}(u_h^+ + u_h^-), \qquad \{\{\sigma_h\}\} := \tfrac{1}{2}(\sigma_h^+ + \sigma_h^-)\ , \\ &\text{the jumps:} \quad [\![u_h]\!]_N := u_h^+ n^+ + u_h^- n^-, \quad [\![\sigma_h]\!]_N := \sigma_h^+\cdot n^+ + \sigma_h^-\cdot n^-\ . \end{aligned}$$

Taking a cue from [4], we can now introduce the *mixed* numerical fluxes: on $\partial K^-\cap\partial K^+\subset\mathcal{F}_h^{\mathcal{I}}$, we define

$$\begin{aligned} \widehat{\sigma}_h &= \{\{\sigma_h\}\} - \alpha\,[\![u_h]\!]_N - \gamma[\![\sigma_h]\!]_N, \\ \widehat{u}_h &= \{\{u_h\}\} + \gamma\cdot[\![u_h]\!]_N - \beta\,[\![\sigma_h]\!]_N, \end{aligned} \tag{6}$$

and on $\partial K\cap\partial\Omega\subset\mathcal{F}_h^{\mathcal{B}}$, we define

$$\begin{aligned} \widehat{\sigma}_h &= \sigma_h - (1-\delta)\left(\sigma_h + u_h\,n - \frac{1}{i\omega}\,g\,n\right), \\ \widehat{u}_h &= u_h - \delta\left(\sigma_h\cdot n + u_h - \frac{1}{i\omega}\,g\right). \end{aligned} \tag{7}$$

These numerical fluxes are consistent and therefore the corresponding method is consistent. Moreover, adjoint consistency is guaranteed, due to symmetry; see [1].

We will assume

$$\alpha = \mathtt{a}/\omega\mathtt{h}\,, \qquad \beta = \mathtt{b}\omega\mathtt{h}\,, \qquad \gamma = 0\,, \qquad \delta = \mathtt{d}\omega\mathtt{h}\,, \tag{8}$$

with functions $\mathtt{a} > 0$ on $\mathcal{F}_h^{\mathcal{I}}$, $\mathtt{b} \geq 0$ on $\mathcal{F}_h^{\mathcal{I}}$ and $\mathtt{d}$ on $\mathcal{F}_h^{\mathcal{B}}$, all bounded from above (and below in the case of $\mathtt{a}$) independent of the mesh size and ω. Moreover, the choice of $\mathtt{d}$ on $\mathcal{F}_h^{\mathcal{B}}$ must ensure that $0 \leq \delta \leq 1$. A further assumption on $\mathtt{d}$ will be stated in Sect. 3.

Remark 1. Whenever $\beta = 0$, it is possible to eliminate the auxiliary variable σ_h from the final system and write the mixed DG methods in primal formulation. On the other hand, also in these cases, the PWDG methods defined in the next section differs from the PWDG methods presented in [8] because the fluxes there were defined using $\nabla_h u_h$ instead of σ_h in (6)–(7).

3 Convergence Analysis of the Mixed PWDG Method

We restrict ourselves to the two-dimensional case and assume that Ω is a *convex* polygon and that $\mathcal{T}_h$ is a triangular mesh with possible hanging nodes satisfying the shape regularity assumption.

We carry our our analysis of the mixed DG method (5) with numerical fluxes defined by (6) and (7), with parameters satisfying (8). In addition, we opt for a Trefftz type DG method: the local test and trial spaces will be spanned by plane wave functions and their gradients, which belong to the kernel of the Helmholtz operator. In particular, writing

$$PW_\omega^p(\mathbb{R}^2) = \left\{ v \in C^\infty(\mathbb{R}^2) : v(x) = \sum_{j=1}^{p} \alpha_j \exp(i\omega d_j \cdot x),\ \alpha_j \in \mathbb{C} \right\}, \tag{9}$$

with even spaced directions

$$d_j = \begin{pmatrix} \cos(\frac{2\pi}{p}(j-1)) \\ \sin(\frac{2\pi}{p}(j-1)) \end{pmatrix}, \quad j = 1, \dots, p\,, \tag{10}$$

we set

$$V_h = \{ v \in L^2(\Omega) : v_{|_K} \in PW_\omega^p(\mathbb{R}^2)\ \forall K \in \mathcal{T}_h \}\,, \qquad \Sigma_h = V_h^2; \tag{11}$$

notice that $\nabla V_h \subseteq \Sigma_h$ and $\nabla \cdot \Sigma_h \subseteq V_h$.

Let $V \subseteq H^2(\Omega)$ be the space containing all possible solutions u to (1) and $\Sigma = \nabla V$, and denote by $\mathfrak{Q}$ the product space $\mathfrak{Q} = \Sigma \times V$; we set

$$\mathfrak{p} := \begin{bmatrix} \sigma \\ u \end{bmatrix}.$$

Similarly, we define $\mathfrak{Q}_h := \Sigma_h \times V_h$ and denote by $\mathfrak{p}_h$ and $\mathfrak{q}_h$ the vectors containing the discrete solution to (5) and the generic test function in $\mathfrak{Q}_h$, namely,

$$\mathfrak{p}_h := \begin{bmatrix} \sigma_h \\ u_h \end{bmatrix}, \qquad \mathfrak{q}_h := \begin{bmatrix} \tau_h \\ v_h \end{bmatrix}.$$

We define the following seminorm and norms in $\mathfrak{Q} + \mathfrak{Q}_h$:

$$\begin{aligned} |\mathfrak{q}|_{DG}^2 &= \omega^2 \|\tau\|_{0,\Omega}^2 + \omega \|\beta^{1/2} [\![\tau]\!]_N\|_{0,\mathcal{F}_h^{\mathcal{I}}}^2 + \omega \|\alpha^{1/2} [\![v]\!]_N\|_{0,\mathcal{F}_h^{\mathcal{I}}}^2 \\ &\quad + \omega \|\delta^{1/2} \tau \cdot n\|_{0,\mathcal{F}_h^{\mathcal{B}}}^2 + \omega \|(1-\delta)^{1/2} v\|_{0,\mathcal{F}_h^{\mathcal{B}}}^2, \\ \|\mathfrak{q}\|_{DG}^2 &= |\mathfrak{q}|_{DG}^2 + \omega^2 \|v\|_{0,\Omega}^2, \\ \|\mathfrak{q}\|_{DG^+}^2 &= \|\mathfrak{q}\|_{DG}^2 + \omega \|\beta^{-1/2} \{\!\{v\}\!\}\|_{0,\mathcal{F}_h^{\mathcal{I}}}^2 + \omega \|\alpha^{-1/2} \{\!\{\tau\}\!\}\|_{0,\mathcal{F}_h^{\mathcal{I}}}^2 + \omega \|\delta^{-1/2} v\|_{0,\mathcal{F}_h^{\mathcal{B}}}^2. \end{aligned}$$

Multiply the first and second equations in (5) by $-i\omega$ and by $i\omega$, respectively, and add over all $K \in \mathcal{T}_h$, integrating the second term in the second equation by parts. Then, add the conjugate of the first equation to the second equation. Finally, replace $\widehat{u}_h$ and $\widehat{\sigma}_h$ with the numerical fluxes according to (6) and (7), and write the mixed PWDG method as follows: find $\mathfrak{p}_h \in \mathfrak{Q}_h$ such that, for all $\mathfrak{q}_h \in \mathfrak{Q}_h$,

$$\mathcal{A}_h(\mathfrak{p}_h, \mathfrak{q}_h) - \omega^2 (u_h, v_h) = (f, v_h) - \int_{\mathcal{F}_h^{\mathcal{B}}} \delta\, \tau_h \cdot n \overline{g} + \int_{\mathcal{F}_h^{\mathcal{B}}} (1-\delta)\, g \overline{v}_h. \tag{12}$$

Here, $\mathcal{A}_h(\cdot,\cdot)$ is the DG-bilinear form on $(\mathfrak{Q} + \mathfrak{Q}_h) \times (\mathfrak{Q} + \mathfrak{Q}_h)$ defined by

$$\begin{aligned} \mathcal{A}_h(\mathfrak{p}, \mathfrak{q}) = \omega^2 (\tau, \sigma) &+ i\omega \int_{\mathcal{F}_h^{\mathcal{I}}} \beta\, [\![\tau]\!]_N [\![\overline{\sigma}]\!]_N + i\omega \int_{\mathcal{F}_h^{\mathcal{B}}} \delta\, \tau \cdot n \overline{\sigma \cdot n} \\ &+ i\omega (\nabla_h \cdot \tau, u) - i\omega \int_{\mathcal{F}_h^{\mathcal{I}}} [\![\tau]\!]_N \{\!\{\overline{u}\}\!\} - i\omega \int_{\mathcal{F}_h^{\mathcal{B}}} (1-\delta)\, \tau \cdot n \overline{u} \\ &- i\omega (\nabla_h \cdot \sigma, v) + i\omega \int_{\mathcal{F}_h^{\mathcal{I}}} [\![\sigma]\!]_N \{\!\{\overline{v}\}\!\} + i\omega \int_{\mathcal{F}_h^{\mathcal{B}}} (1-\delta)\, \sigma \cdot n \overline{v} \\ &+ i\omega \int_{\mathcal{F}_h^{\mathcal{I}}} \alpha\, [\![u]\!]_N \cdot [\![\overline{v}]\!]_N + i\omega \int_{\mathcal{F}_h^{\mathcal{B}}} (1-\delta)\, u \overline{v}. \end{aligned}$$

Notice that

$$|\mathcal{A}_h(\mathfrak{q}_h, \mathfrak{q}_h)| \geq \frac{1}{\sqrt{2}} |\mathfrak{q}_h|_{DG}^2 \quad \forall \mathfrak{q}_h \in \mathfrak{Q}_h. \tag{13}$$

Moreover, the PWDG method (12) is consistent by construction, and thus

$$\mathcal{A}_h(\mathfrak{p}_h, \mathfrak{q}_h) = \mathcal{A}_h(\mathfrak{p}, \mathfrak{q}_h) - \omega^2 (u - u_h, v_h) \quad \forall \mathfrak{q}_h \in \mathfrak{Q}_h. \tag{14}$$

We develop the theoretical analysis of the method (12) by using Schatz' argument (see [14]). We start by stating the following abstract estimate.

Proposition 1. *Assume $0 < \delta < 1/2$. Denoting by Π_h the L^2–projection onto $\mathfrak{Q}_h$, we have*

$$\|\mathfrak{p} - \mathfrak{p}_h\|_{DG} \leq C_{\mathrm{abs}} \|\mathfrak{p} - \Pi_h \mathfrak{p}\|_{DG^+} + (\sqrt{2} + 1) \sup_{0 \neq w_h \in V_h} \frac{\omega\, |(u - u_h, w_h)|}{\|w_h\|_{0,\Omega}},$$

where $C_{\mathrm{abs}} > 0$ is a constant independent of ω and of the mesh size.

Proof. By the triangle inequality, for all $\mathfrak{q}_h \in \mathfrak{Q}_h$, it holds

$$\|\mathfrak{p} - \mathfrak{p}_h\|_{DG} \leq \|\mathfrak{p} - \mathfrak{q}_h\|_{DG} + \|\mathfrak{q}_h - \mathfrak{p}_h\|_{DG}.$$

From the definition of the DG–norm, (13) and (14), we get

$$\begin{aligned}
\|\mathfrak{q}_h - \mathfrak{p}_h\|_{DG}^2 &= |\mathfrak{q}_h - \mathfrak{p}_h|_{DG}^2 + \omega^2 \|v_h - u_h\|_{0,\Omega}^2 \\
&\leq \sqrt{2}\,|\mathcal{A}_h(\mathfrak{q}_h - \mathfrak{p}_h, \mathfrak{q}_h - \mathfrak{p}_h)| + \omega^2 (v_h - u_h, v_h - u_h) \\
&\leq \sqrt{2}\,|\mathcal{A}_h(\mathfrak{q}_h - \mathfrak{p}, \mathfrak{q}_h - \mathfrak{p}_h)| + \sqrt{2}\omega^2 |(u - u_h, v_h - u_h)| \\
&\quad + \omega^2 |(v_h - u, v_h - u_h)| + \omega^2 |(u - u_h, v_h - u_h)| \\
&= \sqrt{2}\,|\mathcal{A}_h(\mathfrak{q}_h - \mathfrak{p}, \mathfrak{q}_h - \mathfrak{p}_h)| + \omega^2 |(v_h - u, v_h - u_h)| \\
&\quad + (\sqrt{2}+1)\,\omega^2 |(u - u_h, v_h - u_h)|.
\end{aligned}$$

Now, select $\mathfrak{q}_h = \Pi_h \mathfrak{p}$, i.e., $\tau_h = \Pi_{\Sigma_h} \sigma$ and $v_h = \Pi_{V_h} u$, with Π_{Σ_h} and Π_{V_h} denoting the L^2–projections onto Σ_h and V_h, respectively. Since, as consequence of $\nabla \cdot \Sigma_h \subseteq V_h$,

$$\begin{aligned}
(\tau_h - \sigma_h, \Pi_{\Sigma_h} \sigma - \sigma) &= 0 \,, \\
\big(\nabla_h \cdot (\tau_h - \sigma_h), \Pi_{V_h} u - u\big) &= 0 \,,
\end{aligned}$$

and, integrating by parts and using $\big(\Pi_{\Sigma_h} \sigma - \sigma, \nabla_h (v_h - u_h)\big) = 0$,

$$\begin{aligned}
(\nabla_h \cdot (\Pi_{\Sigma_h} \sigma - \sigma), v_h - u_h) &= \int_{\mathcal{F}_h^{\mathcal{I}}} \{\{\Pi_{\Sigma_h} \sigma - \sigma\}\} \cdot [\![\overline{v_h - u_h}]\!]_N \\
&\quad + \int_{\mathcal{F}_h^{\mathcal{I}}} [\![\Pi_{\Sigma_h} \sigma - \sigma]\!]_N \{\{\overline{v_h - u_h}\}\} \\
&\quad + \int_{\mathcal{F}_h^{\mathcal{B}}} (\Pi_{\Sigma_h} \sigma - \sigma) \cdot n\, (\overline{v_h - u_h}) \,,
\end{aligned}$$

we immediately have

$$|\mathcal{A}_h(\mathfrak{q}_h - \mathfrak{p}, \mathfrak{q}_h - \mathfrak{p}_h)| \leq C\, \|\mathfrak{p} - \Pi_h \mathfrak{p}\|_{DG^+} \|\mathfrak{q}_h - \mathfrak{p}_h\|_{DG} \,,$$

where $C > 0$ is a constant independent of ω and of the mesh size (also independent of α and β; yet it may depend on δ). Moreover,

$$(v_h - u, v_h - u_h) = (\Pi_{V_h} u - u, v_h - u_h) = 0.$$

Therefore,

$$\begin{aligned}
\|\mathfrak{p} - \mathfrak{p}_h\|_{DG} &\leq (C+1) \|\mathfrak{p} - \Pi_h \mathfrak{p}\|_{DG^+} + (\sqrt{2}+1)\,\omega^2 \,\frac{|(u - u_h, v_h - u_h)|}{\|\mathfrak{p} - \mathfrak{p}_h\|_{DG}} \\
&\leq (C+1) \|\mathfrak{p} - \Pi_h \mathfrak{p}\|_{DG^+} + (\sqrt{2}+1)\,\omega^2 \,\frac{|(u - u_h, v_h - u_h)|}{\omega\, \|v_h - u_h\|_{0,\Omega}},
\end{aligned}$$

from which the result follows.

We bound the term $\sup_{0\neq w_h\in V_h} \dfrac{\omega\,|(u-u_h,w_h)|}{\|w_h\|_{0,\Omega}}$ in the estimate of Proposition (1) by a duality argument. To this end, we assume $0<\delta<1/2$.

We will make use of the following theorem proved in [13]. Its original statement makes use of the following weighted norm on $H^1(\Omega)$:

$$\|v\|^2_{1,\omega,\Omega} = |v|^2_{1,\Omega} + \omega^2\|v\|^2_{0,\Omega}\ . \tag{15}$$

Theorem 1. *[13, Propostion 8.1.4] Let Ω be a bounded convex domain (or smooth and star-shaped). Consider the adjoint problem to* (1) *with right-hand side $w\in L^2(\Omega)$:*

$$\begin{aligned} -\Delta\varphi-\omega^2\varphi &= w \qquad \text{in } \Omega,\\ -\nabla\varphi\cdot n+i\omega\,\varphi &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{16}$$

Then, $\varphi\in H^2(\Omega)$, and there are constants $C_1(\Omega), C_2(\Omega)>0$ such that

$$\begin{aligned} \|\varphi\|_{1,\omega,\Omega} &\le C_1(\Omega)\|w\|_{0,\Omega}\ ,\\ |\varphi|_{2,\Omega} &\le C_2(\Omega)\,(1+\omega)\,\|w\|_{0,\Omega}\ . \end{aligned} \tag{17}$$

As a consequence of Theorem 1, we have the following bounds:

$$|\varphi|_{2,\Omega} + \omega^2\|\varphi\|_{0,\Omega} \le C\,(1+\omega)\|w\|_{0,\Omega}\ , \tag{18}$$

and, setting $\Phi = \nabla\varphi/i\omega$,

$$\omega|\Phi|_{1,\Omega} + \omega^2\|\Phi\|_{0,\Omega} \le C\,(1+\omega)\|w\|_{0,\Omega}\ . \tag{19}$$

The next lemma summarizes the results in Propositions 4.12, 4.13, 4.14 and Lemma 5.6 of [8].

Lemma 1. *Let v be in $H^2(\Omega)$. Then,*

$$\begin{aligned} \|v-\Pi_{V_h}v\|_{0,\Omega} &\le C\,h^2(|v|_{2,\Omega}+\omega^2\|v\|_{0,\Omega})\ ,\\ |v-\Pi_{V_h}v|_{1,\Omega} &\le C\,h\,(\omega h+1)(|v|_{2,\Omega}+\omega^2\|v\|_{0,\Omega})\ ,\\ |\Pi_{V_h}v|_{2,\Omega} &\le C\,(\omega h+1)^2(|v|_{2,\Omega}+\omega^2\|v\|_{0,\Omega})\ ,\\ \|v-\Pi_{V_h}v\|_{0,\mathcal{F}_h} &\le C\,h^{3/2}(\omega h+1)^{1/2}\,(|v|_{2,\Omega}+\omega^2\|v\|_{0,\Omega})\ ,\\ \|\nabla_h(v-\Pi_{V_h}v)\|_{0,\mathcal{F}_h} &\le C\,h^{1/2}(\omega h+1)^{3/2}(|v|_{2,\Omega}+\omega^2\|v\|_{0,\Omega})\ , \end{aligned}$$

with a constant $C>0$ depending only on the bound for the minimal angle of elements and the domain Ω.

For functions which are only in $H^1(\Omega)$, we have the following bounds.

Lemma 2. *Let v be in $H^1(\Omega)$. Then,*

$$\begin{aligned} \|v-\Pi_{V_h}v\|_{0,\Omega} &\le C\,h\,\max\{\omega^{-1},h\}\,(\omega|v|_{1,\Omega}+\omega^2\|v\|_{0,\Omega})\ ,\\ |\Pi_{V_h}v|_{1,\Omega} &\le C\,(\omega h+1)\,\max\{\omega^{-1},h\}\,(\omega|v|_{1,\Omega}+\omega^2\|v\|_{0,\Omega})\ ,\\ \|v-\Pi_{V_h}v\|_{0,\mathcal{F}_h} &\le C\,h^{1/2}(\omega h+1)^{1/2}\max\{\omega^{-1},h\}\,(\omega|v|_{1,\Omega}+\omega^2\|v\|_{0,\Omega})\ , \end{aligned}$$

with a constant $C > 0$ depending only on the bound for the minimal angle of elements and the domain Ω. In particular, as soon as $\omega h < 1$,

$$\begin{aligned} \|v - \Pi_{V_h} v\|_{0,\Omega} &\le C\,\omega^{-1} h\,(\omega |v|_{1,\Omega} + \omega^2 \|v\|_{0,\Omega})\,, \\ |\Pi_{V_h} v|_{1,\Omega} &\le C\,\omega^{-1} (\omega h + 1)\,(\omega |v|_{1,\Omega} + \omega^2 \|v\|_{0,\Omega})\,, \\ \|v - \Pi_{V_h} v\|_{0,\mathcal{F}_h} &\le C\,\omega^{-1} h^{1/2} (\omega h + 1)^{1/2} (\omega |v|_{1,\Omega} + \omega^2 \|v\|_{0,\Omega})\,. \end{aligned}$$

Proof. The proof can be carried out by proceeding as in the proofs of Proposition 4.12, Proposition 4.13 and Lemma 5.6 of [8].

Proposition 2. *Let the assumptions of Theorem* 1 *hold true. With the choice of the flux parameters in* (8) *and $0 < \delta < 1/2$, the following estimate holds true:*

$$\sup_{0 \neq w_h \in V_h} \frac{\omega\,|(u - u_h, w_h)|}{\|w_h\|_{0,\Omega}} \le C_{\mathrm{dual}}\,\omega\,h\,(1+\omega)(\|\mathfrak{p} - \mathfrak{p}_h\|_{DG} + h\|f - \Pi_{V_h} f\|_{0,\Omega}),$$

with a constant $C_{\mathrm{dual}} > 0$ independent of the mesh and ω.

Proof. Consider the adjoint problem to (1) with right-hand side $w_h \in V_h$:

$$\begin{aligned} -\Delta\varphi - \omega^2 \varphi &= w_h \qquad \text{in } \Omega, \\ -\nabla\varphi \cdot n + i\omega\,\varphi &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{20}$$

Then, from Theorem 1, we have that $\varphi \in H^2(\Omega)$, $\|\varphi\|_{1,\omega,\Omega} \le C_1(\Omega)\|w_h\|_{0,\Omega}$ and $|\varphi|_{2,\Omega} \le C_2(\Omega)\,(1+\omega)\,\|w_h\|_{0,\Omega}$, with $C_1(\Omega), C_2(\Omega) > 0$.

Define $\Phi := \nabla\varphi / i\omega$; setting

$$\mathfrak{s} := \begin{bmatrix} \Phi \\ \varphi \end{bmatrix}, \qquad \mathfrak{t} := \begin{bmatrix} \Psi \\ \psi \end{bmatrix}, \qquad \mathfrak{t}_h := \begin{bmatrix} \Psi_h \\ \psi_h \end{bmatrix},$$

the solution $\mathfrak{s}$ satisfies

$$\mathcal{A}_h(\mathfrak{t}, \mathfrak{s}) - \omega^2 (\psi, \varphi) = (\psi, w_h) \quad \forall \mathfrak{t} \in \mathfrak{Q}.$$

The adjoint consistency of the DG method implies that

$$\mathcal{A}_h(\mathfrak{t}_h, \mathfrak{s}) - \omega^2 (\psi_h, \varphi) = (\psi_h, w_h) \quad \forall \mathfrak{t}_h \in \mathfrak{Q}_h.$$

Taking into account adjoint consistency and consistency, we have

$$\begin{aligned} (u - u_h, w_h) &= (u - w_h) - (u_h, w_h) \\ &= \mathcal{A}_h(\mathfrak{p} - \mathfrak{p}_h, \mathfrak{s}) - \omega^2 (u - u_h, \varphi) \\ &= \mathcal{A}_h(\mathfrak{p} - \mathfrak{p}_h, \mathfrak{s} - \mathfrak{t}_h) + \mathcal{A}_h(\mathfrak{p} - \mathfrak{p}_h, \mathfrak{t}_h) - \omega^2 (u - u_h, \varphi) \\ &= \mathcal{A}_h(\mathfrak{p} - \mathfrak{p}_h, \mathfrak{s} - \mathfrak{t}_h) + \omega^2 (u - u_h, \psi_h) - \omega^2 (u - u_h, \varphi) \\ &= \mathcal{A}_h(\mathfrak{p} - \mathfrak{p}_h, \mathfrak{s} - \mathfrak{t}_h) - \omega^2 (u - u_h, \varphi - \psi_h). \end{aligned} \tag{21}$$

From $i\omega\nabla\cdot\sigma+\omega^2 u=f$, we have the identity

$$i\omega(\nabla_h\cdot(\sigma-\sigma_h),\varphi-\psi_h)+\omega^2(u-u_h,\varphi-\psi_h)= \\ (f,\varphi-\psi_h)-(i\omega\nabla_h\cdot\sigma_h+\omega^2u_h,\varphi-\psi_h). \quad (22)$$

Moreover, integrating by parts and using the definition of σ, we obtain the identity

$$\begin{aligned} & i\omega(\nabla_h\cdot(\Phi-\Psi_h),u-u_h)-i\omega\int_{\mathcal{F}_h^{\mathcal{I}}}[\![\Phi-\Psi_h]\!]_N\{\{\overline{u-u_h}\}\} \\ & -i\omega\int_{\mathcal{F}_h^{\mathcal{B}}}(\Phi-\Psi_h)\cdot n(\overline{u-u_h}) \\ & =-\omega^2(\Phi-\Psi_h,\sigma-\sigma_h)-i\omega(\Phi-\Psi_h,i\omega\sigma_h-\nabla_hu_h) \\ & \quad +i\omega\int_{\mathcal{F}_h^{\mathcal{I}}}\{\{\Phi-\Psi_h\}\}\cdot[\![\overline{u-u_h}]\!]_N. \end{aligned} \quad (23)$$

Using (22) and (23), equation (21) becomes

$$\begin{aligned} (u-u_h,w_h)= & \, i\omega\int_{\mathcal{F}_h^{\mathcal{I}}}\beta\,[\![\Phi-\Psi_h]\!]_N\,[\![\overline{\sigma-\sigma_h}]\!]_N+i\omega\int_{\mathcal{F}_h^{\mathcal{B}}}\delta\,(\Phi-\Psi_h)\cdot n\,(\overline{\sigma-\sigma_h\cdot n}) \\ & -i\omega(\Phi-\Psi_h,i\omega\sigma_h-\nabla_hu_h)+i\omega\int_{\mathcal{F}_h^{\mathcal{I}}}\{\{\Phi-\Psi_h\}\}\cdot[\![\overline{u-u}_h]\!]_N \\ & +i\omega\int_{\mathcal{F}_h^{\mathcal{B}}}\delta\,(\Phi-\Psi_h)\cdot n\,(\overline{u-u_h})+i\omega\int_{\mathcal{F}_h^{\mathcal{I}}}[\![\sigma-\sigma_h]\!]_N\{\{\overline{\varphi-\psi_h}\}\} \\ & +i\omega\int_{\mathcal{F}_h^{\mathcal{B}}}(1-\delta)\,(\sigma-\sigma_h)\cdot n\,(\overline{\varphi-\psi_h}) \\ & +i\omega\int_{\mathcal{F}_h^{\mathcal{I}}}\alpha\,[\![u-u_h]\!]_N\cdot[\![\overline{\varphi-\psi_h}]\!]_N+i\omega\int_{\mathcal{F}_h^{\mathcal{B}}}(1-\delta)\,(u-u_h)(\overline{\varphi-\psi_h}) \\ & -(f,\varphi-\psi_h)+(i\omega\nabla_h\cdot\sigma_h+\omega^2u_h,\varphi-\psi_h). \end{aligned}$$

Form the Cauchy-Schwarz inequality, since $0<\delta<1/2$, we have

$$\begin{aligned} \omega|(u-u_h,w_h)|\leq & \, C\,\|\mathfrak{p}-\mathfrak{p}_h\|_{DG}\,\omega^{3/2}\Big(\|\beta^{1/2}[\![\Phi-\Psi_h]\!]_N\|_{0,\mathcal{F}_h^{\mathcal{I}}} \\ & \quad +\|\alpha^{-1/2}\{\{\Phi-\Psi_h\}\}\|_{0,\mathcal{F}_h^{\mathcal{I}}}+\|\delta^{1/2}(\Phi-\Psi_h)\cdot n\|_{0,\mathcal{F}_h^{\mathcal{B}}} \\ & \quad +\|\alpha^{1/2}[\![\varphi-\psi_h]\!]_N\|_{0,\mathcal{F}_h^{\mathcal{I}}}+\|\beta^{-1/2}\{\{\varphi-\psi_h\}\}\|_{0,\mathcal{F}_h^{\mathcal{I}}} \\ & \quad +\|\delta^{-1/2}\varphi-\psi_h\|_{0,\mathcal{F}_h^{\mathcal{B}}}\Big) \\ & +\omega\,|(f,\varphi-\psi_h)|+\omega^2|(\Phi-\Psi_h,i\omega\sigma_h-\nabla_hu_h)| \\ & +\omega|(i\omega\nabla_h\cdot\sigma_h+\omega^2u_h,\varphi-\psi_h)|. \end{aligned}$$

We choose $\psi_h=\Pi_{V_h}\varphi$ and $\Psi_h=\Pi_{\Sigma_h}\Phi$. We immediately have

$$\begin{aligned} & \omega^2|(\Phi-\Psi_h,i\omega\sigma_h-\nabla_hu_h)|=0\,, \\ & \omega|(i\omega\nabla_h\cdot\sigma_h+\omega^2u_h,\varphi-\psi_h)|=0\,, \end{aligned}$$

and, from Lemma 1 and (18),

$$\omega\,|(f,\varphi-\psi_h)| = \|f-\Pi_{V_h}f\|_{0,\Omega}\,\omega\|\varphi-\psi_h\|_{0,\Omega}$$
$$\leq C\,\omega\,h^2(1+\omega)\,\|f-\Pi_{V_h}f\|_{0,\Omega}\,\|w_h\|_{0,\Omega}\ .$$

We estimate all the interelement terms containing $(\varphi-\psi_h)$ and those containing $(\Phi-\Psi_h)$ by using Lemma 1 and (18), and Lemma 2 and (19), respectively. Taking the definitions of the flux parameters into account, we obtain

$$\omega^{3/2}(\text{interelement terms}) \leq C\,\omega\,h\,(1+\omega)\,\|w_h\|_{0,\Omega}\ .$$

The result readily follows.

The following estimate of the L^2–projection error of $\mathfrak{p}$ is a consequence of Lemma 1 and Lemma 2.

Lemma 3. *For any* $\mathfrak{p}\in H^2(\Omega)\times H^1(\Omega)$*, as soon as* $\omega h<1$*, we have*

$$\|\mathfrak{p}-\Pi_h\mathfrak{p}\|_{DG^+} \leq C\,h\,\left(|u|_{2,\Omega}+\omega^2\|u\|_{0,\Omega}+\omega|\sigma|_{1,\Omega}+\omega^2\|\sigma\|_{0,\Omega}\right).$$

The complete error estimate is a straightforward consequence of Proposition 1, Proposition 2 and Lemma 3.

Theorem 2. *Let the assumptions of Theorem* 1 *hold true and assume that the analytical solution to* (1) *belongs to* $H^2(\Omega)$*. With the choice of the flux parameters in* (8) *and* $0<\delta<1/2$*, provided that* $\omega h<1$ *and*

$$\omega\,h\,(1+\omega) < \frac{1}{(\sqrt{2}+1)\,C_{\text{dual}}}\ , \tag{24}$$

the following estimate holds true:

$$\|\mathfrak{p}-\mathfrak{p}_h\|_{DG} \leq C\,h\,(|u|_{2,\Omega}+\omega^2\|u\|_{0,\Omega}+\omega|\sigma|_{1,\Omega}+\omega^2\|\sigma\|_{0,\Omega}$$
$$+\,\omega\,h\,(1+\omega)\|f-\Pi_{V_h}f\|_{0,\Omega})\ ,$$

with a constant $C>0$ *independent of of the mesh and* ω*.*

Remark 2. In the relevant case of $\omega>1$, in order to satisfy the threshold condition (24), we need to require $\omega^2 h$ to be sufficiently small, instead of the milder condition $\omega\,h$ sufficiently small, as required for the best approximation estimates. This reflects the fact that, like for the PWDG methods of [8], the mixed PWDG methods also suffer from a *pollution effect*.

Remark 3. Like for the PWDG methods of [8], the presence of a source term $f\neq 0$ prevents the methods from being higher order convergent when increasing the number of elemental plane waves used in the approximation.

Remark 4. By proceeding like in the proof of Theorem 5.13 of [8], one can prove that, under the threshold conditions of Theorem 2,

$$\|u-u_h\|_{0,\Omega} \leq C\,h^{3/2}\,(|u|_{2,\Omega}+\omega^2\|u\|_{0,\Omega}+\omega|\sigma|_{1,\Omega}+\omega^2\|\sigma\|_{0,\Omega}$$
$$+\,\omega\,h\,(1+\omega)\|f-\Pi_{V_h}f\|_{0,\Omega})\ ,$$

with a constant $C>0$ independent of the mesh and ω.

4 Conclusion

The *h*-version of the plane wave discontinuous Galerkin method has been shown to converge asymptotically optimally. However, as all other local discretizations of the Helmholtz equation, this method is also affected by numerical dispersion. This is reflected by a threshold condition of the form "$\omega^2 h$ sufficiently small" for the onset of asymptotic convergence. For a primal plane wave DG method, numerical experiments in [8] demonstrate that this condition is essential. There is no reason to believe that the mixed method analyzed in this paper behaves differently.

Yet, in practical applications of the UWVF one rather tries to raise the number of plane waves than to refine the mesh. Hence, it is the *p*-version of the plane wave DG method that deserves more attention than the *h*-version.

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